Introduction to Semigroups for Evolution Equations

1. Abstract Initial Value Problems

We have considered initial boundary value problems (IBVP's) of the form

$$\begin{aligned} \partial_t u(x,t) + Lu(x,t) &= f(x,t) && x \in U,\ t \in (0,T) \\ u(x,0) &= u_0(x) && x \in U, \\ u(x,t) &= 0, && x \in \partial U,\ 0 < t < T, \end{aligned}$$

where L denotes an elliptic operator of order 2. This problem can be written abstractly as

$$\begin{aligned} &u'(t) + Au(t) = f(t), \qquad t \in (0,T) \\ &u(0) = u_0, \end{aligned}$$

where

$$(Au,v)_H = B[u,v] \qquad \forall u,v \in V \subset H \subset V'.$$

We showed that if the bilinear form B is coercive then the linear mapping A is an isomorphism from V onto V'. Recall that there exists a family of H-orthonormal eigenfunctions for A such that

$$\|Aw_k\|_H = |\lambda_k|\,\|w_k\|_H = |\lambda_k| \to \infty \quad as \quad k \to \infty.$$

Then A is clearly not bounded as a mapping from H into H. If we define

$$D_A = \{u \in H : Au \in H\}$$

then $\qquad H_0^1(U) \cap H^2(U) \subset D_A \quad and \quad A : D_A \to H$

Here we are saying that A restricted to D_A takes its values in H rather than V' but A is still not bounded in the norm of H. Note, however, that if

$$\{u_n\} \subset D_A, \qquad u_n \to u \ in\ H, \qquad and \quad Au_n \to v \ in\ H$$

then $\qquad u_n \to u \ in\ D'(U), \quad and \quad Au_n \to v \ in\ D'(U).$

This means $v = Au$ in the sense of distributions which implies in turn that

$$(Au,\phi) = (v,\phi) \qquad \forall \phi \in D(U)$$

and since the test functions are dense in H, it follows that $v = Au \ in\ H,$ so $u \in D_A \ and\ Au = v.$ Any operator with the property that

$$\left\{ \begin{array}{l} \{u_n\} \subset D_A, \qquad u_n \to u \ in\ H, \\ \qquad\quad and \quad Au_n \to v \ in\ H \end{array} \right\} implies \quad \{u \in D_A \ and\ Au = v\}$$

is said to be a **closed** operator. Clearly any bounded/continuous linear operator is closed but, as we have just seen, the converse is, in general false. In our example, $H_0^1(U) \cap H^2(U) \subset D_A$ implies that D_A must be dense in $H = L^2(U)$.

Typically, we will consider problems

$$\begin{aligned} &u'(t) + Au(t) = f(t), \qquad t \in (0,T) \qquad (IVP) \\ &u(0) = u_0, \end{aligned}$$

with the assumption that A is closed and densely defined on H. We will also suppose

$$(Ax,x)_H \geq 0 \quad \forall x \in D_A \subset H.$$

Then A is said to be **accretive**. Note that this is not the same as coercive. For example,

$$A = -\nabla^2 \quad on \quad D_A = H_0^1(U) \cap H^2(U)$$

satisfies

$$(Au,u)_H = -\int_U u\nabla^2 u\,dx = \int_U \nabla u \bullet \nabla u\,dx \geq 0 \quad \forall u \in D_A$$

i.e., the Laplacian is both coercive and accretive. More generally, if A is any elliptic operator of order 2, we can change the dependent variable in the initial value problem to obtain a new problem with A replaced by $A+\mu I,$ which is coercive for μ sufficiently large. Then all parabolic IBVP's can be assumed without loss of generality to involve an operator A that is coercive and hence accretive. On the other hand,

$$A = -\tfrac{d}{dx} \quad on \quad D_A = H_0^1(0,1)$$

satisfies

$$(Au,u)_H = -\int_0^1 u'(x)\,u(x)\,dx =$$
$$= -\int_0^1 \tfrac{1}{2}\,\tfrac{d}{dx}\,u^2(x)\,dx = \tfrac{1}{2}[u(0)^2 - u(1)^2] = 0$$

so this operator is accretive but not coercive.

Lemma 1 If A is closed, densely defined and accretive, then (IVP) has at most one solution.

Proof- Suppose $u(t)$ solves (IVP) with $u_0 = 0 = f.$ Then

$$\tfrac{d}{dt}\|u(t)\|_H^2 = 2(u'(t),u(t))_H = -2(Au(t),u(t))_H \leq 0,$$

hence

$$\|u(t)\|_H \leq \|u(0)\|_H = 0 \quad for\ all\ t > 0.\blacksquare$$

Note that if $u(t)$ solves (IVP) for an accretive operator A, then it is necessarily the case that $\|u(t)\|_H \leq \|u(0)\|_H.$

2. The Linear Space L(H)

If A and B are bounded linear operators on Hilbert space H then

$$(aA+bB)(x) = aA(x) + bB(x) \qquad \forall x \in H$$

defines another bounded linear operator on H; i.e., L(H) is a linear space. We can define a norm on L(H) by

$$\|A\|_{L(H)} = \sup\{\|Ax\|_H : \|x\|_H = 1\}.$$

Then it is an exercise to show that $L(H)$ is a complete normed linear space for this norm; i.e., if $\{A_n\}$ is a Cauchy sequence in L(H) then there exists an A in L(H) such that $\|A_n - A\|_{L(H)} \to 0$ as $n \to \infty.$ We say in this case that A_n **converges in the operator norm** to the limit, $A.$ A sequence $\{A_n\} \subset L(H)$ such that for some $A \in L(H)$ we have $\|A_nx - Ax\|_H \to 0\ as\ n \to \infty \ \ \forall x \in H,$ is said to **converge strongly** to the limit A. Convergence in the operator norm implies strong convergence but not conversely.

We now state two lemmas that will be needed later.

Lemma 2-(**Neumann Series**) If $B \in L(H)$ with $\|B\|_{L(H)} < 1$, then $(I-B)^{-1} \in L(H)$ and

$$(I-B)^{-1} = \sum_{n=0}^{\infty} B^n$$

The proof of this result is a complete analogue of the proof of the formula for the sum a convergent geometric series, where absolute values are replaced by operator norms.

Lemma 3- Suppose $A : D_A \to H \;\; and \;\; (\mu I - A)^{-1} \in L(H)$ for some complex number, μ. Then for $\lambda \in C$, $(\lambda I - A)^{-1} \in L(H)$ if and only if $\left[I - (\mu-\lambda)(\mu I - A)^{-1}\right]^{-1} \in L(H)$ and in this case

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1}\left[I - (\mu-\lambda)(\mu I - A)^{-1}\right]^{-1}$$

Proof- Let $B = I - (\mu-\lambda)(\mu I - A)^{-1}$. If $B^{-1} \in L(H)$ then

$$(\lambda I - A)(\mu I - A)^{-1}B^{-1} = (\lambda I - \mu I + \mu I - A)(\mu I - A)^{-1}B^{-1}$$
$$= \left((\lambda-\mu)(\mu I - A)^{-1} + I\right)B^{-1} = I \qquad on\ H$$

and

$$(\mu I - A)^{-1}B^{-1}(\lambda I - A) = (\mu I - A)^{-1}B^{-1}(\lambda I - \mu I + \mu I - A)$$
$$= (\mu I - A)^{-1}B^{-1}\left[(\lambda-\mu)(\mu I - A)^{-1} + I\right](\mu I - A) = I \quad on\ D_A.$$

The converse is proved similarly.■

3. **Semigroups of Solution Operators**

We return to the initial value problem

$$u'(t) + Au(t) = 0, \qquad t \in (0,T) \qquad (IVP)$$
$$u(0) = u_0,$$

assuming that A is closed, accretive and densely defined on H. Suppose that for every $u_0 \in D_A$ there exists a unique solution $u(t)$, and let us denote the dependence of $u(t)$ on u_0 by writing $u(t) = S(t)[u_0]$. Then, integrating the differential equation shows

$$u(t) = S(t)[u_0] = u_0 + \int_0^t -Au(\tau)\,d\tau.$$

It is evident that $S(t) : H \to H$ defines a linear operator on H. In addition, $\forall u_0 \in D_A$

i) $S(0)[u_0] = u_0$; $i.e.,\ S(0) = I$

ii) $S(t+\tau)[u_0] = S(t)[S(\tau)[u_0]] = S(\tau)[S(t)[u_0]]$;

i.e.,

$$S(t+\tau) = S(t) \circ S(\tau) = S(\tau) \circ S(t) \ for\ t,\tau \geq 0.$$

The second result follows from the uniqueness of the solution. We summarize these two results by saying that $\{S(t) : t \geq 0\}$ is a semigroup of solution operators on H. If, in addition,

$$S(t)[u_0] \to S(t_1)[u_0] \ in\ H\ as\ t \to t_1 \geq 0, \quad \forall u_0 \in D_A,$$

then we say that $S(t)$ is a **strongly continuous semigroup**, a $C^0-semigroup.$ Note that since A is accretive,

$$\frac{d}{dt}\|u(t)\|_H^2 = 2(-Au(t), u(t))_H \leq 0,$$

hence,

$$\|S(t)[u_0]\|_H = \|u(t)\|_H \leq \|u(0)\|_H = \|u_0\|_H;$$

i.e.,

$$\|S(t)\|_{L(H)} \leq 1 \qquad \forall t \geq 0.$$

In this case, we say $\{S(t) : t \geq 0\}$ is a *semigroup of contractions*.

Associated with any $C^0-semigroup$ there is a linear operator, $B,$ called the generator of the semigroup; B is defined by

$$D_B = \left\{x \in H : \lim_{h\to 0} \frac{S(h)-I}{h}x \text{ exists and belongs to H}\right\}$$

$$Bx := \lim_{h\to 0} \frac{S(h)-I}{h}x \qquad \forall x \in D_B.$$

It will be shown now that if $S(t)$ is the semigroup associated with the IVP involving operator A, then B is an extension of $-A.$ If A is accretive, then the semigroup generated by B will be a contraction semigroup.

Remark: $f(t) = e^{at}$ is defined by $\quad i) \quad f(t)f(\tau) = f(t+\tau)$

$ii) \quad f(0) = 1$

$iii) \quad a = \lim_{h\to 0} \frac{f(h)-1}{h}.$

i.e., Assuming only that f is continuous and has the semigroup property implies that f is differentiable.

Properties of $S(t)$.

Lemma 4 Suppose $S(t)$ *is a* $C^0-semigroup$ and $u_0 \in D_B.$ Then

i) $S(t)[u_0] \in D_B \quad \forall t \geq 0$

ii) $BS(t)[u_0] = S(t)[Bu_0] \qquad \forall t \geq 0$

iii) $(0,\infty) \ni t \rightsquigarrow S(t)[u_0]$ is differentiable for $t > 0$

and $\quad \frac{d}{dt}\left(S(t)[u_0]\right) = BS(t)[u_0] \quad \forall t > 0$

iv) $S(t)[u_0] - u_0 = \int_0^t BS(\tau)[u_0]\,d\tau = \int_0^t S(\tau)[Bu_0]\,d\tau.$

Proof- For $u_0 \in D_B$

$$\frac{S(h)S(t)[u_0] - S(t)[u_0]}{h} = \frac{S(t)(S(h)[u_0] - u_0)}{h}$$

$$\begin{array}{ccc} \frac{S(h)-I}{h} S(t)[u_0] & = & S(t)\frac{S(h)-I}{h} u_0 \\ \downarrow & & \downarrow \quad as\ h \to 0 \\ BS(t)[u_0] & = & S(t)[Bu_0] \end{array}$$

The limit on the right side of this equality exists because we have assumed that $u_0 \in D_B$. Then the limit on the left side exists, which implies that $S(t)[u_0] \in D_B$, and, moreover, ii) holds. Note further that

$$\begin{array}{ccccc} \frac{S(t+h)[u_0]-S(t)[u_0]}{h} & = & \frac{S(t)\,S(h)[u_0]-S(t)[u_0]}{h} & = & \frac{S(h)-I}{h} S(t)[u_0] \\ \downarrow & & as\ h \to 0+ & & \downarrow \end{array}$$

$\frac{d}{dt}(S(t)[u_0])$ (derivative from the right) $= B(S(t)[u_0])$

For $0 < h < t,$ we have

$$\begin{array}{ccc} \frac{S(t)[u_0]-S(t-h)[u_0]}{h} & = & \frac{S(h)-I}{h} S(t-h)[u_0] \\ \downarrow & as\ h \to 0+ & \downarrow \end{array}$$

(derivative from the left) $\frac{d}{dt}(S(t)[u_0]) \; = B(S(t)[u_0]);$

i.e., for $u_0 \in D_B,\ t > 0,\ S(t)[u_0]$ is differentiable with $\frac{d}{dt}(S(t)[u_0]) \; = B(S(t)[u_0]).$
Now integrating both sides of this expression from 0 to $t > 0,$

$$S(t)[u_0]-u_0 = \int_0^t B(S(\tau)[u_0])\,d\tau = \int_0^t S(\tau)[Bu_0]\,d\tau \quad \forall u_0 \in D_B \blacksquare \qquad (1)$$

Properties of B

Lemma 5- Suppose B is the generator of a strongly continuous semigroup on H. Then

i) D_B is dense in H
ii) B is closed

Proof- For an arbitrary $x \in H,\ let \quad x_t = \int_0^t S(\tau)[x]\,d\tau.$ Then

$$\begin{aligned} \frac{S(h)[x_t]-x_t}{h} &= \frac{\int_0^t S(\tau+h)[x]\,d\tau] - \int_0^t S(\tau)[x]\,d\tau}{h} \\ &= \frac{\int_h^{t+h} S(\tau)[x]\,d\tau] - \int_0^t S(\tau)[x]\,d\tau}{h} \\ &= \frac{\int_t^{t+h} S(\tau)[x]\,d\tau] - \int_0^h S(\tau)[x]\,d\tau}{h}, \end{aligned}$$

and, as $h \to 0+,$ we get $Bx_t = S(t)x - x.$ Since the limit on the right side exists, it follows that the limit on the left exists, which is to say, $x_t \in D_B,\ for\ t > 0.$ Moreover, $\frac{1}{t} x_t \to x\ in\ H\ as\ t \to 0+$ hence D_B is dense in H. Note that we have proved,

$$S(t)x - x = B\int_0^t S(\tau)[x]\,d\tau \qquad \forall x \in H,\ t \geq 0. \qquad (2)$$

To prove B is closed, suppose

$$D_B \ni u_n \to u \text{ (in } H) \text{ and } Bu_n \to v \text{ (in } H) \text{ as } n \to \infty.$$

Then $S(t)[u_n] - u_n = \int_0^t (S(\tau)[Bu_n])\,d\tau$ and as $n \to \infty$ we get,

$$S(t)[u] - u = \int_0^t (S(\tau)[v])\,d\tau.$$

But then,

$$\begin{array}{ccc} \dfrac{S(t)[u]-u}{t} & = & \dfrac{1}{t}\int_0^t (S(\tau)[v])\,d\tau. \\ \downarrow & & \downarrow \quad as\ t \to 0+ \\ Bu & = & v \end{array}$$

Since the limit on the right side exists, the limit on the left must exist, which implies $u \in D_B$ *with* $Bu = v$; *i.e.*, B is closed.■

We have seen that an initial value problem may generate a semigroup and that associated with any strongly continuous semigroup is a generator. Now we will see the connection between the IVP and the generator of its semigroup.

Theorem 1 Suppose $A : D_A \to H$ is closed and densely defined. Suppose also that for any $u_0 \in D_A$ there is a unique $u(t)$ such that

1. $u \in C^0([0,\infty) : H) \cap C^1((0,\infty) : H)$
2. $u'(t) + Au(t) = 0 \quad t > 0$
3. $u(0) = u_0$

Then $u(t) := S(t)[u_0]$ defines a strongly continuous semigroup (of contractions if A is accretive) on H. The generator, B, of this semigroup is then an extension of $-A$.

Proof- We have already noted that $u(t) := S(t)[u_0]$ defines a strongly continuous semigroup on H. Let the generator be denoted by B. Then it follows from 1) and 2) that for all $u_0 \in D_A$,

$$u(t) = S(t)[u_0] = u_0 + \int_0^t -Au(\tau)\,d\tau.$$

Then

$$\begin{array}{ccc} \dfrac{S(t)[u_0]-u_0}{t} & = & \dfrac{1}{t}\int_0^t -Au(\tau)\,d\tau. \\ \downarrow & as\ t \to 0+ & \downarrow \\ Bu_0 & = & -Au(0) = -Au_0. \end{array}$$

The limit on the right exists by properties of the integral, hence the limit on the left also exists. But then $u_0 \in D_A$ implies $u_0 \in D_B$ *and* $Bu_0 = -Au_0$ *for all* $u_0 \in D_A$. This proves B is an extension of $-A$.■

Examples:

1) Consider $u'(t) = b\,u(t), \quad u(0) = u_0$ *for* $b < 0$.
Then

$$u(t) = e^{bt}u_0 = S(t)[u_0],$$

and

$$|S(t)[u_0]| = |e^{bt}u_0| \le |u_0| \text{ since } b < 0.$$

Note that $a = -b > 0$ *satisfies* $\left(au, u \right)_H = au^2 \geq 0 \quad H = R^1.$

2) Consider $\quad u'(t) + Au(t) = 0, \qquad u(0) = u_0$
where

$$A = -\nabla^2 \text{ with } D_A = H_0^1(U) \cap H^2(U) \subset H = H^0(U).$$

$i.e.,$ This is the heat equation with homogeneous Dirichlet boundary conditions and we recall that A is closed, densely defined and accretive. Also, as we have seen often before,

$$u(t) = \sum_{n=1}^{\infty} (u_0, \phi_n)_H e^{-\lambda_n t} \phi_n(x) = S(t)[u_0].$$

In fact if we denote by T the isomorphism

$$H \ni u_0 \longrightarrow \{(u_0, \phi_n)_H\} \in \ell_2,$$

then $\quad S(t)[\bullet] = T^{-1}\{e^{-\lambda_n t} T(\bullet)\}$ and $B = -A = \nabla^2$ is the generator of the semigroup.