Problems

Problems from the text:
page 12, .problem 3
page 290 problems 5,6,7,8
page 291 problems 10,13,14,16,17
page 292 problem 18
page 345 1,2,3,4,7
page 425 1 through 10 (Hint for 2 and 6: expand u(x,t) in terms of an appropriate family of eigenfunctions $\{\phi_n(x)\}$)

1. (equivalence of definition of weak derivative)
   Show that for $u \in L^2(R^1)$, the following are equivalent:

   (a) $\partial_x u \in L^2(R^1)$
   (b) $z\hat{u} \in L^2(R^1)$
   (c) $\frac{u(x+h)-u(x)}{h}$ converges in $L^2(R^1)$ as $h \to 0$
   (d) there exists a sequence of test functions ϕ_n, such that ϕ_n converges to f in $L^2(R^1)$ and $\partial_x\phi_n$ converges in $L^2(R^1)$

2. For what values of s is the characteristic function of $I = [0,1]$ in $H^s(R)$?
   For what values of s is the characteristic function of $I^2 = [0,1]^2$ in $H^s(R^2)$?

3. To which spaces $H^m(U)$ do the following functions u belong?

$$(a) \qquad u(x) = \begin{cases} 0 & if\ \ 0 < x < 1 \\ x-1 & if\ \ 1 \le x < 2 \\ x^3 - x^2 - 3 & if\ \ 2 \le x < 3 \end{cases}$$

$$(b) \quad u(x,y) = \begin{cases} xy & 0 < x < 1, 0 < y < 1 \\ x(2-y) & 0 < x < 1, 1 < y < 2 \\ x & 0 < x < .5,\ y = 1 \\ 4 & x = .5,\ y = 1 \\ x & .5 < x < 1, y = 1 \end{cases}$$

4) Let $u(x) = \left\{ \begin{array}{ll} x & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \end{array} \right\}$ and $v(x) = Sin\pi x$

a) Determine whether u and v are orthogonal in $L^2(0,2)$.
   Are they orthogonal in $H^1(0,2)$?
b) Find the distance from u to v in $L^2(0,2)$ and in $H^1(0,2)$.

5. Use the Sobolev embedding theorem to show that the function

$$u(x,y) = \left\{ \begin{array}{ll} x^2y^2 & \text{if } x > 0, y > 0 \\ 0 & \text{otherwise} \end{array} \right\}$$

is continuous on $\Omega = (-1,1) \times (-1,1)$

6. Suppose $u \in H^4(U)$, *and* $v \in H^2(U)$. Show that

$$\int_U \nabla^2 u \nabla^2 v dx = \int (\nabla^4 u) v dx + \int_{\partial U} [(\nabla^2 u)\partial_N v - \partial_N(\nabla^2 u) v] dS$$

Here $\partial_N = n \bullet \nabla$ *where* $n =$ outward unit normal to ∂U

7. By evaluating the relevant integrals, show that:

x and $x \ln x \in H^1(0,1)$,
$\ln x \in L^2(0,1)$ but $x^{-1} \notin L^2(0,1)$
$x^{-1} \in H^{-1}(0,1)$ (a function belongs to $\in H^{-1}(0,1)$ if it is the weak derivative of an element of $L^2(0,1)$)

8. Suppose $u(x)$ is continuous for all x in R. Show that for all test functions $\phi(x)$,

$$\lim_{h \to 0} \int \phi(x)[u(x+h) - u(x)]/h dx = -\int u(x)\phi'(x) dx$$

9. Let $U = (a,b)$ denote a bounded open set in R.

a) Show that if $u \in H_0^1(U)$, then u is absolutely continuous and $u(a) = u(b) = 0$.
b) Show that $H^1(U)$ is the direct sum of $H_0^1(U)$ and the space of functions $v(x) = Ae^x + Be^{-x}$ for arbitrary constants A, B; *i..e*, show that any v of this form is orthogonal to every function in $H_0^1(U)$.
c) Show that for every u in $C_0^\infty(U)$,

$$u(x)^2 = \int_a^x 2u'(z)u(z)dz$$

and then use the C-S inequality to show that for some constant C $>$ 0,

$$\int_a^b u(z)^2 dz \le C \int_a^b u'(z)^2 dz$$

d) Show that the previous inequality continues to hold for all u in $H_0^1(U)$.

10. Let

$$u_1(x) = \left\{ \begin{array}{ll} 1-|x| & if\ |x|<1 \\ 0 & if\ \ |x|>1 \end{array} \right\}$$

$$u_2(x) = \left\{ \begin{array}{c} x^2\ \ if\ |x|<1, \\ 0\ \ if\ |x|>1 \end{array} \right\}$$

Find $u_1'(x)$, the weak derivative of $u_1(x)$ and show that it is a locally integrable function. Show that the weak derivative of $u_2(x)$ is not locally integrable

11. Show that for a bounded linear operator A on H, if $\|Ax\|_H \ge c\|x\|_H$ for all x in H, then the range of A is a closed subset of H.

12. Suppose $u(x,y) \in H^m(R_+^2)$ and let

$$v(x,y) = \left\{ \begin{array}{ll} u(x,y) & if\quad y>0 \\ \sum_{k=1}^{m} a_k u(x,-ky) & if\ \ y<0 \end{array} \right\}$$

where

$$\sum_{k=1}^{m} (-k)^s a_k = 1 \quad for \ \ 0 \le s \le m-1.$$

Show that

$$\partial_x^p \partial_y^q v(x,y) = \left\{ \begin{array}{ll} \partial_x^p \partial_y^q u(x,y) & if\quad y>0 \\ \sum_{k=1}^{m} (-k)^q a_k \partial_x^p \partial_y^q u(x,-ky) & if\ \ y<0 \end{array} \right\}$$

and that this implies

$$\lim_{y\to 0_-} \partial_y^q v(x,y) = \lim_{y \to 0+} \partial_y^q u(x,y) \quad for \ \ 0 \le q \le m-1.$$

Does this imply that the mapping defined by $v(x,y) = Eu(x,y)$ maps $H^m(R_+^2)$ continuously into $H^m(R^2)$?