Analytic Semigroups

The operator $A = \partial_x \ on \ D_A = H^1(R) \subset H^0(R) = H$ is closed and densely defined and generates a strongly continuous semigroup of contractions on H,

$$S(t)u_0(x) = u_0(x-t) \qquad \forall u_0 \in D_A.$$

Moreover, for any $u_0 \in D_A$,

$$A\,S(t)u_0(x) = \partial_x u_0(x-t) = u_0'(x-t) \in H$$

and $u(t) = S(t)u_0$ is the unique solution of $u'(t) + Au(t) = 0, \quad u(0) = u_0$. Note that it is necessary to have $u_0 \in D_A$ in order for this to be true. On the other hand, $A = -\nabla^2 \ on \ D_A = H^2(U) \cap H_0^1(U) \subset H^0(U) = H$ is also closed and densely defined and this operator generates the following strongly continuous semigroup

$$S(t)u_0 = \sum_n e^{-\lambda_n t}(u_0, \phi_n)_H \phi_n \qquad .$$

Then $S(t)u_0$ is the unique solution to the problem $u'(t) + Au(t) = 0, \quad u(0) = u_0$, which in this case has the realization

$$\begin{aligned} &\partial_t u - \nabla^2 u = 0 \quad in \quad U \times (0,T) \\ &\quad u = 0 \qquad\quad on \quad \Gamma \times (0,T), \\ &u(\bullet, 0) = u_0 \qquad in \ U. \end{aligned}$$

Note, however, that

$$A\,S(t)u_0(x) = \sum_n e^{-\lambda_n t}(u_0, \phi_n)_H A\phi_n = \sum_n e^{-\lambda_n t}(u_0, \phi_n)_H \lambda_n \phi_n$$

and recall, $\quad A\,S(t)u_0 \in H \quad \Leftrightarrow \{e^{-\lambda_n t}(u_0, \phi_n)_H \lambda_n\} \in \ell_2$. Then it follows that for $t > 0$, $A\,S(t)u_0 \in H$ for all $u_0 \in H$. In fact,

for $t > 0, \quad A^m S(t)u_0 \in H$ for all $u_0 \in H \ and \ all \ m > 0;$

i.e., this semigroup maps H into $D(A^m)$ for every m whereas the previous semigroup mapped $D_A \ into \ D_A$. The difference arises from the fact that while both A's are accretive, only the second example operator A is coercive; i.e. for some $a_0 > 0$

$$(Au, u)_H \geq a_0 \|u\|_H^2 \qquad \forall u \in D_A \subset H$$

In fact let us recall that for an elliptic operator $Lu = -\nabla(K(x)\nabla u(x))$ we can define Hilbert spaces $V \ and \ H = H^0(U)$ such that $V \hookrightarrow H$ and there is an associated bilinear form $a(u,v) : V \times V \rightarrow R$ satisfying

i) $|a(u,v)| \leq C\|u\|_V\|v\|_V$

ii) $a(u,u) \geq c\|u\|_V^2.$

Here the bilinear form is defined from $a(u,v) = (Lu,v)_H \; for \; u,v \in C_c^\infty(U)$ and is extended to $V \times V$ by continuity. This induces a bounded linear operator (we can call it L) acting from $V \; into \; the \; dual \; V'$. We can also define an operator A with domain dense in H such that A is closed but not bounded on H. That is,

$$D_A = \{u \in V : Au \in H\}$$

and

$$(Au,v)_H = a(u,v) \qquad \text{for all } u \in D_A \text{ and } v \in V,$$

Then A is a closed operator whose domain contains $H^2(U) \cap V$. and whose domain is contained in V. Since L is bounded from $V \; to \; V'$ it is not correct to say that L is an extension of A.

The semigroup theory we have developed asserts that the operator $-A$ generates a $C^0 - s/g$ of contractions on H. In addition, the Hille-Yosida theorem implies that

$i) \;\; \forall \, \lambda > 0, \;\; (\lambda + A) : D_A \to H$ is bijective

and $\quad ii) \;\; \|\lambda(\lambda + A)^{-1}\|_{L(H)} \leq 1 \qquad \forall \, \lambda > 0.$

Recall that it is sufficient for A to be accretive in order for these conditions to hold. For the operator A that is associated with a coercive bilinear form $a(u,v)$, the condition i) holds not just for all λ on the positive real axis, but for all λ in a sectorial subset of the complex plane. More precisely, it can be shown that for an operator A associated with a bilinear form satisfying,

$$a(u,u) \geq c\|u\|_V^2 - \mu\,\|u\|_H^2 \qquad (1)$$

there exists $\theta, \; 0 < \theta < \pi/2$, such that i) holds for all $\lambda \in \sum(\,\theta + \pi/2)$ where

$$\sum(\alpha) =: \{\lambda \in C : 0 \leq |\arg\lambda| < \alpha\};$$

i.e., $\sum(\alpha)$ denotes a sector which is symmetric about the positive real axis and has central angle equal to 2α. For example, when the operator A is symmetric, the eigenvalues of A all lie on the positive real axis. This means that the only λ for which $\lambda + A$ fails to be invertible arethose λ lying on the negative real axis. Then in this case, i) holds for all $\lambda \in \sum(\,\theta + \pi/2)$ where θ can be chosen as close as we like to $\pi/2$. In general, the eigenvalues of A are confined to a sector about the real axis (depending on c and μ) and then θ must be chosen correspondingly smaller. In general, for an operator A associated with a bilinear form satisfying (1), we can show

$\exists\theta_0,\ 0 < \theta_0 < \frac{\pi}{4},\ such\ that$

i) $(\lambda I + A)^{-1} \in L(H) \quad \forall\lambda \in \sum(\theta_0 + \pi/2)$ (2)

ii) $||(\lambda I + A)^{-1}||_{L(H)} \le \frac{1}{|\lambda|} \qquad \forall\lambda \in \sum(\theta_0 + \pi/2)$

The possibility to allow λ to move off the positive real axis has unexpectedly strong consequences. Recall that we can write

$$(\lambda + A)^{-1}x = \int_0^\infty e^{-\lambda t}S(t)x\,dt = \pounds\{S(t)\} \quad \text{for } x \in H$$

and, (formally), using the inversion formula for the Laplace transform,

$$S(t) = \pounds^{-1}\{(\lambda + A)^{-1}\} = \frac{1}{2\pi i}\lim_{b\to\infty}\int_{a-ib}^{a+ib} e^{\lambda t}(\lambda + A)^{-1}d\lambda \quad \text{for } t \ge a. \qquad (3)$$

Now let T(t) be defined by

$$T(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}(\lambda + A)^{-1}dt \qquad (4)$$

where C denotes a deformation of the Laplace inversion contour lying in $\sum(\theta + \pi/2)$, with $\theta \le \theta_0$, as shown.

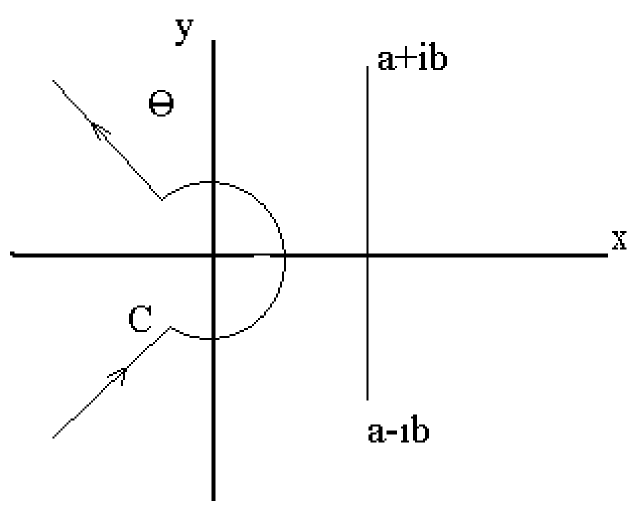

Figure 1

The estimates on $\|(\lambda + A)^{-1}\|_{L(H)}$ can be used to show that this integral converges in the uniform operator topology to T(t) in L(H). Then T(t) can be shown to have the following properties:

(a) $T(t)$ is an analytic function of t for $t \in \sum(\omega)$, for $\omega < \theta_0$.

(b) $T'(t) = -AT(t)$ for all $t \in \sum(\omega)$,

(c) $\|T^{(n)}(t)\|_{L(H)} \leq C_n(\omega)|t|^{-n}$ for all $t \in \sum(\omega)$,

(d) $\|T(t)x - x\|_H \leq C\,|t|\,\|Ax\|_H \quad \forall x \in D_A$

These results have a number of implications. First, it follows from (d) that $T(t)$ converges to the identity, I, in the strong operator topology as t decreases to zero. Also, since the hypotheses imply that $-A$ generates a $C^0 - s/g$, $S(t)$, and $S'(t)u = -AS(t)u$ for all $u \in D_A$, it follows from uniqueness that $T(t) = S(t)$. Evidently, when $S(t)$ is generated by an operator $-A$ as described, the semi-group is analytic. There are several important differences between analytic semigroups and the semigroups that are only strongly continuous. For example, for $t > 0$, we can use the definition (4) for T(t) to show that

$$\lim_{h\to 0} \frac{T(t+h) - T(t)}{h} x = T'(t)x \qquad \forall x \in H.$$

But the semigroup properties of T(t) imply

$$\lim_{h\to 0} \frac{T(h) - I}{h} T(t)x = -AT(t)x.$$

The result is that for all postive t, $T'(t)x = -AT(t)x$ for all x in H. In the case of strongly continuous semigroups this equality holds only for x in D_A. Evidently, for an analytic semigroup, T(t) maps H into D_A for $t > 0$. This leads to the following additional result

$\forall x \in H$ and every $t > 0$, $\quad T(t)x \in D(A^n) \ \ \forall n \geq 1$ and $\quad T^{(n)}(t) = (-A)^n T(t)x$

The proof of this result is by induction. For $n = 1$ this is just the basic s/g definition. Then suppose the result holds up to $n = N - 1$ and write

$$t > 0, \qquad T^{(N)}(t)x = \left(T^{(N-1)}(t)x\right)' = \left((-A)^{N-1}T(t)x\right)' = \left((-A)^{N-1}T(t-s)T(s)x\right)' \qquad 0 < s < t,$$

$$= \left(T(t-s)(-A)^{N-1}T(s)x\right)' = (-A)T(t-s)(-A)^{N-1}T(s)x, \qquad 0 < s < t,$$

$$= (-A)^N T(t)x.$$

Then the result follows by induction.

The extra regularity of the analytic semigroup leads to the following existence result for the abstract initial value problem.

Thereom Suppose $-A$ generates an analytic semigroup $S(t)$ on H. Then for all $u_0 \in H$ there exists a unique solution to the IVP $u'(t) + Au(t) = 0,\ u(0) = u_0$. this solution is given

by $u(t) = S(t)u_0$, and for each $t > 0$, $S(t) : H \to D(A^m)$ *for every* $m > 0$.
If $f \in C([0,\infty) : H)$ is such that for some $\alpha,\ 0 < \alpha \leq 1,$ and $C > 0$,

$$\|f(t) - f(s)\|_H \leq C|t-s|^{\alpha} \qquad 0 < t < s,$$

there exists a unique solution
$u(t) \in C^0([0,\infty) : H) \cap C^1[(0,\infty) : H]$ *for* $u'(t) + Au(t) = f(t),\ u(0) = 0$. This solution is given by

$$u(t) = \int_0^t S(t-\tau)f(\tau)\,d\tau.$$

These exceptional regularity results are illustrated in the following examples:

For $H = L^2(R),\ -A = \partial_{xx}$ with $D_A = H^2(R) \hookrightarrow C^0(R) \hookrightarrow H$ the operator $-A$ generates an analytic semigroup. Then for any $u_0 \in H$, the unique solution to the abstract IVP is given by $S(t)u_0$; in this case that is,

$$S(t)u_0(x) = (4\pi t)^{-1/2}\int_R e^{-|x-y|^2/4t}u_0(y)dy.$$

Note that for any $u_0 \in H$, $S(t)u_0(x) = u(x,t)$ is infinitely differentiable with respect to both x and t for all x and all $t > 0$. This smoothing is also present in the example where $H = L^2(U)$ for U a bounded open set in R^n and $-A = div(M(x)\nabla u)$ with $D_A = H^2(U) \cap H_0^1(U)$. In this case

$$S(t)u_0(x) = \sum_{n>0}(u,w_n)_H\ e^{-\lambda_n t}w_n(x)$$

and it is evident that for any $u_0 \in H$ and any $t > 0$, $S(t)u_0(x)$ belongs to $C^\infty((0,\infty) : D(A^n))$ for all n.