Eigenvalues problems in inverse scattering theory

Samuel Cogar

Department of Mathematical Sciences
University of Delaware

Advisors: David Colton and Peter Monk
Email: cogar@udel.edu
Website: sites.udel.edu/cogar

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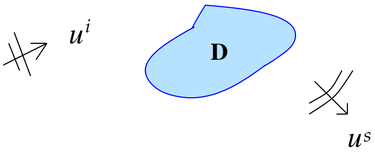
Outline

Scattering by an inhomogeneous medium



$$\begin{aligned} \Delta u^s + k^2 u^s &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \nabla \cdot A \nabla u + k^2 n u &= 0 && \text{in } D, \\ u &= u^s + u^i && \text{on } \partial D, \\ \frac{\partial u}{\partial \nu_A} &= \frac{\partial (u^s + u^i)}{\partial \nu} && \text{on } \partial D, \\ \lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) &= 0, && r = |x| . \end{aligned}$$

- A is a 3×3 matrix function with $L^\infty(D)$ entries which represents the anisotropic properties of the material, $\Re(A)$ positive-definite, $\Im(A)$ nonpositive
- $n \in L^\infty(D)$ is the index of refraction, $\Re(n) > 0$, $\Im(n) \geq 0$
- u^i is an incident field which satisfies the Helmholtz equation in $\mathbb{R}^3$ (except for possibly at one point)

The scattered field u^s

- The Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

is assumed to hold uniformly in all directions, and it ensures that the scattered field u^s is outgoing rather than incoming.

- For the plane wave incident field $u^i(x) = e^{ikx\cdot d}$ with $|d| = 1$, the scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}, d) + \mathcal{O}\left(\frac{1}{|x|^2}\right)$$

as $|x| \to \infty$, where $\hat{x} := x/|x|$ and $u_\infty(\hat{x}, d)$ is the **far field pattern**.

The far field operator

- We define the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d) ds(d), \ \hat{x} \in \mathbb{S}^2.$$

- With the Herglotz wave function v_g defined by

$$v_g(x) := \int_{\mathbb{S}^2} e^{ikx \cdot d} g(d) ds(d), \ x \in \mathbb{R}^3,$$

  linearity of the scattering problem implies that Fg is the far field pattern for the incident field $u^i = v_g$.

- The operator F is compact and has infinitely many eigenvalues.

Nondestructive testing of materials

- A problem in the field of nondestructive testing is to detect a flaw in a material using only its measured scattering data $u_\infty(\hat{x}, d)$ for some observation directions $\hat{x}$ and incident directions d.
- **Theorem:** If $A = I$, then the refractive index n is uniquely determined by a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \mathbb{S}^2$ and a fixed wave number k.

D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd edition, Springer, New York, 2013.

- It has been shown that there exist anisotropic materials ($A \neq I$) for which the far field data does not uniquely determine A and n!

F. Gylys-Colwell, An inverse problem for the Helmholtz equation, *Inverse Problems*, 1996.

Outline

Target signatures

- Since we only wish to detect a *change* in the material properties, we use the idea of a *target signature*: a set of numbers which corresponds to a material.
- A good target signature satisfies the following properties:
    1. the target signature should exist for any given material;
    2. one should be able to compute the target signature from scattering data;
    3. minimal data collection should be required, preferably for a single fixed frequency.
- With a good choice of target signature, we hope to infer changes in a material compared to some reference configuration from shifts in the target signature.

Scattering resonances

- Values of the wave number k for which there exist nontrivial solutions of the scattering problem corresponding to $u^i = 0$ are called *scattering resonances*.
- The study of scattering resonances has produced beautiful mathematics and provided insight into both direct and inverse scattering theory.

S. Dyatlov and M. Zworski, *Mathematical Theory of Scattering Resonances*, in preparation.

- However, every resonance has negative imaginary part and hence their detection from far field data becomes problematic.

Transmission eigenvalues

Theorem: The far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is injective with dense range if and only if there does not exist a nontrivial solution to the *transmission eigenvalue problem* of finding $w, v \in H^1(D)$ satisfying

$$\begin{aligned} \nabla \cdot A \nabla w + k^2 n w &= 0 \quad \text{in } D, \\ \Delta v + k^2 v &= 0 \quad \text{in } D, \\ w - v &= 0 \quad \text{on } \partial D, \\ \frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial D, \end{aligned}$$

such that v is a Herglotz wave function.

Definition: A value of the wave number $k \in \mathbb{C}$ for which this problem has nontrivial solutions is called a *transmission eigenvalue*.

Transmission eigenvalues

- A transmission eigenvalue may be viewed as a value of the wave number for which no scattering occurs for special incident fields.
- Transmission eigenvalues have received considerable attention since their introduction by Kirsch (1986) and Colton and Monk (1988). A great deal of effort has been spent in studying their discreteness, existence, and distribution in the complex plane.

F. Cakoni, D. Colton, and H. Haddar, *Inverse Scattering Theory and Transmission Eigenvalues*, SIAM, Philadelphia, 2016.

- However, only real transmission eigenvalues may be detected from far field data, no real transmission eigenvalues exist for absorbing media, and the detection of transmission eigenvalues requires collecting data for multiple frequencies in a predetermined range.

A modified far field operator

- Many of the problems faced by using scattering resonances or transmission eigenvalues involve their relationship to the physical parameter of frequency.
- We can overcome this issue by modifying the far field operator and generating new eigenvalue problems whose eigenparameters are entirely artificial.
- We consider an *auxiliary scattering problem* depending on a parameter η with scattered field u_0^s, and we let $F_0 : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ be the corresponding *auxiliary far field operator*.
- We construct the *modified far field operator* as $\mathcal{F} := F - F_0$, which may be written explicitly as

$$(\mathcal{F}g)(\hat{x}) = \int_{\mathbb{S}^2} \left[u_\infty(\hat{x}, d) - u_{0,\infty}(\hat{x}, d) \right] g(d) ds(d), \ \hat{x} \in \mathbb{S}^2.$$

Modified transmission eigenvalues

We choose some $\gamma > 0$, and we consider the transmission auxiliary problem

$$
\begin{aligned}
\Delta u_0^s + k^2 u_0^s &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \\
\frac{1}{\gamma}\Delta u_0 + k^2 \eta u_0 &= 0 \quad \text{in } B, \\
u_0 - (u_0^s + u^i) &= 0 \quad \text{on } \partial B, \\
\frac{1}{\gamma}\frac{\partial u_0}{\partial \nu} - \frac{\partial (u_0^s + u^i)}{\partial \nu} &= 0 \quad \text{on } \partial B, \\
\lim_{r\to\infty} r\left(\frac{\partial u_0^s}{\partial r} - iku_0^s\right) &= 0 \quad r = |x|\,.
\end{aligned}
$$

Modified transmission eigenvalues

In this case, the modified far field operator $\mathcal{F}$ is injective with dense range provided that there exist no nontrivial solutions $(w, v) \in H^1(B) \times H^1(B)$ of the *modified transmission eigenvalue problem*

$$\begin{aligned} \nabla \cdot A \nabla w + k^2 n w &= 0 \quad \text{in } B, \\ \frac{1}{\gamma} \Delta v + k^2 \eta v &= 0 \quad \text{in } B, \\ w - v &= 0 \quad \text{on } \partial B, \\ \frac{\partial w}{\partial \nu} - \frac{1}{\gamma} \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial B. \end{aligned}$$

Definition: A value of η for which there exist nontrivial solutions of this problem is called a *modified transmission eigenvalue*.

Modified transmission eigenvalues

- The modified transmission eigenvalue problem possesses many desirable properties, including discreteness, existence even for complex-valued $n \in C^\infty(\overline{B})$, and the ability to detect eigenvalues using only data for a single $k > 0$.
- For $A = I$, the choice $\gamma \neq 1$ is sufficient for the modified transmission eigenvalue problem to be of Fredholm type.
- The choice of γ can dramatically affect the sensitivity of the eigenvalues to changes in n, in some cases improving the sensitivity by an order of magnitude.

S. C., D. Colton, S. Meng, and P. Monk, Modified transmission eigenvalues in inverse scattering theory, *Inverse Problems*, 2017.

A variational formulation

Find $(w, v) \in \mathcal{H} := \{(\varphi, \psi) \in H^1(B) \times H^1(B) \mid \varphi - \psi \in H_0^1(B)\}$ which satisfies

$$(A\nabla w, \nabla w')_B - \frac{1}{\gamma}(\nabla v, \nabla v')_B - k^2(nw, w')_B + k^2\eta(v, v')_B = 0$$

for all $(w', v') \in \mathcal{H}$.

A variational formulation

Find $(w,v) \in \mathcal{H} := \{(\varphi,\psi) \in H^1(B) \times H^1(B) \mid \varphi - \psi \in H^1_0(B)\}$ which satisfies

$$\Big[(A\nabla w, \nabla w')_B - \frac{1}{\gamma}(\nabla v, \nabla v')_B + k^2\alpha(w,w')_B - k^2\beta(v,v')_B\Big] + \Big[-k^2((n+\alpha)w,w')_B + k^2(\eta+\beta)(v,v')_B\Big] = 0$$

for all $(w',v') \in \mathcal{H}$. If $\hat{\mathbb{A}}, \mathbb{B}_\eta : \mathcal{H} \to \mathcal{H}$ represent the first and second bracketed expressions, respectively, then we arrive at the operator equation

$$(\hat{\mathbb{A}} + \mathbb{B}_\eta)(w,v) = 0.$$

Theorem: The operator $\mathbb{B}_\eta$ is compact. Under certain conditions on γ, the operator $\hat{\mathbb{A}}$ is invertible.

Generalized linear sampling method (GLSM)

- We define the GLSM cost functional

$$J_\alpha(g) := \alpha \left|(F_0 g, g)_{L^2(\mathbb{S}^2)}\right| + \|\mathcal{F}g - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)}$$

and we let $\{g_z^\alpha\}$ be the minimizing sequence satisfying

$$J_\alpha(g_z^\alpha) \leq \left(\inf_{g \in L^2(\mathbb{S}^2)} J_\alpha(g)\right) + C\alpha.$$

Theorem: $(F_0 g_z^\alpha, g_z^\alpha)_{L^2(\mathbb{S}^2)}$ is bounded as $\alpha \to 0$ if and only if η is not a modified transmission eigenvalue.

Note: In practice we use a regularized cost function $J_\alpha^\delta(\cdot)$ and minimize it using a suitable optimization scheme.

L. Audibert, F. Cakoni, H. Haddar, New sets of eigenvalues in inverse scattering for inhomogeneous media and their determination from scattering data, *Inverse Problems*, 2017.

Outline

The test domain

- We perform tests in two dimensions using an L-shaped domain with $n = 4$ unless otherwise stated.
- We choose B to be the ball of radius 1.5 centered at the origin.
- We test both $\gamma = 0.5$ and $\gamma = 2$.

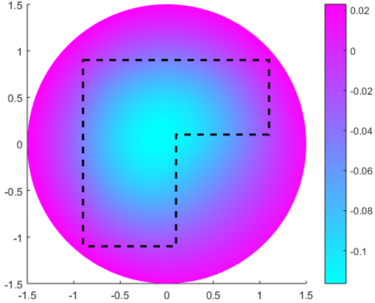


Figure: Eigenfunction corresponding to $\eta = -12.3088$

Sensitivity of the eigenvalues to changes in the location of a circular flaw

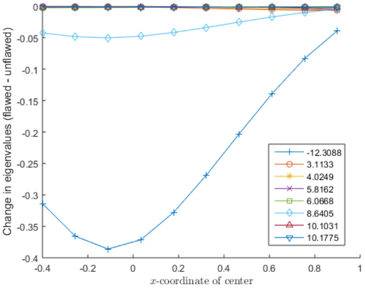


Figure: $\gamma = 0.5$

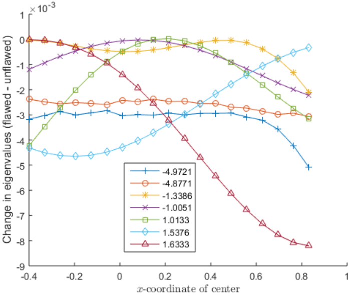


Figure: $\gamma = 2$

Sensitivity of the eigenvalues to changes in the size of a circular flaw

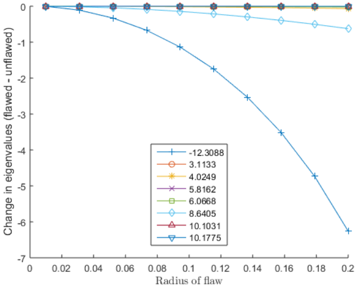


Figure: $\gamma = 0.5$

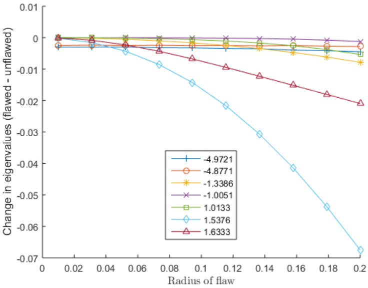


Figure: $\gamma = 2$

Detecting eigenvalues from far field data

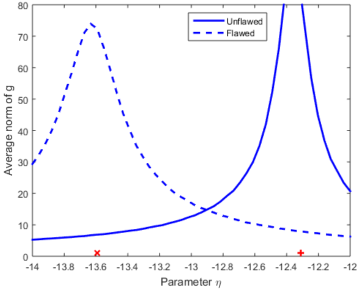


Figure: $\gamma = 0.5$, $n = 4$

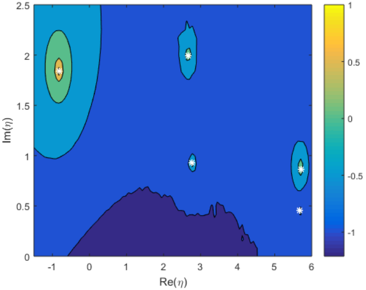


Figure: $\gamma = 0.5$, $n = 4 + 4i$

Outline

Related Work

$$\nabla \cdot A\nabla w + k^2 n w = 0, \quad \gamma\Delta v + k^2\eta v \text{ in } \mathbb{R}^3 \setminus \overline{B}$$
$$w - v = 0, \quad \frac{\partial w}{\partial \nu_A} - \gamma\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B$$
$$\lim_{r\to\infty} r\left(\frac{\partial w}{\partial r} - ikw\right) = 0, \quad \lim_{r\to\infty} r\left(\frac{\partial v}{\partial r} - ikv\right) = 0$$

S. C., D. Colton, and P. Monk, Using eigenvalues to detect anomalies in the exterior of a cavity, *Inverse Problems* (accepted).

$$\Delta w + k^2 w = 0 \text{ in } B \setminus \overline{\Gamma}, \quad \gamma\Delta v + k^2\eta v = 0 \text{ in } B,$$
$$w^- = 0, \quad \frac{\partial w^+}{\partial \nu} + i\sigma w^+ = 0 \text{ on } \Gamma,$$
$$w - v = 0, \quad \frac{\partial w}{\partial \nu} - \gamma\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B$$

S. C., A modified transmission eigenvalue problem for scattering by a partially coated crack, *under review*.

Future Work

- Develop a modified transmission eigenvalue problem for electromagnetic scattering (and possibly elastic scattering)
  - For a constant permittivity ϵ with contrast supported in the unit ball, the electromagnetic modified transmission eigenvalues accumulate at both $+\infty$ and ϵ. No compactness!
- Design more exotic auxiliary problems which may improve detection of eigenvalues with limited aperture data or for non-smooth domains
- Investigate the precise effect of changes in a material on the eigenvalues