Field patterns: A new type of wave

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Joint work with Graeme W. Milton

Waves and Particles in Random Media: Theory and Applications

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Formulation of the problem

- Generic wave equation:

$$\frac{\partial}{\partial x}\left(\alpha(x,t)\frac{\partial u(x,t)}{\partial x}\right)-\frac{\partial}{\partial t}\left(\beta(x,t)\frac{\partial u(x,t)}{\partial t}\right)=0$$

- Initial conditions:

$$u(x,0)=g(x)$$
$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}=f(x)$$

- Boundary conditions: The medium is infinite in the x–direction

The coefficients are time – dependent $\rightarrow$ DYNAMIC MATERIALS [see, e.g., Lurie, 2007]

Realization of dynamic materials

- Liquid crystals
- Ferroelectric, ferromagnetic materials
- Pump wave + small amplitudes waves [e.g. Louisell & Quate (1958)]
- Trasmission line with modulated inductance [e.g. Cullen (1958)]
- Experiments and more references in [Honey & Jones (1958)]
- Dynamic modulation + photons [e.g. Fang et al. (2012), Boada et al. (2012), Celi et al. (2014), Yuan et al. (2016)]

...

- Walking droplets [e.g. Couder et al. (2005), Couder & Fort (2006), Bush (2015)]
- Time reversal [e.g. Fink (2016), Goussev et al. (2016)]

2D Conductivity problem for a two-component composite

$$\mathbf{j}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \text{where} \quad \nabla \cdot \mathbf{j} = 0, \quad \mathbf{e} = -\nabla V,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{pmatrix} \alpha(\mathbf{x}) & 0 \\ 0 & -\beta(\mathbf{x}) \end{pmatrix}, \quad \begin{matrix} \text{material } 1 & \rightarrow & \alpha_1, \beta_1 \\ \text{material } 2 & \rightarrow & \alpha_2, \beta_2 \end{matrix}$$

$$\boxed{\frac{\partial}{\partial x_1}\left(\alpha(x_1, x_2)\frac{\partial V(x_1, x_2)}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\beta(x_1, x_2)\frac{\partial V(x_1, x_2)}{\partial x_2}\right) = 0}$$

N.B. Hyperbolic materials!! [see, e.g., the review by Poddubny, Iorsh, Belov, Kivshar, 2013]

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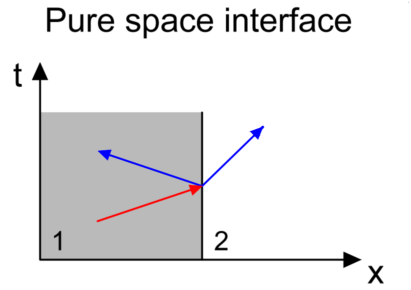
$$\boxed{\alpha_i \frac{\partial^2 V_i}{\partial x^2} - \beta_i \frac{\partial^2 V_i}{\partial t^2} = 0, \qquad i = 1, 2}$$

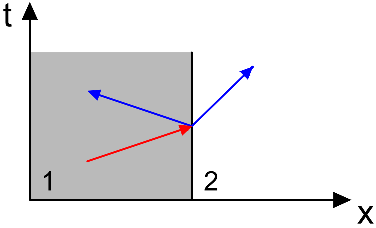
D′Alembert solution : $\quad V_i(x, t) = V_i^+(x - c_i t) + V_i^-(x + c_i t) \qquad c_i = \sqrt{\dfrac{\alpha_i}{\beta_i}}$

Dynamic composites



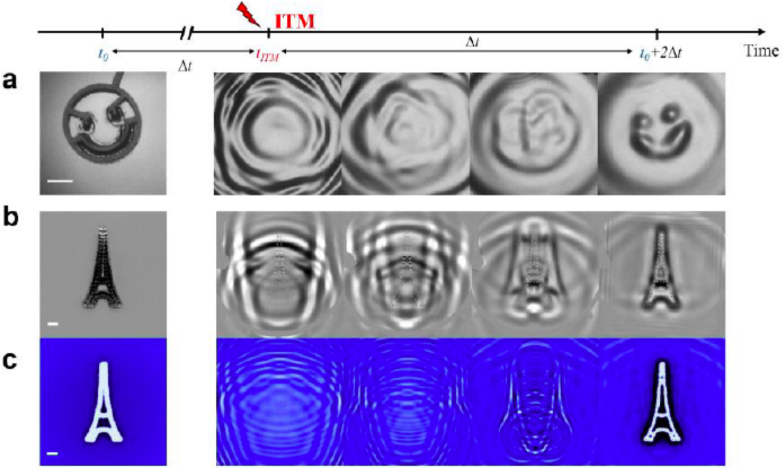
Dynamic composites

Pure space interface



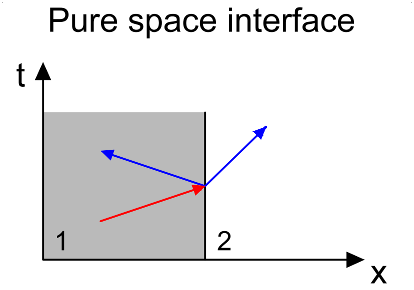
Pure time interface

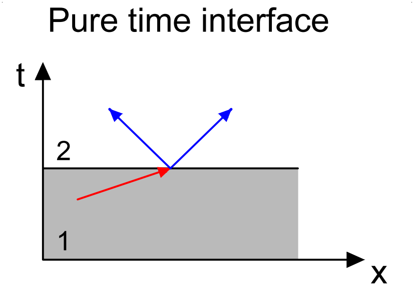
What happens at a time-interface?



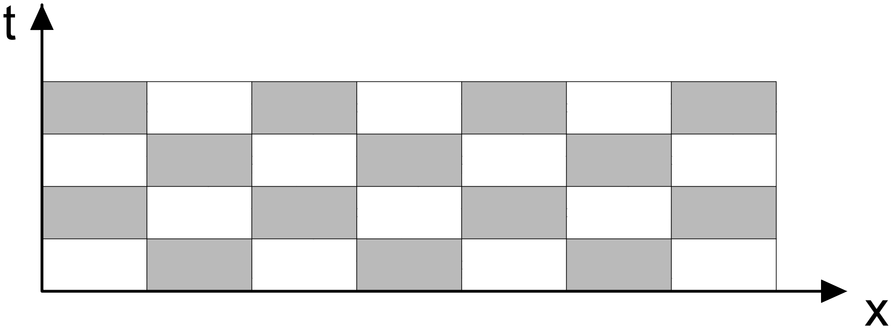
[Bacot, Labousse, Eddi, Fink, and Fort, *Nature*, 2016]

Dynamic composites





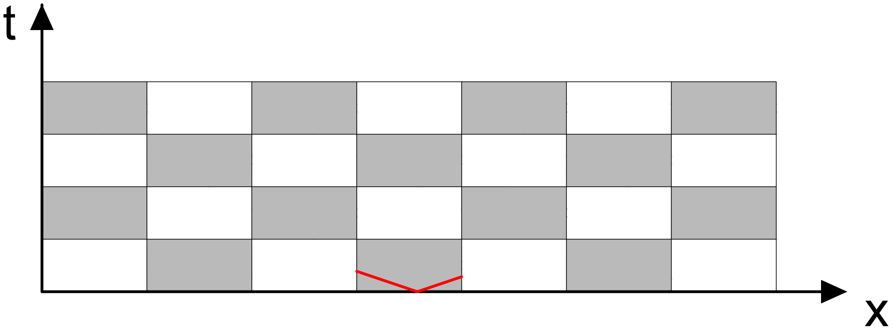
Evolution of a disturbance in a space-time checkerboard



Initial conditions:

$$\begin{cases} V(x,0) = H(x-a) \\ j_t(x,0) = j_0\delta(x-a) \end{cases}$$

Evolution of a disturbance in a space-time checkerboard



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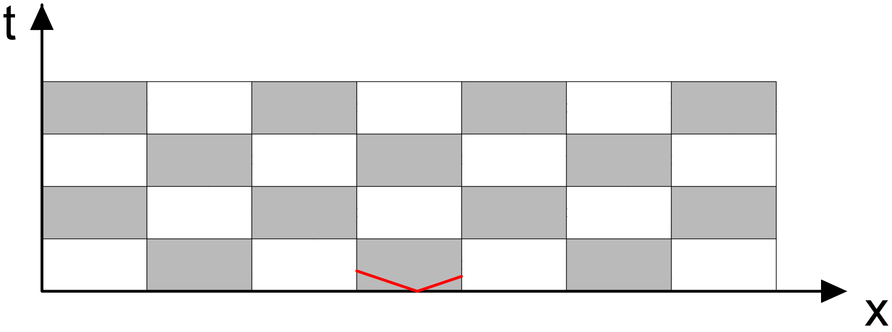

Transmission conditions:

$$\begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \sigma_1 \nabla V_1 = \mathbf{n} \cdot \sigma_2 \nabla V_2 \end{cases}$$

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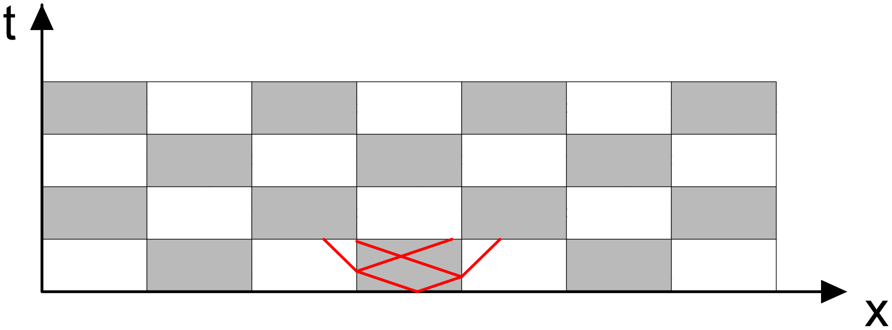

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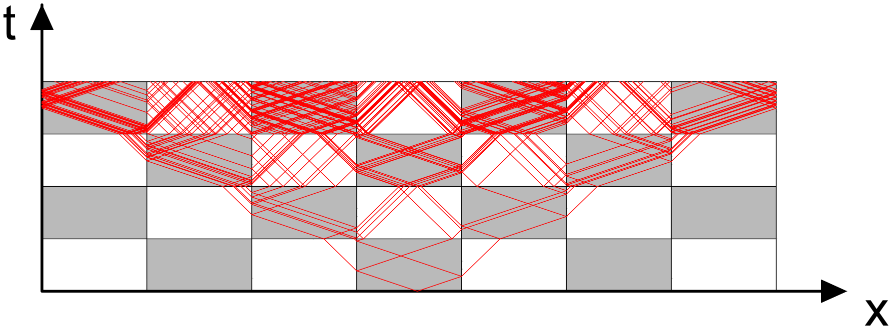

Transmission conditions:

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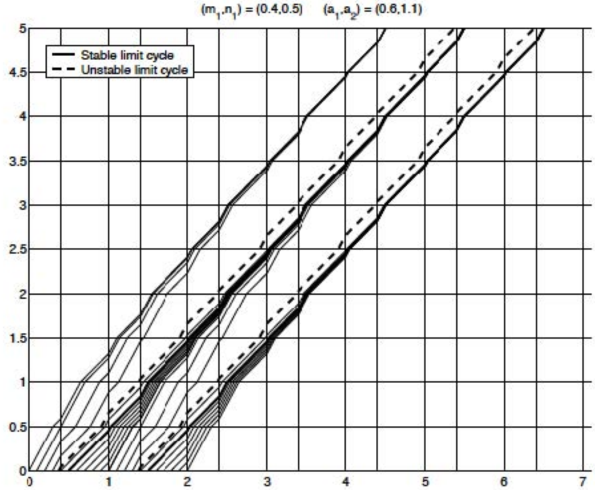
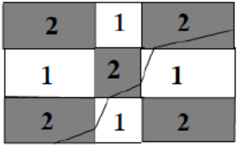
How to avoid this complicated cascade?



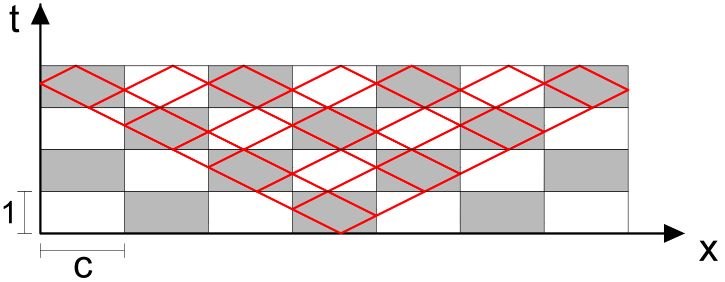


[Lurie, Onofrei, Weeks, 2016]

A bit like a shock but in a linear medium!!

Field patterns in a space-time checkerboard

$$c_1 = c_2 = c \quad \Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$



Field patterns are a new type of wave propagating along orderly patterns of characteristic lines!!!

Families of field patterns

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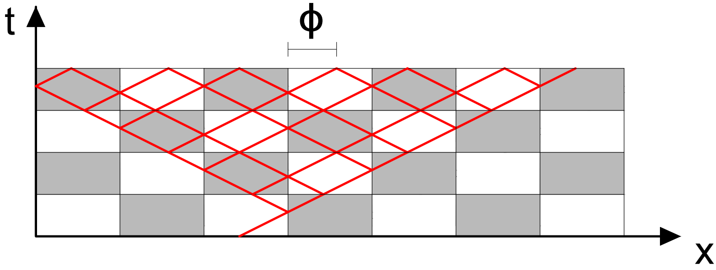

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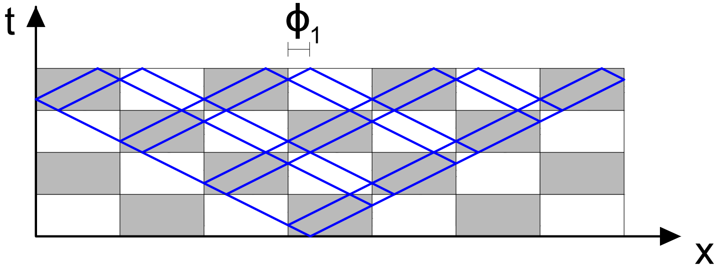
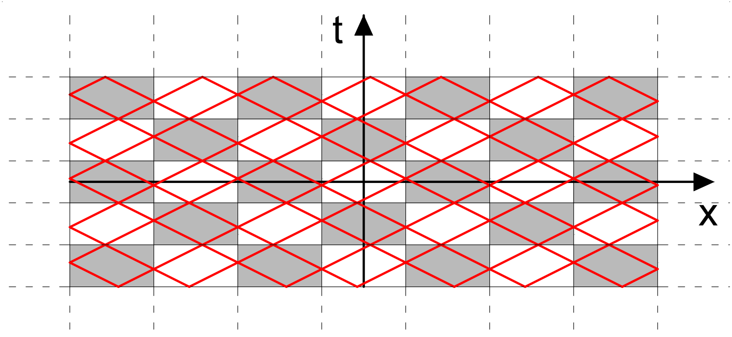


Field patterns are a new type of wave propagating along orderly patterns of characteristic lines!!!

PT–symmetry of field patterns

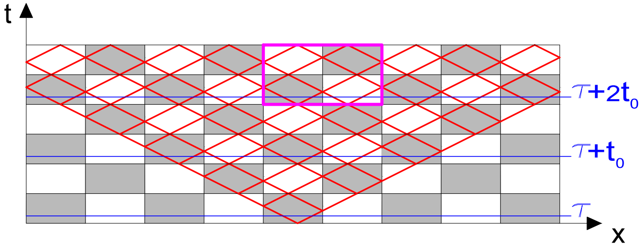


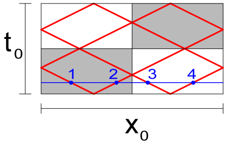
[Quantum physics, e.g., Bender and Boettcher, 1998,
Optics, e.g., Zyablovsky et al., 2014]

Unbroken PT–symmetry → real eigenvalues

Broken PT–symmetry → complex conjugate eigenvalues

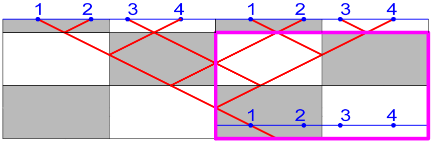
The transfer matrix





$$j(k,m,n+1) = \sum_{k',m'} T_{(k,m),(k,'m')} j(k',m',n)$$

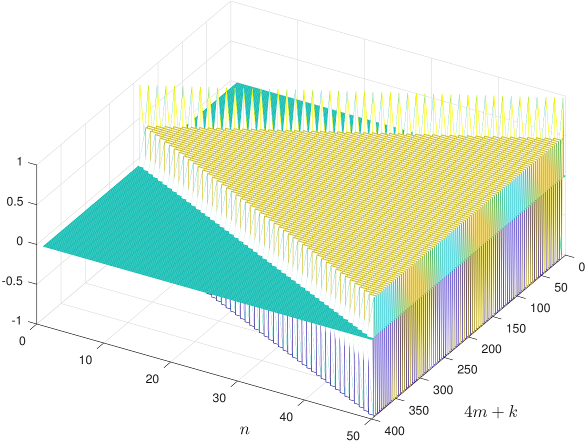
$$T_{(k,m),(k,'m')} = G_{k,k'}(m-m')$$



T depends only on γ_i

T is PT–symmetric

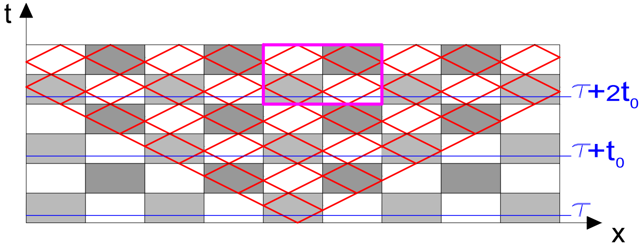
Evolution in time of the current distribution

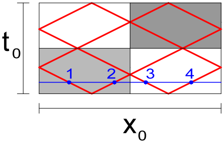


Oscillations continue after the wavefront!!!

Three–component space–time checkerboard

$$c_1 = c_2 = c_3$$

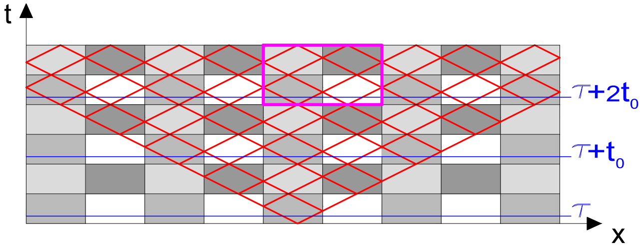


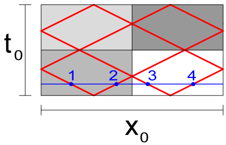


UNBROKEN PT–symmetry $\Rightarrow$ Only propagating modes

Four–component space–time checkerboard

$$c_1 = c_2 = c_3 = c_4$$

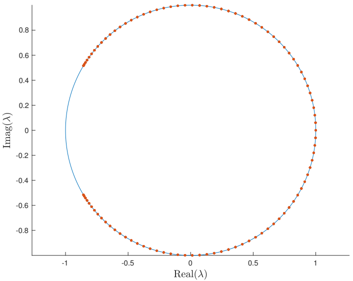


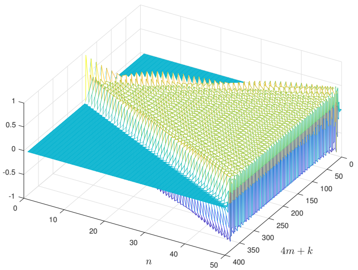


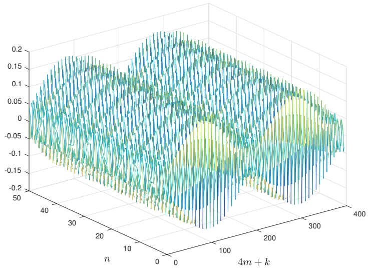
For some combinations of γ_1, γ_2, γ_3, γ_4 : UNBROKEN PT–symmetry

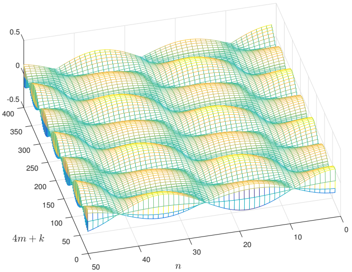
For other combinations of γ_1, γ_2, γ_3, γ_4 : BROKEN PT–symmetry

Unbroken PT–symmetry for the four–phase checkerboard

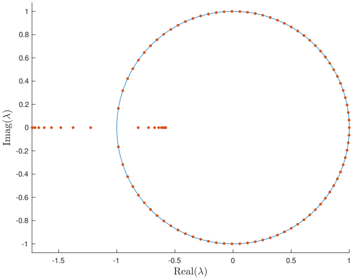


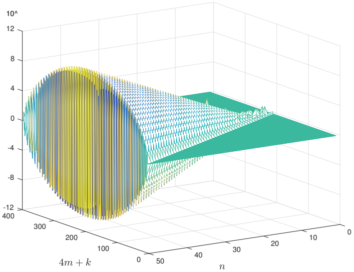






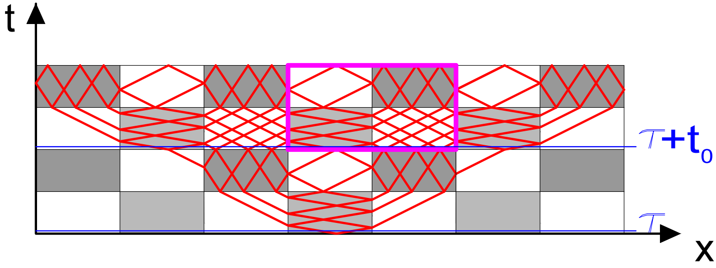
Broken PT–symmetry for the four–phase checkerboard





Three–phase space–time checkerboard

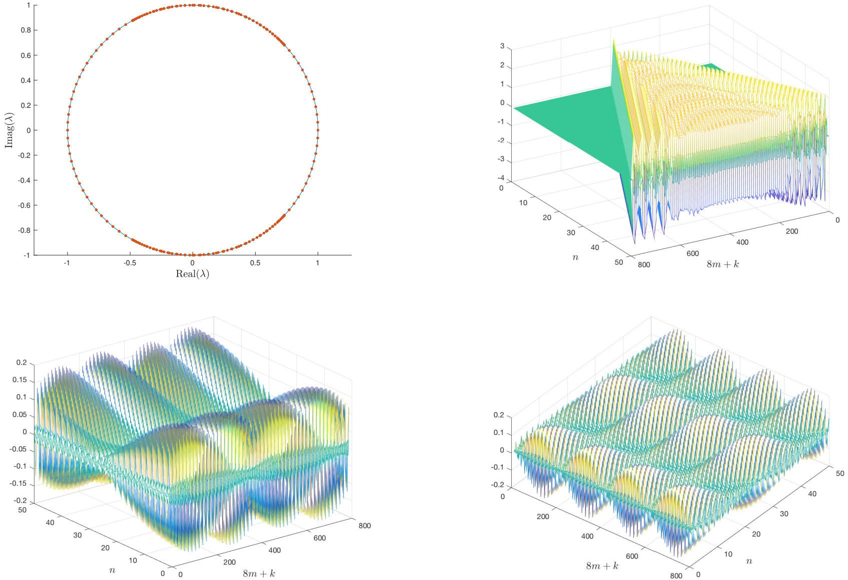
$$c_2/c_1 = c_1/c_3 = 3$$



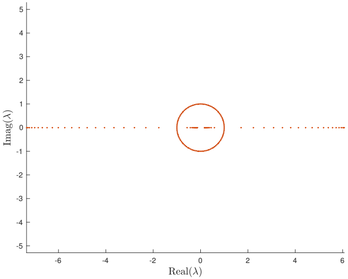
For some combinations of γ_1, γ_2, γ_3 : UNBROKEN PT–symmetry

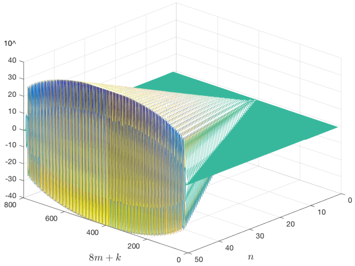
For other combinations of γ_1, γ_2, γ_3 : BROKEN PT–symmetry

Unbroken PT–symmetry for the three–phase checkerboard



Broken PT–symmetry for the three–phase checkerboard

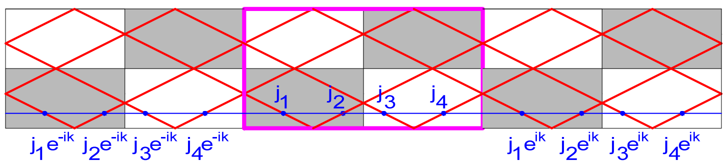




Bloch–Floquet theory applied to field patterns

Periodicity with respect to x:

$$j(l, m+s, n) = \exp(\mathrm{i}ks)\, j(l, m, n)$$



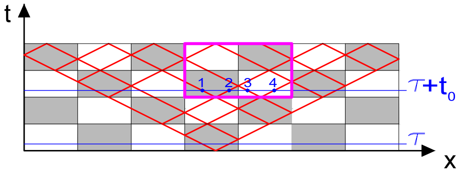
Periodicity with respect to t:

$$j(l, m, n+q) = \exp(\mathrm{i}\,\omega\, q) j(l, m, n)$$

Recall: $j(l, m, n+q) = \lambda^q\, j(l, m, n)$, then

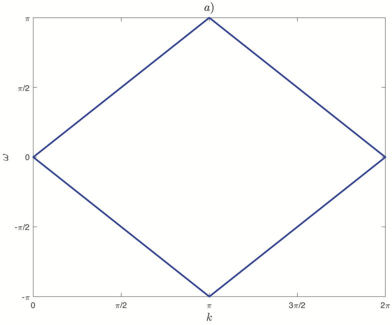
Dispersion relation : $\boxed{\lambda(k) = \exp(\mathrm{i}\,\omega)}$

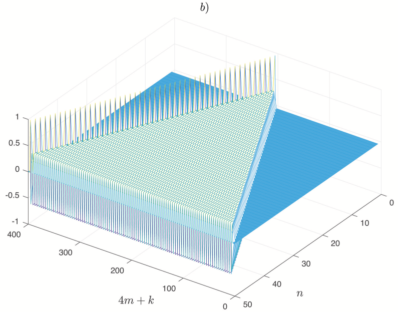
Dispersion diagram for the two–phase checkerboard



$\lambda(k) = \exp(\pm ik)$

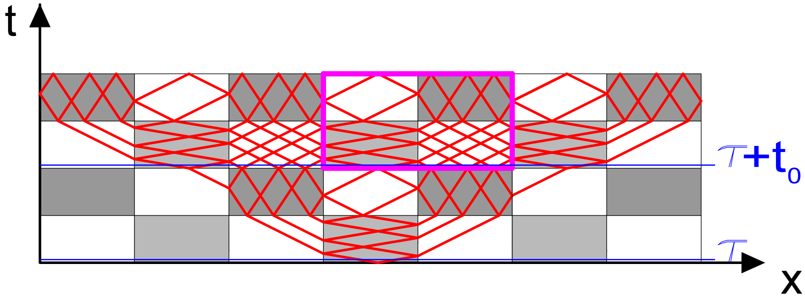
Dispersion relation: $\omega = \pm k$



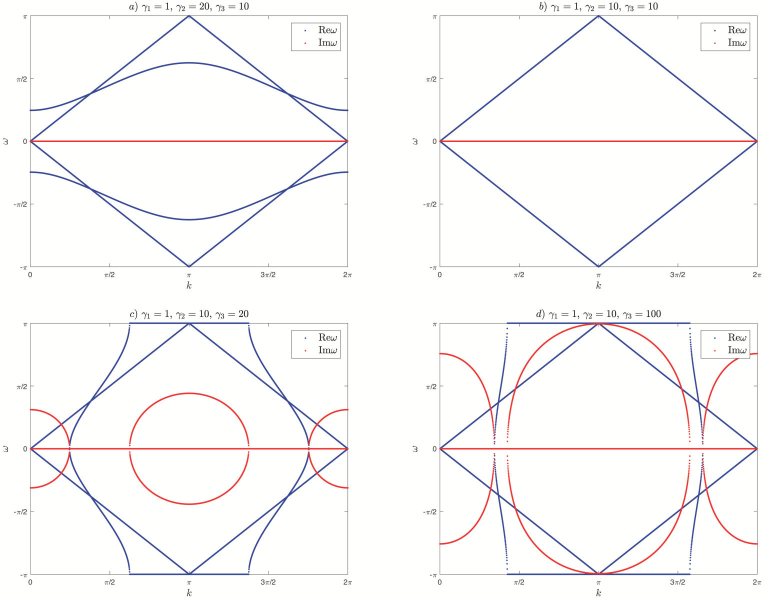


Bloch waves are infinitely degenerate!!!

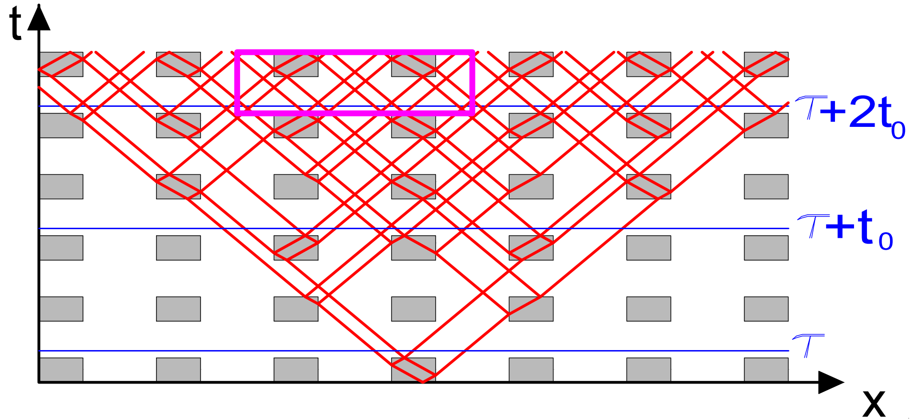
Three–phase checkerboard with phases having speed in a certain ratio



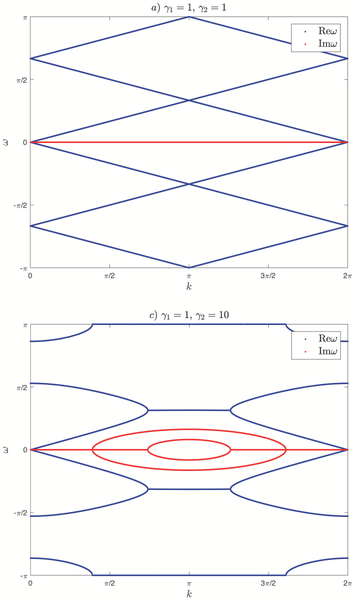
Dispersion diagrams for the three–phase checkerboard



Space–time microstructure with rectangular inclusions



Dispersion diagrams for the microstructure with inclusions



Thank you for your attention!!

- Milton GW, Mattei O, 2017. *Field patterns: a new mathematical object.* Proc R Soc A. 473:20160819.

- Mattei O, Milton GW, 2017. *Field patterns without blow up*. New J Phys. 19, 093022.

- Mattei O, Milton GW, 2017. *Field patterns: a new type of wave with infinitely degenerate band structure*. Europhys. Lett. 120(5), 54003.