Spectral Properties of Elliptic Operators

In previous work we have replaced the strong version of an elliptic boundary value problem

$$L[u(x)] = f(x) \qquad x \in U$$
$$BC[u(x)] = g(x) \qquad x \in \Gamma$$

with the weak problem

find $u \in V \;\; such \;\; that \quad B[u,v] = f(v) \quad \forall v \in V.$

The connection between theweak problem and the strong problem, (and in particular the boundary conditions), was established by making use of the abstract Green's formula

$$B[u,v] = (L[u],v)_H + \langle T_0 v, T_1 u\rangle \qquad \forall u,v \in V.$$

If the bilinear form B is $V-elliptic, \quad B[u,u] \geq c_0\|u\|_V^2 \;\; \forall u \in V.$ then it follows immediately from the Lax-Milgram lemma that the weak problem has a unique solution for every f. This is the same thing as saying that L is an isomorphism from V onto H.

On the other hand, if B is only $V-H-coercive, \quad B[u,u] \geq c_0\|u\|_V^2 \; - c_1\|u\|_H^2 \;\; \forall u \in V,$ then we are not able to apply this lemma directly to the weak problem. Instead, we replace the weak problem with

find $u \in V \;\; such \;\; that \quad B_\mu[u,v] = f(v) + \mu(u,v)_H \quad \forall v \in V.$

For μ sufficiently large, B_μ is V-elliptic and the Lax Milgram lemma implies the existence of a solution to this perturbed problem. Then L_μ is an isomorphism from V onto H, hence there is a bounded inverse, L_μ^{-1} from H to V. Then, since the embedding of V into H is compact by the Rellich lemma, we have that $i \circ L_\mu^{-1} : H \to H$ is compact. That is,

$$L_\mu u = f + \mu u \quad \Leftrightarrow u = L_\mu^{-1} f + \mu L_\mu^{-1} u \Leftrightarrow (I-K)u = F$$

where $Ku = \mu L_\mu^{-1} u, \; and \; F = L_\mu^{-1} f.$ The operator $I-K$ is an operator of Fredholm type for which there is a theorem, known as the Fredholm alternative theorem, stating conditions under which there is a unique solution to $(I-K)u = F$. By translating this theorem into the notation and terminology of the weak boundary value problem, we obtain a similar existence-uniqueness theorem for the weak boundary value problem.

Now we are going to apply a similar approach in considering the eigenvalue problem for an elliptic operator. We suppose that

$$L[u(x)] = -div(A(x)\nabla u(x)) + c(x)u(x) \qquad x \in U$$

is uniformly elliptic on U, and we will consider the eigenvalue problem of finding values of the scalar λ, such that

$$\begin{aligned} L[u(x)] &= \lambda u(x) \qquad x \in U \\ u(x) &= 0 \qquad\quad\; x \in \Gamma \end{aligned}$$

has nontrivial solutions. Note that in the abscence of first order terms,

$$(L[\phi],\psi)_0 = (\phi, L[\psi])_0 \qquad \forall \phi,\psi \in C_0^\infty(U)$$

and we say that L is symmetric. Then it follows that the associated bilinear form B is also symmetric,

$$B[u,v] = B[v,u] \quad \forall u,v \in V = H_0^1(U).$$

Unless $c(x) \geq 0$, the bilinear form B is not V-elliptic, but it is V-H-coercive, which is to say, B_μ is V-elliptic, for μ sufficiently large. Then we recall that $L_\mu : H_0^1(U) \to H^0(U)$ is an isomorphism for μ sufficiently large. Then, instead of considering the eigenvalue problem for L, we will consider the eigenvalue problem for L_μ^{-1} which is a compact operator for which there is a classical theorem. By translating the conclusions of the classical theorem into the terminology of our elliptic operator, we obtain the following theorem.

Theorem Under the assumptions on L, there is a countable collection $\{\lambda_k\}$ of scalars such that

$$\begin{aligned} L[u(x)] &= \lambda u(x) \qquad x \in U \\ u(x) &= 0 \qquad\quad\; x \in \Gamma \end{aligned}$$

has nontrivial solutions. Moreover,

(a) the λ_k are all real and $\gamma < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$

(b) Let $N_k = \{u \in H_0^1(U) : L[u] = \lambda_k u\}$. Then dim $N_k < \infty$ *all* k *and*

$$(u,v)_0 = 0 \quad \textit{if} \quad u \in N_k,\ v \in N_j \qquad j \neq k$$

(c) The weak solutions $\{\phi_k\}$, *for*

$$\begin{aligned} L[\phi(x)] &= \lambda \phi(x) \qquad x \in U \\ \phi(x) &= 0 \qquad\quad\; x \in \Gamma \\ \|\phi\|_0 &= 1, \end{aligned}$$

form an orthonormal basis for $H^0(U)$, and at the same time, an orthogonal (but not orthonormal) basis for $H_0^1(U)$.

Proof- Let γ denote the smallest value such that $L_\gamma : H_0^1(U) \to H^0(U)$ is an isomorphism. Then for any $\mu \geq \gamma$, there is a bounded linear operator $S_\mu : H^0(U) \to H_0^1(U)$, which is inverse to L_μ; i.e., $L_\mu(S_\mu f) = f$ *for all* $f \in H^0(U)$. Let $K = i \circ S_\mu : H^0(U) \to H_0^1(U) \subset H^0(U)$. Then K is a compact linear operator on $H^0(U)$. Note that

$$S_\mu f = u \qquad \Leftrightarrow \qquad B_\mu[u,v] = (f,v)_0 \ \forall v \in H_0^1(U) \qquad \Leftrightarrow \qquad L_\mu u = f$$

For arbitrary $f,g \in H^0(U),\ \textit{let}\ Kf = u,\quad Kg = v.\ \textit{Then}$

$$(f,Kg)_0 = (f,v)_0 = B_\mu[u,v] = B_\mu[v,u] = (g,u)_0 = (u,g)_0 = (Kf,g)_0.$$

Thus the symmetry of the bilinear form implies that K is symmetric, and since K is bounded, in fact compact, it follows that K is self adjoint.
In addition,

$$(Kf,f)_0 = (u,f)_0 = B_\mu[u,u] \geq c_0\|u\|_1^2,$$

so K is a compact self adjoint, positive operator on the Hilbert space $H = H^0(U)$. We have the following theorem about such operators,

Theorem (*Fredholm-Riesz-Schauder*) For K a compact self adjoint, positive operator on the Hilbert space H,

(a) K has a countable collection of positive eigenvalues, $\{\theta_k\}$, accumulating at zero; i.e.,

$$\|K\|_{L(H)} \geq \theta_1 \geq \theta_2 \geq \cdots \to 0$$

(b) Let $N_k = \{u \in H : K[u] = \theta_k u\}$. Then $\dim N_k < \infty$ *all* k *and*

$$(u,v)_H = 0 \quad \textit{if} \quad u \in N_k,\ v \in N_j \qquad j \neq k$$

(c) The normalized eigenfunctions for K form an orthonormal basis for H.

The compact self adjoint, positive operators on the Hilbert space H are the analogues of symmetric positive definite matrices on R^n.
Now

$$K\phi = \theta\phi \qquad \Leftrightarrow \qquad B_\mu[\theta\phi, v] = (\phi, v)_0 \ \forall v \in V = H_0^1(U) \qquad \Leftrightarrow \qquad L_\mu[\theta\phi] = \phi$$

hence

$$L_\mu[\theta\phi] = \phi \qquad \Leftrightarrow \qquad L_\mu[\phi] = \frac{1}{\theta}\phi$$

so the eigenfunctions of K are the eigenfunctions of L_μ but the eigenvalues of L_μ are the reciprocals of the eigenvalues of K. Thus

$$\|K\|_{L(H)} \geq \theta_1 \geq \theta_2 \geq \cdots \to 0 \quad implies \qquad 0 < \frac{1}{\|K\|_{L(H)}} < \alpha_1 \leq \alpha_2 \leq \cdots \to \infty.$$

In addition, $\qquad L_\mu[\phi_k] = \alpha_k \phi_k \Leftrightarrow L[\phi_k] = (\alpha_k - \mu)\phi_k = \lambda_k \phi_k$
with

$$0 < \frac{1}{\|K\|_{L(H)}} - \mu < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

If $\{\phi_k\}$ denote the weak solutions of

$$\begin{aligned} L_\mu[\phi(x)] &= \alpha\phi(x) \qquad x \in U \\ \phi(x) &= 0 \qquad\quad\; x \in \Gamma \\ \|\phi\|_0 &= 1, \end{aligned}$$

then for $u \in V = H_0^1(U),$ *write* $\qquad u = \sum_{k>0}(u,\phi_k)_0\,\phi_k = \sum_{k>0} u_k \phi_k$

Define $\qquad ((u,v))_V = B_\mu[u,v] + (u,v)_H \qquad \forall u,v \in V.$

Note that $\qquad ((u,v))_V = B_\mu\left[\sum_{k>0} u_k\phi_k, \sum_{j>0} v_j\phi_j\right] + \left(\sum_{k>0} u_k\phi_k, \sum_{j>0} v_j\phi_j\right)_H$

$$= \sum_k \sum_j u_k v_j B_\mu[\phi_j,\phi_k] + \sum_k u_k v_k.$$

But $\qquad B_\mu[\phi_j,\phi_k] = (L_\mu\phi_j,\phi_k)_H = \alpha_j(\phi_j,\phi_k)_H = \lambda_j \delta_{jk}$

hence $\qquad ((u,v))_V = \sum_k (\alpha_k + 1) u_k v_k.$

and, in particular, $\qquad ((\phi_j,\phi_k))_V = (\alpha_k + 1)\,\delta_{jk}.$

To see that the eigenfunctions form a complete family for the inner product on V, suppose

$$((u,\phi_k))_V = 0 \qquad \forall k.$$

Then $\qquad 0 = ((u,\phi_k))_V = B_\mu[u,\phi_k] + (u,\phi_k)_H = \alpha_k(u,\phi_k)_H + (u,\phi_k)_H$

i.e., $\qquad (\alpha_k + 1)(u,\phi_k)_H = 0 \qquad \forall k.$

But $\qquad \alpha_k \geq \frac{1}{\|K\|_{L(H)}} > 0 \qquad \forall k$

hence $\qquad (u,\phi_k)_H = 0 \qquad \forall k$ and since the family is complete in H, $u = 0.$■

Note that $u \in V = H_0^1(U)$ *iff* $\|u\|_V^2 = ((u,u))_V = \sum_k(\alpha_k + 1)u_k^2 < \infty$

Similarly, $u \in H = H^0(U)$ *iff* $\|u\|_H^2 = (u,u)_0 = \sum_k u_k^2 < \infty$

For $F \in D'(U),\ \psi \in D(U) \subset H$,

$$F(\psi) = F(\textstyle\sum_k \psi_k \phi_k) = \sum_k \psi_k F(\phi_k) = \sum_k \psi_k F_k$$

and we have $F \in V'$ *iff*

$$|F(\psi)| = \left|\textstyle\sum_k \psi_k F_k\right| = \left|\textstyle\sum_k (1+\alpha_k)^{1/2}\psi_k(1+\alpha_k)^{-1/2}F_k\right|$$

$$\leq \left|\textstyle\sum_k (1+\alpha_k)^{-1}F_k^2\right|^{1/2}\|\psi\|_V^2$$

i.e., $F \in V'$ *iff* $\left\{(1+\alpha_k)^{-1/2}F_k\right\} \in \ell_2$.

In addition, $u \in H$ *iff* $\{u_k\} \in \ell_2$.

$u \in V$ *iff* $\left\{(1+\alpha_k)^{1/2}u_k\right\} \in \ell_2$.

and we can define, more generally,

$$u \in H^s(U) \ \ iff \ \ \left\{(1+\alpha_k)^{s/2}u_k\right\} \in \ell_2,$$

and we have, with this definition, $(H^s(U))' = H^{-s}(U)$. This scale of Hilbert spaces, each with the inner product

$$((u,v))_s = \textstyle\sum_k (1+\alpha_k)^s u_k v_k$$

is completely analogous to the Fourier spaces $H^s(R^n)$ we have defined previously.

Consider now the following examples.

1) $L[u(x)] = -u''(x), \qquad 0 < x < \pi$
$u(0) = u(\pi) = 0$

Here $c(x) = 0$ *so* $\mu = 0$ *and* $\alpha_k = \lambda_k$. For this operator, it is easy to compute the eigenvalues and eigenfunctions

$$\phi_k(x) = \sqrt{\frac{2}{\pi}}\ \sin(kx) \quad and \quad \lambda_k = k^2 \qquad k = 1,2,\ldots$$

Then the function

$$u(x) = \left\{ \begin{array}{ll} \frac{2x}{\pi} & 0 < x < \pi/2 \\ 2\left(1 - \frac{x}{\pi}\right) & \pi/2 < x < \pi \end{array} \right\}$$

has

$$u_k = (u, \phi_k)_0 = -\frac{4}{\pi k^2} \sin\left(\tfrac{k\pi}{2}\right).$$

Now $(1+\lambda_k)^{1/2} \sim k$ so $(1+\lambda_k)^{s/2} u_k \sim k^{s-2} \in \ell_2$ for $s \le 1$;

i.e. $u \in H^1 = H_0^1(U)$.

On the other hand, for the function $v(x) = 1,\ 0 < x < \pi,$

we find $$v_k = \frac{1-(-1)^k}{k} \in \ell_2$$

but $(1+\lambda_k)^{s/2} v_k \sim k^{s-1} \in \ell_2$ for $s \le 0$;

i.e., $v \in H^0 = L^2(U)$.

Finally, for $w(x) = \delta(x - x_0) \quad x_0 \in (0, \pi)$

$$w_k = \sqrt{\frac{2}{\pi}}\ \sin(kx_0) \in \ell_\infty \quad w_k \notin \ell_2$$

and

$$(1+\lambda_k)^{-s/2} w_k \sim k^{-s} \in \ell_2 \quad \text{for } s > 1/2;$$

In particular, $\delta \in H^{-1}(U)$.

2) $$\begin{array}{ll} L[u(x,y)] = -\nabla^2 u(x,y) & in\ U = \{0 < x, y < \pi\} \\ u = 0 & on\ \Gamma \end{array}$$

Then, again, $c = 0 \ \ so \ \ \alpha = \lambda, \ \ and$

$$\phi_{j,k}(x,y) = \frac{2}{\pi} \sin(jx)\sin(ky) \quad \lambda_{j,k} = (j^2 + k^2), \quad j,k = 1,2,\ldots$$

In this case, for $w(x,y) = \delta(x-x_0)\delta(y-y_0) \quad (x_0, y_0) \in U$

we have $$w_{j,k} = \frac{2}{\pi}\sin(jx_0)\sin(ky_0)$$

and $(1+\lambda_{j,k})^{-s/2} w_k \sim (1 + j^2 + k^2)^{-s/2} \in \ell_2$ for $s > 1$

Then $w \notin H^{-1}(U)$.

For the function $$u(x,y) = \left\{ \begin{array}{ll} \frac{2x}{\pi} & 0 < x < \pi/2,\ 0 < y < \pi \\ 2\left(1 - \frac{x}{\pi}\right) & \pi/2 < x < \pi, 0 < y < \pi \end{array} \right\}$$

we have $$u_{j,k} = \frac{c_{j,k}}{j\,k^2} \in \ell_2$$

but $$(1 + j^2 + k^2)^{s/2} u_{j,k} \sim \frac{(1 + j^2 + k^2)^{s/2}}{j\,k^2} \notin \ell_2 \quad \text{if } s = 1$$

Then $u \in H$ *but* $u \notin V$.

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