# Inverse source problem for the wave equation with reduced data: an explicit solution

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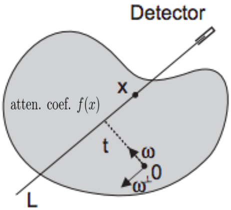
# Inverse problems and an example

Direct problem: solve a given PDE.
Inverse problem: given solution on the boundary, determine coefficient(s) in PDE.
An example: X-rays - send a narrow beams of X-rays with initial intensity $I_{in}$ through the body, measure $I_{out}$.




- $I_{out} = I_{in} \exp(-\int_L f(x)dx)$
- Radon transform $$\mathcal{R}f(t,\omega) := \int_{x\cdot\omega=t} f(x)dx$$
- $\mathcal{R}f(-t,-\omega) = \mathcal{R}f(t,\omega)$

**Inverse problem for X-rays:**

given $\mathcal{R}f(t,\omega)$ for $(t,\omega) \in \mathbb{R} \times S$, reconstruct atten. coef. $f(x)$

# Inverting the Radon transform

**Inversion of the Radon transform is a well-solved problem by now.**

$$f(x) = \frac{1}{4\pi}\mathcal{R}^*\mathcal{H}\frac{\partial}{\partial t}\mathcal{R}f(t,\omega),$$

where the adjoint (backprojection) operator $R^*$is

$$(\mathcal{R}^* h)(x) \equiv \int_{\mathbb{S}^1} h(w, x\cdot\omega)dw,$$

and $\mathcal{H}$ is the Hilbert transform

$$(\mathcal{H}u)(p) \equiv \mathrm{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{u(s)}{p-s}ds.$$

This is the famous filtered backprojection inversion formula.

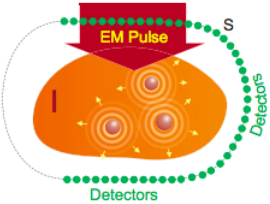
# Hybrid methods: motivation

- Acoustic waves yield high resolution but the contrast is low.
- Conductivity in tumors is much higher than that in healthy tissues $\Longrightarrow$ EM waves or currents yield high contrast. Electrical impedance tomography, optical and microwave tomography lead to strongly non-linear and ill-posed inverse problems

**Idea:** Use hybrid techniques: couple ultrasound with EM field:
Thermo-Acoustic and Photo-Acoustic Tomography (TAT/PAT)
Ultrasound Modulated Optical Tomography (UMOT)
Acousto-Electric Tomography (AET)
Magneto-Acousto-Electric Tomography (MAET)
Magneto-Acoustic Tomography with Magnetic Induction (MAT-MI)

# Thermo- and photo-acoustic tomography (TAT/PAT)

Send a short EM pulse $\Rightarrow$
EM energy will be absorbed$\Rightarrow$
Tissues will heat and expand $\Rightarrow$
Acoustic wave will propagate $\Rightarrow$
Detectors will measure acoustic pressure

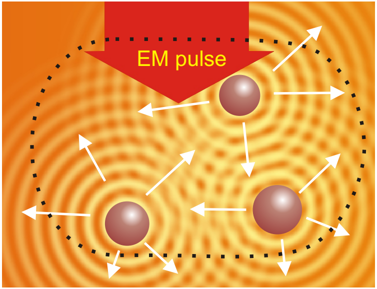



**Inverse source problem of TAT/PAT:**

given pressure on the boundary, find the initial pressure

# Open space TAT

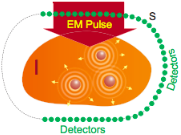
Detectors do not reflect or distort the waves
Waves propagate to infinity
In theory, surface Γ may be closed or open — but in reality, should be open.



# Formulation of the problem

We consider the homogeneous case: constant speed of sound $c(x) \equiv 1$ (acceptable!)
Acoustic pressure $u(t,x)$ satisfies the wave equation

$$\begin{cases} u_{tt} = \Delta u, x \in \mathbb{R}^n \\ u_t(0,x) = 0 \\ u(0,x) = f(x) \end{cases}$$




Initial pressure $f(x)$ is supported within bounded region $\Omega_0 \subseteq \Omega$.
The boundary of $\Omega$ is a closed surface $\Gamma$.
Measurements $g(t,y) \equiv u(t,y)$ are done on the subset $S \subset \Gamma$.

**Inverse source problem:**

Given $g(t,y), (t,y) \in [0, diam(\Omega)] \times S$, reconstruct $f(x)$

This problem was studied intensively.
We are interested in **explicit**, theoretically **exact** inversion formulas with $S$ being a **proper subset** of $\Gamma$.

# Direct problem for TAT/PAT

Acoustic pressure $u(t,x)$ satisfies the wave equation

$$\begin{cases} u_{tt} = \Delta u, x \in \mathbb{R}^n \\ u_t(0,x) = 0 \\ u(0,x) = f(x) \end{cases}$$

Solution of this equation is

$$u(t,y) \equiv \frac{\partial}{\partial t} \int\limits_{\Omega_0} f(x)\Phi_n(t,x-y)dx,$$

where the $\Phi_n(t,x)$ is Green function for free wave equation

$$\Phi_2(t,x) = \frac{H(t-|x|)}{2\pi\sqrt{t^2-|x|^2}}, \Phi_3(t,x) = \frac{\delta(t-|x|)}{4\pi|x|}$$

In particular, measurements $g(t,y) \equiv u(t,y)$ on $S$:

$$g(t,y) = \frac{\partial}{\partial t} G(t,y), G(t,y) \equiv \int\limits_{\Omega_0} f(x)\Phi_n(t,x-y)dx, y \in S \subset \Gamma.$$

# Integral geometry formulation

Circular/Spherical Radon transform $I(t, y)$:

$$I(t, y) \equiv t^{n-1} \int_{\mathbb{S}^{n-1}} f(y + t\hat{\nu}) d\hat{\nu}$$

Data $g(t, y)$ is directly related to the circular Radon transform. In 3D, the relation is very simple

$$g(t, y) = \frac{\partial}{\partial t} \left( \frac{I(t, y)}{4\pi t} \right), y \in \mathbb{S}_2.$$

In 2D, the connection is through the Abel transform:

$$g(t, y) = \frac{\partial}{\partial t} \int_0^t \frac{I(r, y)}{2\pi \sqrt{t^2 - r^2}} dr, y \in \mathbb{S}_1.$$

**Inverse source problem:**

Given spherical data $I(t, y), t \in [0, diam(\Omega)], y \in S$, reconstruct $f(x)$

# Known inversion formulas for various surfaces $S$

$S$ is a plane: multiple works
"Universal formula" in 3D: a sphere, a plane, a cylinder (Xu & Wang)
Spheres (multiple works by Finch et al; Kunyansky; Nguyen)
Ellipsoids and paraboloids (Natterer; Haltmeier; Palamodov;Salman)
Limiting cases of ellipsoids and paraboloids (Haltmeier & Pereverzyev Jr.)
More complicated curves and surfaces (Palamodov)
Triangles, squares, cubes, and some tetrahedra (Kunyansky)
Corner-like domains in 3D, a segment of Coxeter cross in 2D (Kunyansky)
Less explicit: series techniques (Kunyansky; Haltmeier et al)
In **all** of these works either $S$ is **closed** ($S = \Gamma$), or $S = \Gamma$ is **unbounded**

# Motivation, theoretical standpoint, and our goal

**Why reduced data?**

1. A body part can't be surrounded by detectors from all sides
2. An unbounded surface needs to be truncated
3. Acoustic waves deteriorate during propagation

**Theoretical standpoint:**

1. Uniqueness and observation time: If $S$ is a smooth and closed surface bounding domain $\Omega$, then the TAT/PAT data on $S$ collected for time $0 \leq t \leq 0.5 diam(\Omega)$ uniquely determines $f$.
2. Visibility condition: $x_0$ is in the visible region iff any line passing through $x_0$ intersects $S$ at least once.

**Our goal**

given $\mathbf{g(t,y)} = u(t,y), t \in [0,a], a < diam(\Omega)$, $y \in$ open bounded $S$, reconstruct $\mathbf{Rf}(t,\omega)$.

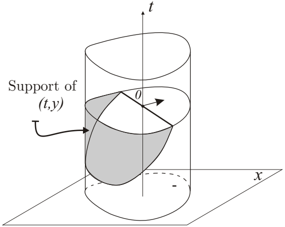
# Representing a plane wave by a single layer potential

Suppose $\delta(t - x \cdot \omega)$ enters $\Omega$ at $T_0(\omega)$ and leaves at $T_1(\omega)$.
Define interval $\mathcal{T}(\omega) \equiv (T_0(\omega), T_1(\omega))$.
We want to represent $\delta(t - \omega \cdot x), x \in \Omega, t \in (T_0(\omega), 0]$ as

$$\delta(\tau - x \cdot \omega) = \int_{T_0(\omega)}^{\tau} \int_{\Gamma} \Phi_n(\tau - t, x - y)\varphi_\omega(t, y) dy dt,$$

where density $\varphi_\omega(t, y)$ is a distribution supported on $\mathcal{T}(\omega) \times \Gamma$.
General scattering theory: this problem is uniquely solvable in the sense of distributions.

More important,
$\varphi_\omega(t, y) = 0$ for $t < x \cdot \omega$.



Measurements $g(t, y) \equiv u(t, y)$ on $S$ are given by:

$$g(t, y) = \frac{\partial}{\partial t} G(t, y), G(t, y) \equiv \int\limits_{\Omega_0} f(x)\Phi_n(t, x - y)dx, y \in S \subset \Gamma.$$

We want to recover the Radon projections of $f(x)$ defined as

$$\mathcal{R}f(\tau, \omega) \equiv \int\limits_{x \cdot \omega = \tau} f(x)dx = \int\limits_{\Omega_0} f(x)\delta(\tau - \omega \cdot x)dx.$$

Let us multiply $G(t, y)$ by $\varphi_\omega(\tau - t, y)$ and integrate over $(0, \tau - T_0(\omega)] \times \Gamma$:

$$\int_0^{\tau-T_0(\omega)} \int_\Gamma G(t,y)\varphi_\omega(\tau-t,y)\,dydt$$

$$= \int_0^{\tau-T_0(\omega)} \int_\Gamma \left[\int_{\Omega_0} f(x)\Phi_n(t,x-y)dx\right] \varphi_\omega(\tau-t,y)\,dydt$$

$$= \int_{\Omega_0} f(x) \left[\int_{T_0(\omega)}^{\tau} \int_\Gamma \Phi_n(\tau-s,x-y)\varphi_\omega(s,y)\,dyds\right] dx$$

$$= \int_{\Omega_0} f(x)\delta(-\omega\cdot x+\tau)dx = \mathcal{R}f(\tau,\omega), \omega \in \mathbb{S}^{n-1}, \tau \in \mathcal{T}(\omega).$$

Similarly,

$$\frac{\partial}{\partial\tau}\mathcal{R}f(\tau,\omega) = \int_0^{\tau-T_0(\omega)} \int_\Gamma g(t,y)\varphi_\omega(\tau-t,y)\,dydt, \omega \in \mathbb{S}^{n-1}, \tau \in \mathcal{T}(\omega).$$

# For circular and spherical geometry

**Theorem (Half-time data)**

*For n=2, 3, given* $\Omega_0 \equiv B_n(0,1), S \equiv \Gamma \equiv \mathbb{S}_{n-1}$, *Radon projections* $\mathcal{R}f(t,\omega)$ *can be fully reconstructed from halftime data* $g(t,y), t \in [0,1]$, $y \in S$.

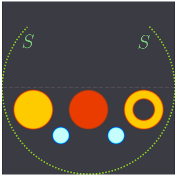
**Theorem (Open observation surface with temporally reduced data)**

*For n=2, 3, given* $\Omega_0$ *to be the lower half of the unit ball* $B_n(0,1)$, *Radon projections* $\mathcal{R}f(t,\omega)$ *can be fully reconstructed from reduced data* $g(t,y), t \in [0, 2-1/\sqrt{2}] \approx [0,1.3]$, $y \in S_n$,
*where* $S_2 = \{(x,y) : x^2+y^2 = 1, |y| \le 1/\sqrt{2}\}$,
$S_3 = \{(x,y,z) : x^2+y^2+z^2 = 1, |z| \le 1/\sqrt{2}\}$.

# Fast algorithm for the circular geometry

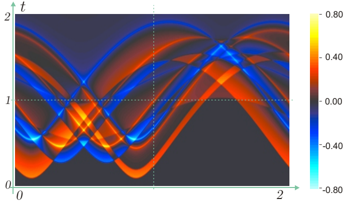
**1.** Expand $\widetilde{g}(t,\hat{y})$ in the Fourier series and Fourier-transform the result to obtain $\widehat{\widetilde{g}}_k(\rho)$ for each $\rho \geq 0$;
**2.** For each grid value of $\rho \geq 0$, compute coefficients $b_k(\rho) \equiv \frac{4}{i}\frac{i^{|k|}}{H^{(1)}_{|k|}(\rho)}\widehat{\widetilde{g}}_k(\rho)$ and extend them to negative $\rho$'s by complex conjugation;
**3.** For each grid value of $\rho$, sum up series $\sum_k b_k(\rho)e^{ik\varpi}$, and apply the inverse Fourier transform in $\rho$ to find $\widehat{\widetilde{\frac{\partial}{\partial\tau}\mathcal{R}f}}(\tau,\omega)$;
**4.** Anti-differentiate $\widehat{\widetilde{\frac{\partial}{\partial\tau}\mathcal{R}f}}(\tau,\omega)$ to find $\widetilde{\mathcal{R}f(\tau,\omega)}$;
**5.** Compute $\mathcal{R}f(\omega,\tau)$ by extracting the correct values of $\widetilde{\mathcal{R}f(\omega,\tau)}$ within the intervals prescribed by the theorem presented above.
**This is fast**: all steps are either FFT's or multiplications;
the total complexity is $O(m^2\log m)$ flops for an $m\times m$ grid (vs. $O(m^3)$ for filtration/backprojection).

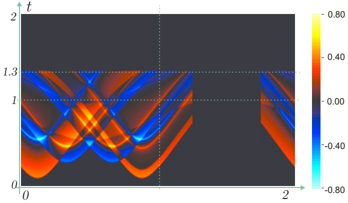
Our phantom is a collection of slightly smoothed characteristic functions of circles.
$S$ is the acquisition surface.



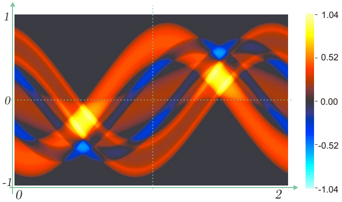

Solve wave equation
find $g(t, \omega(\theta + \pi))$

Truncated $g(t, \omega(\theta + \pi))$

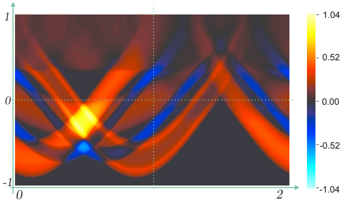


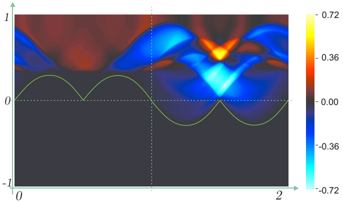
# Reconstruction results, truncated circular geometry



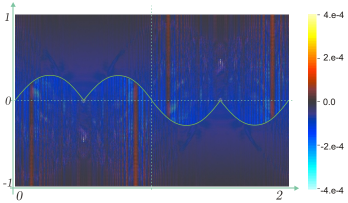

Exact $\mathcal{R}f(\tau,\omega(\theta))$




Reconstructed $\mathcal{R}f(\tau,\omega(\theta))$ on step 5




Reconstruction error after step 5




Reconstruction error after step 6

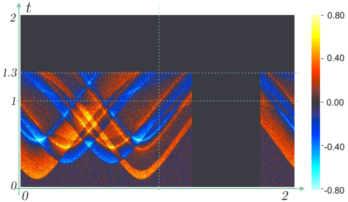
Number of "detectors" $= 512$, number of time samples $= 257$, reconstruction time $= 0.4$ sec., relative $L^\infty$ error $\approx$ 5.E-4.

# Reconstructing $f(x)$, truncated circular geometry

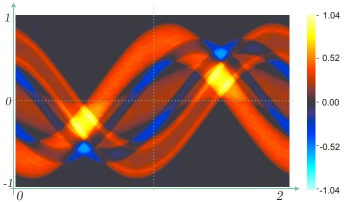
Phantom Reconstruction Error(not to scale)

Relative error in $f(x)$ measured in $L^2(\Omega) \approx 0.6$ %.

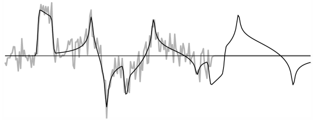
# Next simulation, circular geometry with 50% noise (in $L^2$)



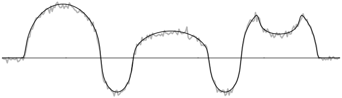

Noisy data $g(t, \omega(\theta + \pi))$




Reconstruction from noisy data



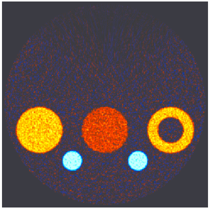
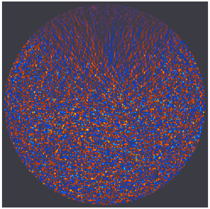
Noisy data $g(t, \omega(0))$ vs exact



Reconstructed $\mathcal{R}f(\tau, \omega(0))$ vs exact

Relative $L^2$ error in the reconstructed $\mathcal{R}f(\tau, \omega)$ is $\approx 7\%$.

# Reconstructing $f(x)$ from data with 50% noise

Phantom Reconstruction Error(not to scale)

Relative error in $f(x)$ measured in $L^2(\Omega) \approx 28\%$

# Fast algorithm for the spherical geometry

**1.** Expand $\widehat{\widetilde{g}}(\rho, \hat{y})$ in spherical harmonics in $\hat{y}$ and compute the Fourier transform in $t$ to obtain $\widehat{\widetilde{g}}_{m,k}(\rho)$, $k = 0, 1, \cdots, m = \overline{-k, k}$;

**2.** For $\rho \geq 0$, compute $b_k(\rho) \equiv \dfrac{4\pi}{i\rho} \dfrac{i^k}{h_k^{(1)}(\rho)} \widehat{\widetilde{g}}_{m,k}(\rho)$, and extend to $\rho < 0$ by complex conjugation;

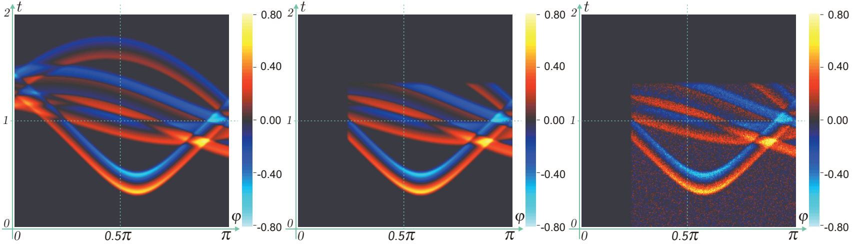
**3.** For each $\rho$ and $\omega$ sum up series $\displaystyle\sum_{k=0}^{\infty} \sum_{m=-k}^{k} b_{m,k}(\rho) \overline{Y_m^k(\omega)}$ and compute the inverse Fourier transform to get $\widetilde{\dfrac{\partial}{\partial \tau} \mathcal{R} f}(\tau, \omega)$;

**4.** Anti-differentiate $\widetilde{\dfrac{\partial}{\partial \tau} \mathcal{R} f}(\tau, \omega)$ to find $\widetilde{\mathcal{R} f}(\tau, \omega)$;

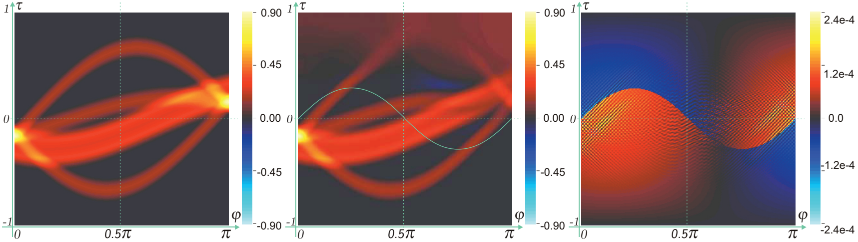
**5.** Compute $\mathcal{R} f(\tau, \omega)$ by extracting the correct values of $\widetilde{\mathcal{R} f(\tau, \omega)}$ within the intervals prescribed by the theorem presented above.

This algorithm is fast: the total complexity is $O(m^4)$ flops for an $m \times m \times m$ grid (vs. $O(m^5)$ for filtration/backprojection)

# Simulation, spherical geometry, 3D




Data $g(t, \hat{y}(\theta_0, \varphi))$, $\theta_0 \approx 69^\circ$ Reduced data $\widetilde{g}(t, \hat{y}(\theta_0, \varphi))$ Reduced noisy data $\widetilde{g}(\dots)$




Exact $\mathcal{R}f(\tau, \omega(\theta_0, \varphi))$, $\theta_0 \approx 69^\circ$ Step 4: $\widetilde{\mathcal{R}f}(\tau, \omega(\theta_0, \varphi))$ Step 5: error in $\mathcal{R}f(\dots)$

Relative $L^\infty$ error is 3.E-4; with 50% noisy data relative $L^2$ error is 0.8%.

Reconstruction of $f(x)$ from 50% noisy data has relative $L^2$ error of 9%.

# Some discussion

The proposed technique is somewhat sub-optimal: generally, $\Omega_0$ could have larger support, still with injectivity/stability.

Good news: our approach is quite general. And this is the only explicit result for open and bounded acquisition surfaces.

More good news: we rely on the scattering problem by **closed** surfaces. For such surfaces there is a significant body of work on finding the density of singular layers and/or solving the scattering problem.

Bad news: our technique is only as explicit as the densities we find.

Fortunately, for certain surfaces this can be done analytically as in the circular/spherical cases.

*Thank you!*