Wave Dynamics in Linear/Nonlinear Photonic Lattices and Topological Insulators

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Outline

- Introduction
- Study light propagation in longitudinal (z) direction with an optical lattice in transverse (x-y) plane
- Systematic method to obtain tight binding approximations for linear/NL lattices with uniform and nonuniform longitudinal structure
- Prototype: honeycomb (HC) photonic lattices
- Uniform longitudinal structure: conical, elliptical, straightline diffraction

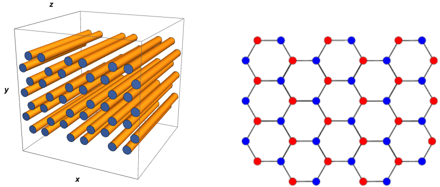
Outline – con't

- Lattices with non-uniform longitudinal structure: 'topological insulator'
- Find linear topologically protected edge waves: they move unidirectionally, do not scatter off defects
- NL problem: find envelope edge solitons satisfying classical 1d-NLS eq; the solitons do no scatter from defects, corners; propagate stably over very long distances
- Conclusion

Refs: MJA, C. Curtis, YP Ma (2013–15); MJA, J. Cole (2017–18)

Introduction

- Investigations of optical lattices extensive
- Paradigm – HC lattice: 'Photonic Graphene' (PG)



Left: Uniform HC lattice: z direc'n; Right: x-y plane : HC lattice

- Segev group 2007–conical diffrct'n; MJA, Y. Zhu, C. Curtis constructed/studied TB models; found conical diffrct'n & various interesting new NL nonlocal eqn's in certain limits (2009-13)

Introduction–con't

- Topological insulators/edge waves were theoretically proposed/observed in magneto-optics, Wang et al 2008-09
- Such top'l waves were found in photonics: HC lattice with longitudinal helical variation, Rechtsman et al 2013
- MJA, YP Ma, C. Curtis (2013–15), studied TB model, developed asymptotic description linear/NL under assumption of rapid helical variation: top'l/non-top'l waves

Introduction–con't

- Leykam et al (2016) studied staggered square lattice with helical variation and phase sh'fts between sublattices
- MJA, J. Cole (2017–18): systematic method to find TB models in lattices with longitudinal structure
- Topological edge/interface/surface waves in physics – very active field of research

Lattice NLS Equation

Maxwell's eq with paraxial approx. => NLS eq with ext pot'l

$$i\frac{\partial\psi}{\partial z} = -\frac{1}{2k_0}\nabla^2\psi + k_0\frac{\Delta n(x,y,z)}{n_0}\psi - \gamma|\psi|^2\psi$$

where: k_0 is input wavenumber

n_0 is the bulk refractive index

$\Delta n/n_0$ is the change of index change relative to n_0

γ is NL index

Non-dimensional NLS Equation

Rescale to non-dimensional form

$$x = \ell x' \ , \ y = \ell y' \ , \ z = z_* z' \ , \ \psi = \sqrt{P_*}\psi'$$

where: ℓ is the lattice scale; P_*: peak input power

Find non-dim NLS eq, $'$: dimensionless:

$$i\frac{\partial \psi'}{\partial z'} + (\nabla')^2\psi' - V(x', y', z')\psi' + \sigma|\psi'|^2\psi' = 0$$

where $z_* = 2k_0\ell^2, \quad V = 2k_0^2\ell^2(\Delta n/n_0), \quad \sigma = 2\gamma k_0 \ell^2 P_*$

Drop $'$ $=>$ normalized lattice NLS eq

$$i\frac{\partial \psi}{\partial z} + \nabla^2\psi - V(x, y, z)\psi + \sigma|\psi|^2\psi = 0$$

Typical Physical Scales

Exp'ts: input wavelength: $\lambda = 630nm$

Bulk index: $n_0 = 1.4$ (silica)

$=>$ input wavenumber $k_0 = \frac{2\pi n_0}{\lambda} = 1.44 \times 10^7 rad/m$

$\ell = 15\mu m => \quad z_* = 2k_0\ell^2 = 6.5 \times 10^{-3} m$

NL Index change: $\Delta n/n_0 = 7 \times 10^{-4}$

$=> \ V = 2k_0^2\ell^2(\Delta n/n_0) \sim 50;$

Later will discuss 'Tight Binding (TB) Approx': $V \gg 1$

Periodic Optical Lattices

Investigate waves in periodic optical lattices in lattice NLS eq.

$$i\psi_z + \nabla^2\psi - V(\mathbf{r}, z)\psi + \sigma|\psi|^2\psi = 0$$

where $V(\mathbf{r}, z)$: in transverse plane 2-d periodic with basis lattice vectors: $\mathbf{v}_1, \mathbf{v}_2$:

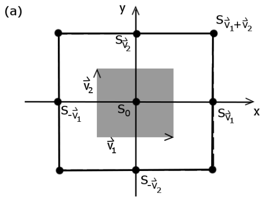
$$V(\mathbf{r} + m\mathbf{v}_1 + n\mathbf{v}_2, z) = V(\mathbf{r}, z), \quad m, n \in \mathbb{Z}$$

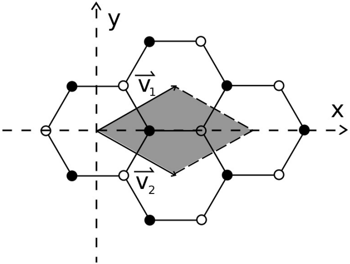
Begin with uniform case: $V(\mathbf{r}, z) = V(\mathbf{r})$

Lattices

Simple lattice: $V(\mathbf{r})$ has one min or 'site' in each unit cell; all sites can be constructed from an initial site: e.g. below: left rectangular lattice

Non-simple HC lattice–below right: two initial sites ("A & B") determine the lattice





Linear problem

Linear problem: $\psi(\mathbf{r}, z) = \varphi(\mathbf{r})e^{-i\mu z}$ with $|\varphi| \ll 1$:

$$(\nabla^2 - V(\mathbf{r}) + \mu)\varphi = 0$$

$V(\mathbf{r})$ is a 2-d periodic potential with lattice vectors: $\mathbf{v}_1, \mathbf{v}_2$

Bloch theory:

$$\varphi(\mathbf{r}; \mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}} U(\mathbf{r}; \mathbf{k}),$$

$U(\mathbf{r}; \mathbf{k})$ is periodic in $\mathbf{r}$ and $\varphi(\cdot, \mathbf{k}), \mu(\mathbf{k})$ are periodic in $\mathbf{k}$ with 'dual' lattice vectors: $\mathbf{k}_1, \mathbf{k}_2$

Dispersion relation: $\mu = \mu(\mathbf{k})$

Potential in TB Limit

Tight binding limit: when $|V| \gg 1$ we approx. the potential

$$V(\mathbf{r}) \approx \sum_{\mathbf{v}} V_0(\mathbf{r} - \mathbf{v})$$

where $V_0(\mathbf{r} - \mathbf{v})$ is the approx. pot'l with minima at site $\mathbf{v}$

Use

$$\left(\nabla^2 - V_0(\mathbf{r})\right) \phi(\mathbf{r}) = -E\phi(\mathbf{r})$$

$\phi(\mathbf{r})$ called an 'orbital'

$|V| \gg 1$: 'tight binding approx.' used widely in physics to study lattices; 'graphene' Wallace 1947

TB Envelope Dynamics

Simple lattices with $\mu(\mathbf{k})$ having a single dispersion relation branch

$$\psi(\mathbf{r}, z, \mathbf{k}) \sim \sum_{\mathbf{v}} a_{\mathbf{v}}(z)\phi(\mathbf{r} - \mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}}$$

$a_{\mathbf{v}}(z)$ represents the Bloch wave envelope at the site $S_{\mathbf{v}}$

$\phi(\mathbf{r})$ orbitals: rapidly decaying

RHS above approximates the 'Wannier functions'

Substitute ψ into lattice NLS, multiply by $\phi(\mathbf{r} - \mathbf{p})$ and integrate:

find discrete NLS eq. at general values of $\mathbf{k}$ in the Brillouin zone: DNLS($\mathbf{k}$)

DNLS($\mathbf{k}$) Eq.

Leading order eq. results from on-site and nearest neighbor interactions; Eq. reduces to DNLS($\mathbf{k}$) eq.

$$\frac{da_{\mathbf{p}}}{dz} + \sum_{<\mathbf{v}>} a_{\mathbf{p}+\mathbf{v}} C_{\mathbf{v}} e^{i\mathbf{k}\cdot\mathbf{v}} + \sigma g |a_{\mathbf{p}}|^2 a_{\mathbf{p}} = 0$$

where $< \mathbf{v} >$ means the sum over $\mathbf{v}$ only takes nearest neighbors; $C_{\mathbf{v}}, g$ are given in terms integrals over orbitals

Continuum limit from DNLS($\mathbf{k}$) yields 2-d NLS($\mathbf{k}$) eq

NLS($\mathbf{k}$) Eq.

As a further limit, assume that the envelope $a_{\mathbf{v}}$ varies slowly over $\mathbf{v}$ with a scale $\mathbf{R} = \nu \mathbf{r}, \quad |\nu| \ll 1$ then

$$a_{\mathbf{p}+\mathbf{v}} \approx a + \nu \mathbf{v} \cdot \tilde{\nabla} a + \cdots$$

Find a continuous 2-d NLS($\mathbf{k}$) eq for $a = a(\mathbf{R}, Z)$

$$i \frac{\partial a}{\partial z} + i\nu \bar{\nabla}\mu \cdot \tilde{\nabla} a + \frac{\nu^2}{2} \sum_{m,n=1}^{2} \bar{\partial}_{m,n}\mu \tilde{\partial}_{m,n} a + \sigma g |a|^2 a = 0$$

where $\tilde{\partial}_m = \frac{\partial}{\partial \mathbf{R}_m}, \quad \bar{\partial}_m = \frac{\partial}{\partial k_m}; \quad \bar{\nabla}\mu$ plays the role of the group velocity

For different $\mathbf{k}$ the dispersive terms may be elliptic, hyperbolic or parabolic

Typical HC Lattice

Typical honeycomb (HC) lattice $V(\mathbf{r})$ from:

$$V(\mathbf{r}) = V_0 \left| e^{ik_0 \mathbf{b}_1 \cdot \mathbf{r}} + \eta e^{ik_0 \mathbf{b}_2 \cdot \mathbf{r}} + \eta e^{ik_0 \mathbf{b}_3 \cdot \mathbf{r}} \right|^2$$

where $\mathbf{b}_1 = (0, 1), \mathbf{b}_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}); \mathbf{b}_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$

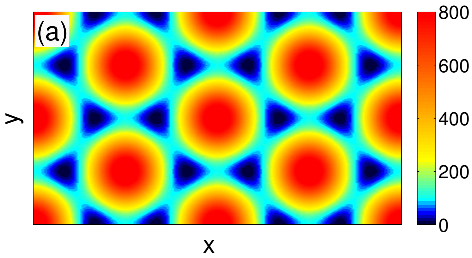
V_0, k_0, η const. $V_0 > 0$ is the lattice intensity;

η corresponds to geometric lattice deformation (strain)

$\eta = 1$ standard/perfect HC lattice

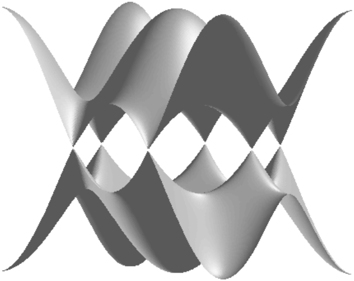
Typical HC–Intensity Plot

HC lattice in physical space: below ($\eta = 1$): intensity plot; local minima form HC lattice



Typical HC–Dispersion Surface

Typical HC dispersion relation $\mu(\mathbf{k})$ first two bands below:



Note: 1st, 2nd bands can touch at certain isolated points, called Dirac points

Dirac points form sites of a hexagonal HC lattice in the **k** plane dual lattice corresponding to the original potential lattice–i.e. have HC structure in the **k** plane.

Honeycomb Lattices

Nonsimple honeycomb (HC) lattices also arise in the study of the 2d material Graphene

Material Graphene: ultra thin carbon material

First demonstrated exp't 2004; nobel prize 2010

Graphene exhibits important properties physically and mathematically

Here we study: Photonic Graphene – photonic analogue of graphene

Uniform/non-uniform lattices:
Segev's group: ('07–), MJA, Zhu,Curtis,Ma,Cole ('09–...), Fefferman,Weinstein ('12–), ...

TB HC Envelope Evolution

For $|V| >> 1$

$$\psi \sim \sum_{\mathbf{v}} a_{\mathbf{v}}(z)\phi_A(\mathbf{r}-\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}} + \sum_{\mathbf{v}} b_{\mathbf{v}}(z)\phi_B(\mathbf{r}-\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}}$$

where the sum is over A, B lattice sites: $\mathbf{v}$

$$\left(\nabla^2 - V_j(\mathbf{r})\right)\phi_j(\mathbf{r}) = -E_j\phi_j(\mathbf{r}); \quad j = A, B$$

$\phi_j(\mathbf{r})$ orbitals; rapidly decaying

Substitute ψ into lattice NLS eq., multiply $\phi_j(\mathbf{r}-\mathbf{p})e^{-i\mathbf{k}\cdot\mathbf{p}}; \quad j = A, B$ and integrate

Discrete HC System

Find discrete system

$$
\begin{aligned}
i\frac{da_{\mathbf{p}}}{dz} + \mathcal{L}^- b_{\mathbf{p}} + \sigma |a_{\mathbf{p}}|^2 a_{\mathbf{p}} = 0 \\
i\frac{db_{\mathbf{p}}}{dz} + \mathcal{L}^+ a_{\mathbf{p}} + \sigma |b_{\mathbf{p}}|^2 b_{\mathbf{p}} = 0
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^- b_{\mathbf{p}} &= b_{\mathbf{p}} + \rho(b_{\mathbf{p}-\mathbf{v}_1} e^{-i\mathbf{k}\cdot\mathbf{v}_1} + b_{\mathbf{p}-\mathbf{v}_2} e^{-i\mathbf{k}\cdot\mathbf{v}_2}) \\
\mathcal{L}^+ a_{\mathbf{p}} &= a_{\mathbf{p}} + \rho(a_{\mathbf{p}+\mathbf{v}_1} e^{i\mathbf{k}\cdot\mathbf{v}_1} + a_{\mathbf{p}+\mathbf{v}_2} e^{i\mathbf{k}\cdot\mathbf{v}_2})
\end{aligned}
$$

ρ: deformation parameter ($\rho = \rho(\eta)$); $\rho = 1$ perfect hexagon

Rigorous analysis of TB: MJA, C. Curtis, Y. Zhu (2012)

Continuous NL Dirac System

When $a_{\mathbf{v}}$ and $b_{\mathbf{v}}$ vary slowly with respect to $\mathbf{v}$ at Dirac point: $\mathbf{k} = K$ find deformed NL Dirac (NLD) system in continuum limit:

$$
\begin{aligned}
i\partial_z a + (\partial_x + i\zeta\partial_y)b + \sigma|a|^2 a = 0 \\
i\partial_z b + (-\partial_x + i\zeta\partial_y)a + \sigma|b|^2 b = 0
\end{aligned}
$$

where $\zeta = \sqrt{\frac{4\rho^2 - 1}{3}}$

When $\rho = 1, \zeta = 1$ conical diffraction

$1/2 < \rho < 1$: elliptical diffraction

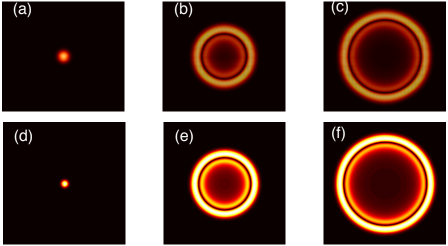
$\rho \to 1/2$: straight line diffraction

When $\sigma = 0$ above system reduces to (deformed) 2+1d wave eq.

Conical Diffraction

Below: simulations of lattice NLS and NLD: $\rho = 1$
Top Fig. lattice NLS

Bottom Fig. NLD system ('a' envelope)
IC: a is a unit Gaussian and $b = 0$



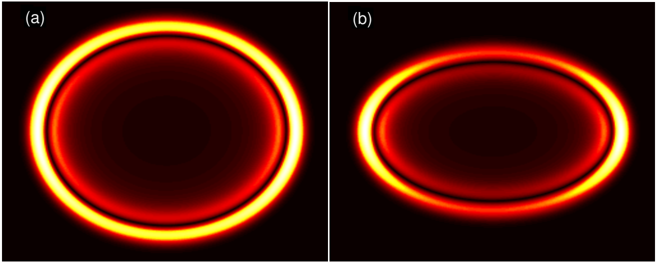
$z = 0$ $\quad z = z_1 > 0$ $\quad z = z_2 > z_1$

NLD system yields conical diffraction–as seen in lattice NLS eq.
conical diffraction observed (Sgev gp 2007)

Elliptical Diffraction–NLD

NLD: The rings in the conical diffraction are elliptic if $\rho \neq 1$, where the ratio of axes is $\zeta = \sqrt{\frac{4\rho^2-1}{3}}$.

Below 2 elliptic rings when (a) $\rho = 0.8$ (b) $\rho = 0.6$



Deformation–con't

- Critical behavior when $\rho \sim 1/2$
- $|2\rho - 1| \ll 1$ find approximate 'straight-line' diffraction; weak transverse variation
- Various small parameters – different balances lead to different eqs: $\beta = 2\rho - 1$, NL: ε, sl varying envelope: ν
- Find numerous new nonlocal nonlinear equations

MJA, Y. Zhu 2013

Deformation Eq: 'NLSKZ'

When $\varepsilon \sim \nu^2, \;\; 0 \ll \nu^2 \ll \beta$

$a \sim \nu F, \; \theta = x - z; \;\; Z = \nu^2 z$:

$$\partial_\theta \left(\partial_Z F - \sigma i |F|^2 F \right) + \partial_y^2 F = 0 \quad \text{NLSKZ}$$

This nonlocal eq is similar is spirit to
'KZ' eq. Khokhlov & Zabolotskaya 1969:

$$\partial_\theta (\partial_t u + u \partial_\theta u) + \partial_y^2 u = 0 \quad \text{KZ}$$

Conclusions–so far

- NL waves in lattice NLS in strong potential/TB limit investigated MJA, Curtis, Zhu ('09–13)
- Simple lattices: discrete and continuous NLS systems
- HC lattices: n'bhd of Dirac points find discrete and continuous NL Dirac (NLD) systems
- NLD exhibits conical-elliptical diffraction when $1/2 < \rho \leq 1$
- When $\rho \to 1/2$ find straight line diffraction; find new reduced asymptotic eq

Longitudinally Varying Waveguides

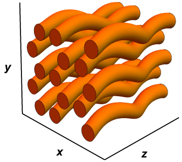
Introduce longitudinally varying waveguides on both sublattice sites; LNLS:

$$i\partial_z \psi = -\nabla^2 \psi + V(\mathbf{r}, z)\psi - \gamma |\psi|^2 \psi$$

Introduce longitudinally varying waveguides (Rechtsman et al '13)

$$x' = x - h_1(z), \; y' = y - h_2(z), \; z' = z$$

$\mathbf{h}(z) = (h_1(z), h_2(z))$: 'path function'– typically periodic in z



Longitudinally Varying Waveguides–con't

Transform LNLS eq. with

$$\psi \to \psi \exp\left[i \int_0^z |\mathbf{A_p}(\xi)|^2 d\xi \right] \text{ with } \mathbf{A_p}(z) = -\mathbf{h}'(z)$$

find lattice NLS with a pseudo-field: $\mathbf{A_p}(z)$: in **transfromed** coordinates

$$i\partial_z \psi = -(\nabla + i\mathbf{A_p}(z))^2 \psi + V(\mathbf{r})\psi - \gamma |\psi|^2 \psi$$

TB approx with pseudo-field $\mathbf{A_p} = \mathbf{A_p}(z)$ yields:

TB with Pseudo-Field: Discrete System

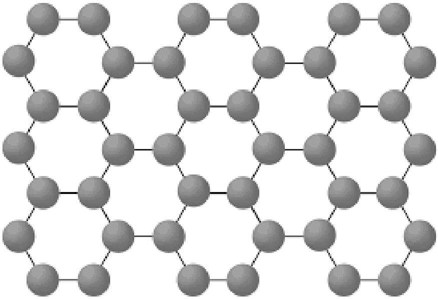
$$i\frac{da_{mn}(z)}{dz} + e^{i\mathbf{d}\cdot\mathbf{A_p}}(\mathcal{L}_- b)_{mn} + \sigma|a_{mn}|^2 a_{mn} = 0$$
$$i\frac{db_{mn}(z)}{dz} + e^{-i\mathbf{d}\cdot\mathbf{A_p}}(\mathcal{L}_+ a)_{mn} + \sigma|b_{mn}|^2 b_{mn} = 0$$

where

$$(\mathcal{L}_- b)_{mn} = b_{mn} + \rho(b_{m-1,n-1}e^{-i\theta_1} + b_{m+1,n-1}e^{-i\theta_2})$$
$$(\mathcal{L}_+ a)_{mn} = a_{mn} + \rho(a_{m+1,n+1}e^{i\theta_1} + a_{m-1,n+1}e^{i\theta_2})$$

and $\theta_j = (\mathbf{d} - \mathbf{v_j}) \cdot \mathbf{A_p}(z)$, $j = 1, 2$, ρ deformation, $\mathbf{d}$ is a vector between adj. horiz. sites, above use m, n row, column format

Zig-Zag-Arm Chair Edges



Zig-Zag (ZZ): Left Right; Armchair: Top, Bottom

BCs – Linear Floquet Bands

Assume $\mathbf{A_p}(z)$ is periodic; e.g. $\mathbf{A_p}(z) = \kappa(\sin\Omega z, \cos\Omega z)$
Look for solutions of the form

$$a_{mn}(z) = a_n(z;\omega)e^{im\omega} \ , \quad b_{mn}(z) = b_n(z;\omega)e^{im\omega} \ ,$$

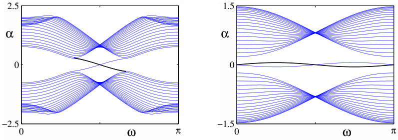
Find linear difference eq with periodic coef; use Floquet thy:

$$f(z+T) = e^{-i\alpha(\omega)z} f(z), \quad T = 2\pi/\Omega$$

$\alpha(\omega)$ Floquet coef; also called the edge dispersion relation

Linear Problem–Dispersion Relations

Dispersion relations (helical waveguides): thin curves are 'bulk' modes; lines in the gap are **edge modes**:



$\rho = 1$ $\rho = 0.4$

Left Fig: **Toplogical Floquet insulator** (Segev gp '13)

Right Fig: allows left and right going waves

In general: number of intersections: $\mathcal{I}$ with $\alpha = 0$ $\mathcal{I} = 0, 1, 2$

$(0 \le \omega < \pi)$ left fig $\mathcal{I} = 1$ (topological)

right fig $\mathcal{I} = 2$ (nontopological)

Analysis: Rapidly Varying Helical lattice

Let: $a_{mn} = a_n e^{im\omega}, b_{mn} = b_n e^{im\omega}$ find

$$i\partial_z a_n + e^{i\mathbf{d}\cdot\mathbf{A_p}}\left(b_n + \rho\gamma^* b_{n-1}\right) + \sigma|a_n|^2 a_n = 0$$
$$i\partial_z b_n + e^{-i\mathbf{d}\cdot\mathbf{A_p}}\left(a_n + \rho\gamma a_{n+1}\right) + \sigma|b_n|^2 b_n = 0$$

where $\gamma = \gamma(\omega, \mathbf{A_p}(\mathbf{z}))$

Take: **$\mathbf{A_p}$ periodic & rapidly varying**

$\mathbf{A_p} = \mathbf{A_p}(\frac{z}{\varepsilon}), |\varepsilon| \ll 1;$

e.g. $\mathbf{A_p} = \kappa(\sin\frac{\mathbf{z}}{\varepsilon}, \cos\frac{\mathbf{z}}{\varepsilon})$: 'helical waveguides'

Expt's Segev gp (2013)

Edge Modes: ZZ

Multiple scales:
$a_n = a_n(z,\zeta)$; $b_n = b_n(z,\zeta)$; $\zeta = \frac{z}{\varepsilon}$; $\partial_z = \frac{1}{\varepsilon}\partial_\zeta + \partial_z$
Expand a_n, b_n in powers of ε;

Apply BCs (ZZ) find **Edge Modes (ZZ)** (exp decay):

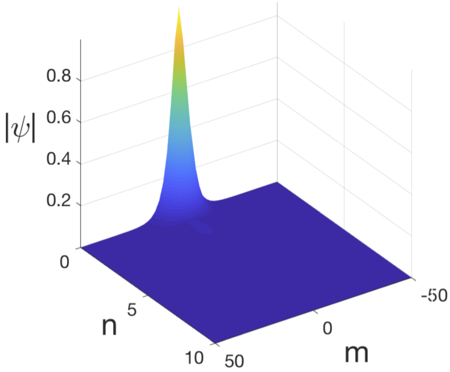
$$a_n \sim 0, \quad b_n \sim C(z,\omega) r^n, \quad |r| = |r(\omega, \rho, \mathbf{A_p})| < 1$$

Linear problem (first order):

$$C(z,\omega) = C_0 \exp\left(-i\alpha(\omega) z\right),$$

C_0 const. $\alpha(\omega) \equiv \alpha(\omega, \rho; \mathbf{A_p})$: 'dispersion relation' (Floquet coef): obtain explicit formulae

Typical Edge Mode



$$b_{mn} \sim C_m e^{(i\omega_0 m - i\alpha(\omega_0)z)} r^n$$

Nonlinear Edge Wave Envelope Evolution Eq

Discrete edge mode: $a_{mn} \sim 0$

$$b_{mn} \sim C(z, y) e^{i\omega_0 m} r^n, \; |r| < 1$$

where slowly varying ($|\nu| \ll 1$) edge mode envelope satisfies

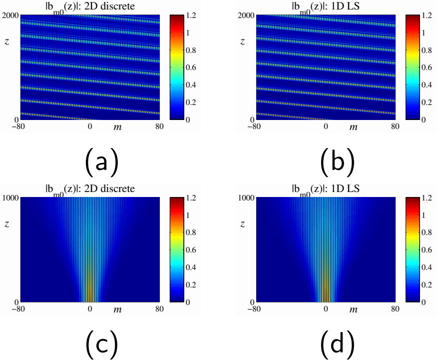
$$i\partial_Z C = \alpha_0 C - i\alpha_0' \nu C_y - \frac{\alpha_0''}{2}\nu^2 C_{yy} + \frac{i\alpha_0'''}{6}\nu^3 C_{yyy} - \alpha_{nl,0}|C|^2 C + \cdots$$

where $\alpha_0 = \alpha(\omega_0), \alpha_0' = \partial_\omega \alpha(\omega_0)$, etc.

May transform to standard NLS

If $\mathbf{A_p} = \mathbf{0}$ then $\alpha = 0$: stationary mode

Typical Linear Edge Wave Evolution



(a) (b) (c) (d)

Left: Linear discrete; Right linear Schrödinger (LS) eq

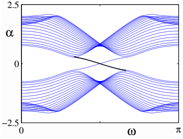
Fig (a-b): $\rho = 1$: Topological Floquet Insulator:

$\alpha_0' \neq 0; {\alpha_0}'' = 0, {\alpha_0}''' \neq 0, \mathcal{I} = 1$

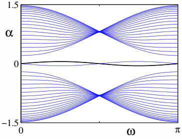
Fig. (c-d): $\rho = 0.4$: at $\alpha_0' = 0; {\alpha_0}'' \neq 0$: $\mathcal{I} = 2$ (nontopological)

Recall: Dispersion Relations of Linear Problem

Dispersion relations (helical waveguides): thin curves are 'bulk' modes; lines in the gap are **edge modes**:

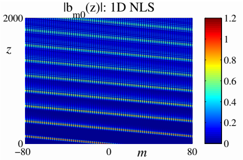


$\rho = 1$



$\rho = 0.4$

Typical NL Edge Wave Evolution



Here: $\alpha_0' \neq 0; \alpha_0'' \neq 0, \alpha_0''' = 0$
Fig: $\rho = 1$ Solitons

Continuous theory agrees with discrete eq.

NL problem inherits Topological Insulator properties

Typical Linear Edge Wave Evolution–Defects

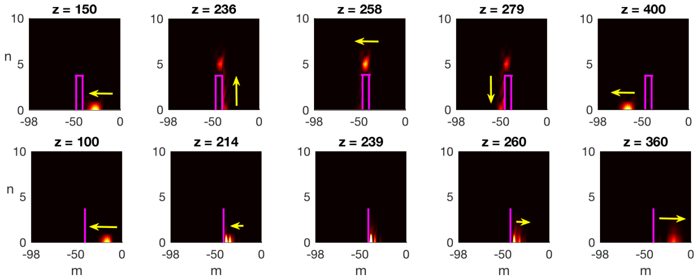


Fig: propagation across defect: left to right

Top fig: Topological mode – wave propagates unidirectionally without losing significant power ($\rho = 1, \omega = \pi/2, \alpha_{nl} = 0$)

Bottom fig: Nontopological mode – wave reflects, broadens/loses significant power ($\rho = 0.4, \omega = \pi/2, \alpha_{nl} = 0$)

NL Edge Wave Propagation Around Defects

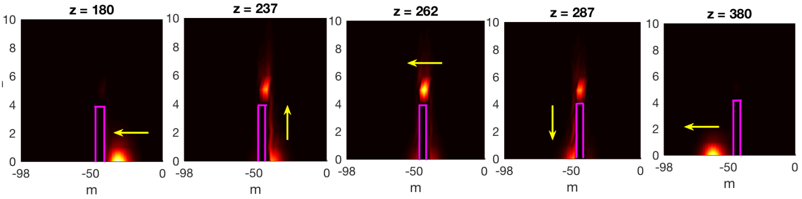
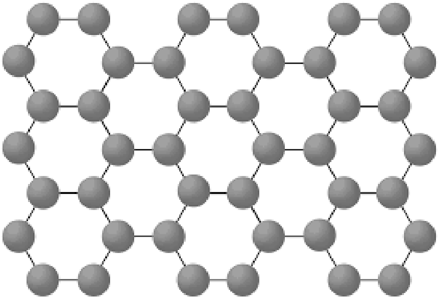


Fig: NL propagation across defect: left to right

NL topological edge wave ($\rho = 1, \alpha_0'' > 0, \alpha_{nl} \neq 0$)) propagates without losing significant power

NL edge solitons: unidirectional, propagates across defects

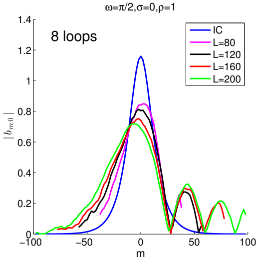
Bounded Photonic Graphene

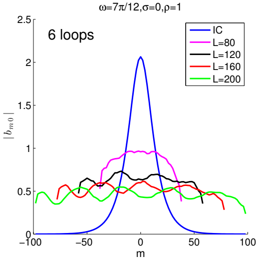


Zig-Zag (ZZ): Left Right; Armchair: Top, Bottom

Mode Propagation–Linear

Linear propagation $\rho = 1$: topological case; different points on the dispersion curve





Left: Linear $\omega = \pi/2$

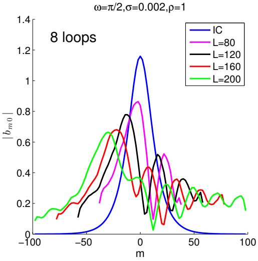
$\alpha'' = 0, \alpha''' \neq 0$

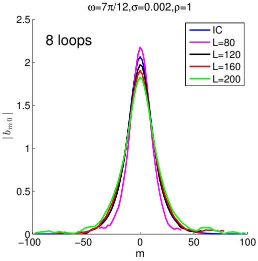
Right: Linear $\omega = 7\pi/12$

$\alpha'' \neq 0$

Mode Propagation–NL

NL propagation $\rho = 1$: topological case:; different points on the dispersion curve





Left: NL $\omega = \pi/2$
$\alpha'' = 0, \alpha''' \neq 0$

Right: NL $\omega = 7\pi/12$: NLS eq
$\alpha'' \neq 0$

General Longitudinal Variation

Typical case nonsimple lattice with two sublattices

$$V_1 = V_1(\mathbf{r} - \mathbf{h}_1(z)), \quad V_2 = V_2(\mathbf{r} - \mathbf{h}_2(z))$$

in nb'hd of sublattices $1, 2$ and $\mathbf{h}_j(z),\ j = 1, 2$ are prescribed (smooth) functions

Simple case, helical variation

$$\mathbf{h}_j(z) = \eta_j \left(\cos\left(\frac{z}{\varepsilon_j} + \chi_j \right),\ \sin\left(\frac{z}{\varepsilon_j} + \tilde{\chi}_j \right) \right), \quad j = 1, 2$$

Rotating frame

Move to coordinate frame co-moving with with the $V_1(\mathbf{r}, z)$ sublattice,

$$\mathbf{r}' = \mathbf{r} - \mathbf{h}_1(z) \quad , \quad z' = z$$

which after the phase transformation

$$\psi = \psi' \exp\left[-i\int_0^z |\mathbf{A}(\xi)|^2 d\xi\right] \text{ with } \mathbf{A}(z) = -\mathbf{h_1}'(z)$$

find lattice NLS with a pseudo-field $\mathbf{A}(z)$ -dropping $'$:

$$i\partial_z\psi + (\nabla + i\mathbf{A}(z))^2\psi - V(\mathbf{r}, z)\psi + \sigma|\psi|^2\psi = 0$$

$V_1(r, z) = V_1(\mathbf{r})$, $V_2(r, z) = V_2(\mathbf{r} - \Delta\mathbf{h}_{21}(z))$, near sites $1, 2$ with $\Delta\mathbf{h}_{21}(z) = \mathbf{h}_2(z) - \mathbf{h}_1(z)$

NL HC Representation

In non-dim NLS eq using HC lattice with $|V| >> 1$ substitute

$$\psi(\mathbf{r},z) \sim \sum_v \left[a_v(z)\phi_{1,v}(\mathbf{r},z) + b_v(z)\phi_{2,v}(\mathbf{r},z)\right]$$

where

$$\left(\nabla^2 - \tilde{V}_j(\mathbf{r},z)\right)\phi_{j,v}(\mathbf{r},z) = -E_j\phi_{j,v}(\mathbf{r},z); \quad j = 1,2$$

$\phi_{j,v}$ are termed orbitals

Substitute ψ into NLS eq. with pseudo-field, multiply $\phi_j(\mathbf{r}-\mathbf{p})e^{-i\mathbf{k}\cdot\mathbf{p}}; \quad j = 1,2$ and integrate

Discrete HC System

$$
\begin{aligned}
i\frac{da_{mn}}{dz} + e^{i\varphi(z)}\left(\mathcal{L}_-(z)b\right)_{mn} + \sigma|a_{mn}|^2 a_{mn} = 0 \\
i\frac{db_{mn}}{dz} + e^{-i\varphi(z)}\left(\mathcal{L}_+(z)a\right)_{mn} + \sigma|b_{mn}|^2 b_{mn} = 0
\end{aligned}
$$

$$
(\mathcal{L}_-(z)b)_{mn} = L_0(z)b_{mn} + L_1(z)b_{m-1,n-1}e^{-i\theta_1(z)} + L_2(z)b_{m+1,n-1}e^{-i\theta_2(z)}
$$

$$
(\mathcal{L}_+(z)a)_{mn} = \tilde{L}_0(z)a_{mn} + \tilde{L}_1(z)a_{m+1,n+1}e^{i\theta_1(z)} + \tilde{L}_2(z)a_{m-1,n+1}e^{i\theta_2(z)},
$$

where $\varphi(z), \theta_j(z), L_j(z), \tilde{L}_j(z) \in \mathbb{R}, \; j = 1, 2, 3$ known

Typical Rotation Patterns for Sublattices

- Same rotation, same or different radii:

$$\mathbf{h}_2(z) = R_a \mathbf{h}_1(z) = R_a \eta \left(\cos\left(\frac{z}{\varepsilon}\right),\ \sin\left(\frac{z}{\varepsilon}\right) \right)$$

- π-Phase offset rotation

$$\mathbf{h}_2(z) = \mathbf{h}_1(z + \varepsilon\pi) = -\eta \left(\cos\left(\frac{z}{\varepsilon}\right),\ \sin\left(\frac{z}{\varepsilon}\right) \right)$$

- Different frequencies

$$\mathbf{h}_j(z) = \eta \left(\cos\left(\frac{z}{\varepsilon_j}\right),\ \sin\left(\frac{z}{\varepsilon_j}\right) \right),\quad j = 1, 2$$

BCs – Linear Floquet Bands

Look for solutions of the form

$$a_{mn}(z) = a_n(z;\omega)e^{im\omega} \ , \quad b_{mn}(z) = b_n(z;\omega)e^{im\omega}$$

Find linear difference eq with periodic coef; Floquet thy:

$$f(z+T) = e^{-i\alpha(\omega)z} f(z)$$

HC Floquet Bands

HC lattice linear band structure–typical parameters

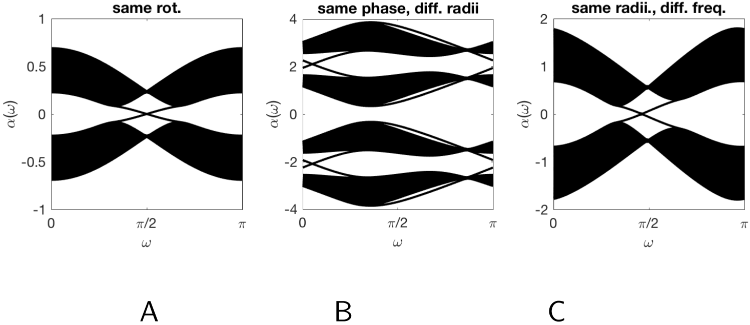
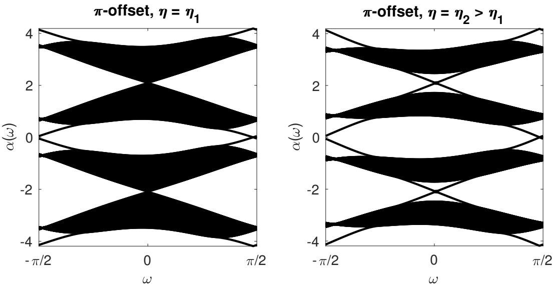


Fig A: same freq, same radii
Fig B: same freq, different radii ($R_2 = R_1/2$)
Fig C: diff freq ($1/\varepsilon_2 = \omega_2 = 2\omega_1 = 1/\varepsilon_1$), same radii

HC Floquet Bands –con't

HC lattice linear band structure–typical parameters



A B

Figs A & B: π offset, same rotation

Fig B vs Fig A: radius $\eta_2 > \eta_1$

Linear HC Edge Mode Dynamics

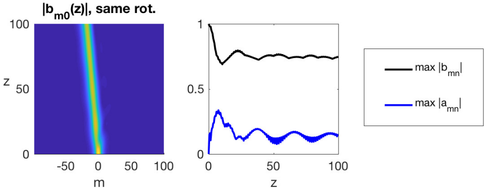


Fig Above: Same rotation, same radii

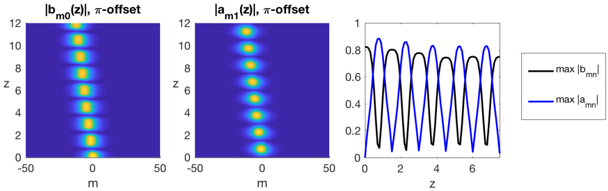


Fig Above: π offset, different radii

Adiabatic HC Lattice

Take HC lattice, uniform rotation, and $\mathbf{A} = \mathbf{A}(Z)$, where $Z = \varepsilon z$

In lattice system: $a_n = a_n(z, Z), \; b_n = b_n(z, Z)$

Multiple scales asymptotics (ZZ BC):

$$a_n \sim 0; \quad b_n \sim C(Z, \omega) b_n^S(Z)$$

where $b_n^S(Z) = \{r^n(Z); \;\; |r| < 1; \;\; r = r(\omega, \rho; \mathbf{A}(Z)), \; n \geq 0\}$

In general edge mode existence ($|r| < 1$) depends on ω, ρ, Z

Modes can 'disintegrate' under evolution

Can find an NLS type eq for envelope C whose coef. depend on Z

Numerics: discrete and continuous models: very good agreement

Conclusion–Topological Edge States

Photonic lattices with longitudinal variation

- Systematic method to find tight binding (TB) discrete eqs for same rotation and complex longitudinally driven lattices; special case: honeycomb lattices
- Find Floquet bands; they indicate topological/nontopological edge waves
- Topological waves: no backscatter, propagate stably around defects, corners

Conclusion: HC Edge States

Same rotation:

- Construct asymptotic theory for rapid helical variation
- Envelope of edge modes satisfy standard NLS eq
- NLS solitons topological case: unidirectional, propagate stably around defects, corners; 'solitons inherit topological properties'

Generalized longitudinal rotation

- Find TB eq; Floquet bands, large number of new/novel topological modes
- Can do this for many lattices: HC, staggered sq, Lieb, Kagome...

Ref.: MJA, C. Curtis, Y-P Ma 2013-15; MJA, J. Cole: 2017-18