Existence and Uniqueness of Weak Solutions to Parabolic IVP's

For $T > 0$, fixed, and $U \subset R^n$ a bounded open set, let $U_T = \{(x,t) : x \in U,\ 0 < t < T\}$. Then consider the following initial boundary value problem,

$$\begin{aligned} \partial_t u(x,t) + Lu(x,t) &= f(x,t) \quad in\ U_T \\ u(x,0) &= u_0(x) \qquad x \in U, \\ u(x,t) &= 0, \qquad x \in \partial U,\ 0 < t < T. \end{aligned} \tag{1}$$

where $Lu(x,t) = -\sum_{i,j=1}^{n} \partial_i([a_{ij}(x,t)]\,\partial_j u) + \sum_{j=1}^{n} b_j(x,t)\,\partial_j u + c(x,t)u(x,t)$

$$= -div([a(x,t)]\nabla u) + \vec{b}(x,t) \cdot \nabla u + c(x,t)u(x,t)$$

with

$$a_{ij},\, b_j,\, c \in L^\infty(U_T)$$

and for $a_0 > 0$, $\vec{z} \cdot [a_{ij}(x,t)]\vec{z} \geq a_0|\vec{z}|^2 \qquad \forall \vec{z} \quad \forall (x,t) \in U_T.$

Then L is said to be uniformly elliptic on U_T. We recall that the assumptions on the coefficients $a_{ij}, b_j, c,$ imply the existence of positive constants, a_1, a_0 and μ_0 such that for all $u,v \in H^1(U)$, and $\mu > \mu_0$,

$$|B[u,v,t]| \leq a_1\|u\|_1\|v\|_1 \qquad a.e.\ t \in (0,T)$$

$$B[u,u,t] + \mu(u,u)_0 \geq a_0\|u\|_1^2 \qquad a.e.\ t \in (0,T)$$

where

$$B[u,v,t] = \int_U \left\{ \sum_{i,j=1}^{n} \partial_i v[a_{ij}(x,t)]\,\partial_j u + v\left(\sum_{j=1}^{n} b_j(x,t)\,\partial_j u + c(x,t) \right) \right\} dx$$

Then the mapping $L_\mu = L + \mu I : H_0^1(U) \to H^{-1}(U)$

is an isomorphism, and the IBVP (1) is said to be a parabolic problem. For convenience, we will let

$$V = H_0^1(U), \qquad H = H^0(U), \text{ and } \qquad V' = H^{-1}(U).$$

Then $V \subset H = H' \subset V'$ and $\langle f,v \rangle_{V' \times V} = (f,v)_H$ for $f \in V', v \in V$.

We recall that

$$W[0,T] = \{u \in L_2[0,T : V] : \partial_t u \in L_2[0,T : V']\}$$

and that $W[0,T] \subset C[0,T : H]$.

Then we define a weak solution of (1) to be a function $u \in W[0,T]$ satisfying

$$i) \qquad (\partial_t u, v)_H + B[u,v,t] = (f,v)_H \qquad \forall v \in V\ a.e.\ t \in (0,T), \tag{2}$$

$$ii) \qquad u(0) = u_0.$$

Equivalently, a weak solution satisfies

$$i) \qquad \int_0^T (\partial_t u, v)_H dt + \int_0^T B[u,v,t]dt = \int_0^T (f,v)_H dt \qquad \forall v \in W[0,T]$$

$$ii) \qquad u(0) = u_0.$$

We point out that in a parabolic problem, we can without loss of generality assume that

the bilinear form $B[u,v]$ is coercive. To see this, we simply observe that when L is uniformly elliptic, the associated bilinear form is at least $V-H-coercive;$ i.e.,
$B_\mu[u,v] = B[u,v]+\mu(u,v)_H$ is coercive for $\mu > \mu_0 > 0$. If u is a weak solution for (2) then it is easy to show that $U(x,t) = e^{\mu t}u(x,t)$ solves

$$i) \qquad (\partial_t U,v)_H + B_\mu[U,v,t] = (e^{-\mu t}f,v)_H \qquad \forall v \in V \;\; a.e.\;\, t \in (0,T),$$

$$ii) \qquad U(0) = u_0.$$

Then existence of U is equivalent to existence of u and the problem for U involves a coercive bilinear form. Now we have,

Theorem (*Existence-uniqueness-continuous dependence*) Suppose L is uniformly elliptic on U_T with coefficients in $L^\infty(U_T)$. Then for every $f \in L_2[0,T:H^{-1}(U)]$ and each $u_0 \in H^0(U)$, there exists a unique $u \in W[0,T]$ satisfying (2).Moreover, the linear mapping

$$L_2[0,T:H^{-1}(U)]\times H^0(U):(f,u_0)\rightarrow u\in W[0,T] \qquad \text{is continuous.}$$

Proof- (uniqueness)
Suppose $u \in W[0,T]$ solves (2.2) in the case $f = u_0 = 0$. Then

$$\int_0^T(\partial_t u,u)_H dt + \int_0^T B[u,u,t]dt = 0.$$

But

$$\int_0^T(\partial_t u,u)_H dt = \frac{1}{2}\int_0^T(\partial_t u,u)_H dt = \frac{1}{2}\frac{d}{dt}\int_0^T(u,u)_H dt$$
$$= \frac{1}{2}\left(\|u(T)\|_H^2 - \|u(0)\|_H^2\right).$$

Then

$$\frac{1}{2}\|u(T)\|_H^2 + \int_0^T B[u,u,t]dt = 0,$$

and, using the fact that B is coercive,

$$\frac{1}{2}\|u(T)\|_H^2 + a_0\int_0^T\|u(t)\|_H^2 dt \leq \frac{1}{2}\|u(T)\|_H^2 + \int_0^T B[u,u,t]dt = 0$$

Since $u \in W[0,T]$ is continuous in t with values in $H = H^0(U)$, it follows that $u(t) = 0$. This proves that the solution is unique, if it exists. The proof of existence will be carried out in steps.

1) Existence of Approximate Solutions

Let $\{w_k\}$denote an orthonormal basis for H which is, at the same time, an orthogonal basis for V. For N a positive integer, let

$$u_N(t) = \sum_{j=1}^N C_{jN}(t)\, w_j$$

where the coefficients $C_{jN}(t)$ are required to satisfy

$$i) \qquad (\partial_t u_N,w_k)_H + B[u_N,w_k,t] = (f,w_k)_H \qquad k = 1,..,N \;\; a.e.\;\, t \in (0,T), \qquad (3)$$

$$ii) \qquad (u_N(0),w_k)_H = (u_0,w_k),.for\;\; k = 1,..,N$$

Since $\{w_k\}$denotes an orthonormal basis for $H,$ this reduces to,

$$\frac{d}{dt}C_{kN}(t) + \sum_{j=1}^{N} C_{jN}(t)B[w_j, w_k, t] = f_k(t) \qquad for\ \ k = 1,..,N$$

$$C_{kN}(0) = (u_0, w_k), .for\ \ k = 1,..,N$$

This is a system of linear ordinary differential equations in N unknowns where the coefficient matrix is positive definite on $[0,T]$ hence it follows that for every $f \in L_2[0,T : H^{-1}(U)]$ and each $u_0 \in H^0(U),$ there exists a unique solution $\left\{C_{kN}(t) : for\ \ k = 1,..,N\right\} \in C[0,T]^N$ for the system ofODE's and a corresponding approximate solution $u_N(t)$ for (3).

2) Energy Estimates

We will show that there exist constants C_1, C_2, C_3 depending only on U, T and the coefficients in $L,$ such that for each $N,$

1. **a**. $\|u_N\|_{C[0,T:H]} \le C_1\left(\|u_0\|_H + \|f\|_{L_2\left[0,T:V'\right]}\right)$

b. $\|u_N\|_{L_2[0,T:V]} \le C_2\left(\|u_0\|_H + \|f\|_{L_2\left[0,T:V'\right]}\right)$

c. $\|u_N'\|_{L_2[0,T:V']} \le C_3\left(\|u_0\|_H + \|f\|_{L_2\left[0,T:V'\right]}\right)$

We note first that

$$\|u_N(0)\|_H^2 = \sum_{j=1}^{N}(u_0, w_j)_H^2 \le \|u_0\|_H^2$$

and $\qquad 2|(f, u_N)_H| = 2|\langle f, u_N\rangle_{V'\times V}| \le 2\|f(t)\|_{V'}\|u_N(t)\|_V$

$$\le \frac{1}{\alpha}\|f(t)\|_{V'}^2 + \alpha\|u_N(t)\|_V^2$$

Now it follows from (3) that

$$(\partial_t u_N, u_N)_H + B[u_N, u_N, t] = (f, u_N)_H$$

and we have that,

$$(\partial_t u_N, u_N)_H = \frac{1}{2}\frac{d}{dt}(u_N, u_N)_H = \frac{1}{2}\frac{d}{dt}\|u_N\|_H^2$$

which leads to,

$$\frac{d}{dt}\|u_N\|_H^2 + 2B[u_N, u_N, t] \le \frac{1}{\alpha}\|f(t)\|_{V'}^2 + \alpha\|u_N(t)\|_V^2$$

i.e., $\qquad \frac{d}{dt}\|u_N\|_H^2 + 2\alpha\|u_N(t)\|_V^2 \le \frac{1}{\alpha}\|f(t)\|_{V'}^2 + \alpha\|u_N(t)\|_V^2$

Then,

$$\frac{d}{dt}\|u_N\|_H^2 + \alpha\|u_N(t)\|_V^2 \le \frac{1}{\alpha}\|f(t)\|_{V'}^2$$

and since $\qquad \|u_N(t)\|_H^2 \le \|u_N(t)\|_V^2,$

$$\frac{d}{dt}\|u_N\|_H^2 + \alpha\|u_N(t)\|_H^2 \leq \frac{1}{\alpha}\|f(t)\|_{V'}^2$$

We rewrite this as,

$$\frac{d}{dt}\left(\|u_N\|_H^2 e^{\alpha t}\right)e^{-\alpha t} \leq \frac{1}{\alpha}\|f(t)\|_{V'}^2$$

from which it follows by integrating from 0 to t,

$$\|u_N(t)\|_H^2 \leq e^{-\alpha t}\left(\|u_N(0)\|_H^2 + \frac{1}{\alpha}\int_0^T e^{\alpha t}\|f(t)\|_{V'}^2 dt\right)$$

Since this holds for every $t \in [0,T]$, it follows.that

$$\sup_{0\leq t\leq T}\|u_N(t)\|_H^2 \leq C_1\left(\|u_0\|_H^2 + \|f\|_{L_2[0,T:V']}^2\right)$$

proving a).

To prove b), we return to the estimate,

$$\frac{d}{dt}\|u_N\|_H^2 + \alpha\|u_N(t)\|_V^2 \leq \frac{1}{\alpha}\|f(t)\|_{V'}^2$$

and we integrate with respect to t over $[0,T]$ to get

$$\|u_N(T)\|_H^2 - \|u_N(0)\|_H^2 + \alpha\int_0^T\|u_N(t)\|_V^2 dt \leq \frac{1}{\alpha}\int_0^T\|f(t)\|_{V'}^2 dt$$

Then

$$\int_0^T\|u_N(t)\|_V^2 d \leq \frac{1}{\alpha}\|u_N(0)\|_H^2 + \frac{1}{\alpha^2}\int_0^T\|f(t)\|_{V'}^2 dt$$

which asserts that

$$\|u_N\|_{L_2[0,T:V]}^2 \leq C_2\left(\|u_0\|_H^2 + \|f\|_{L_2[0,T:V']}^2\right)$$

Finally, to prove c), let $V_N = span\{w_1,\dots,w_N\}$. Then, for each $v \in V$ with $\|v\|_V = 1$, we can use the fact that $\{w_j\}$ is an ON basis in H to write $v = v_1 + v_2$ where $v_1 \in V_N$ and $(v_2, w_j)_H = 0$ for $j = 1,\dots,N$. Since $\{w_j\}$ is also an orthogonal basis in V, we have

$$\|v_1\|_V^2 = (v_1,v_1)_V = \sum_{j=1}^{N}(v_1,w_j)_V^2\;\|w_j\|_V^2 \leq \|v\|_V^2 \leq 1$$

Now, it follows from (3) that

$$(\partial_t u_N, v_1)_H + B[u_N, v_1, t] = (f, v_1)_H$$

and

$$(\partial_t u_N, v)_H = (\partial_t u_N, v_1 + v_2)_H = (\partial_t u_N, v_1)_H$$

since

$$\partial_t u_N(t) = \sum_{j=1}^{N} C'_{jN}(t)\,w_j \qquad and \quad (v_2,w_j)_H = 0 \;\; for \;\; j = 1,\dots,N.$$

Then,

$$(\partial_t u_N, v)_H = (\partial_t u_N, v_1)_H = (f, v_1)_H - B[u_N, v_1, t]$$

and

$$|(\partial_t u_N, v)_H| \leq \|f\|_{V'} \|v_1\|_V + a_1 \|u_N\|_V \|v_1\|_V$$
$$\leq \|f\|_{V'} + a_1 \|u_N\|_V \qquad \text{since } \|v_1\|_V \leq 1$$

Then

$$\sup_{\|v\|_V \leq 1} |\langle \partial_t u_N, v\rangle_{V' \times V}| = \|u_N'(t)\|_{V'} \leq \|f\|_{V'} + a_1 \|u_N\|_V$$

Integrating this expression from 0 to $T,$ and using $b)$ leads to $c)$.

3) Existence of Weak Solutions

We see that b) implies $\{u_N(t)\}$ is bounded in $L_2[0,T : V]$

c) implies $\{\partial_t u_N(t)\}$ is bounded in $L_2[0,T : V']$

Then there exists a subsequence, $\{u_M(t)\} \subset \{u_N(t)\}$ such that

$u_M(t) \rightharpoonup u(t)$ weakly in $L_2[0,T : V]$

$\partial_t u_M(t) \rightharpoonup v(t)$ weakly in $L_2[0,T : V']$

But then v must equal $u'(t)$ since weak convergence in in $L_2[0,T : V]$ implies convergence in $D'[0,T : V],$ which implies convergence in $D'[0,T : V']$ and $u_M(t) \rightharpoonup u(t)$ in $D'[0,T : V']$ implies $\partial_t u_M(t) \rightharpoonup u'(t)$ in $D'[0,T : V']$. Similarly, weak convergence in $L_2[0,T : V']$ implies convergence in $D'[0,T : V']$ so $\partial_t u_M(t) \rightharpoonup v(t)$ in $D'[0,T : V']$ and $v = u'$. Now we have,

$u_M(t) \rightharpoonup u(t)$ weakly in $L_2[0,T : V]$

$\partial_t u_M(t) \rightharpoonup \partial_t u(t)$ weakly in $L_2[0,T : V']$

which implies that $u = w - \lim\, u_M$ belongs to $W[0,T]$. To see that this weak limit is a weak solution to the IVP, let

$$V_p = \left\{ v(t) = \sum_{j=1}^{p} d_j(t)\, w_j \, : \, d_j \in C^1[0,T],\ 1 \leq j \leq p \right\}.$$

Then it follows from (3) that for each M and all $p \leq M,$

$$\int_0^T (\partial_t u_M, v(t))_H dt + \int_0^T B[u_M, v(t), t] dt = \int_0^T (f, v(t))_H dt \qquad \forall v \in V_p$$

Now, the weak convergence results imply that we can let M tend to infinity to get,

$$\int_0^T (\partial_t u, v(t))_H dt + \int_0^T B[u, v(t), t] dt = \int_0^T (f, v(t))_H dt$$

and this holds for all $v \in V_p$, for arbitrarily large p. But it is evident that

$$\bigcup_{p>0} V_p \text{ is dense in } L_2[0,T : V]$$

which means that the last result extends to all $v \in L_2[0,T : V]$. Then u is a weak solution of the equation in (3). To see that $u(0) = u_0,$ note that for arbitrary $v \in C^1[0,T : V]$ such that $v(T) = 0,$

$$\int_0^T -(u,\partial_t v(t))_H dt + \int_0^T B[u,v(t),t]dt = \int_0^T (f,v(t))_H dt + (u(0),v(0))_H$$

Similarly, for each M,

$$\int_0^T -(u_M,\partial_t v(t))_H dt + \int_0^T B[u_M,v(t),t]dt = \int_0^T (f,v(t))_H dt + (u_M(0),v(0))_H$$

and if we subtract the latter equation from the former, we get

$$\int_0^T -(u-u_M,\partial_t v(t))_H dt + \int_0^T B[u-u_M,v(t),t]dt = (u(0)-u_M(0),v(0))_H.$$

Now we let M tend to infinity, and obtain

$$0 = \left(u(0) - \lim_M u_M(0), v(0)\right)_H \qquad \forall v \in C^1[0,T:V].$$

But $\lim_M u_M(0) = u_0$ and $C^1[0,T:V] \subset C[0,T:H]$ and it follows that $u(0) = u_0$. Then u is a weak solution of the IVP. Note that it now follows that every weakly convergent subsequence must converge to a weak solution of the IVP and since the weak solution has been shown to be unique, all subsequences must have the same weak limit. But in that case, the sequence $\{u_N\}$ itself must converge weakly to $u \in W[0,T]$.

By using a $v \in C^1[0,T:V]$ such that $v(0) = 0$, we can show in the same way that $\lim_M u_M(T) = u(T)$ in H. Then we can show that $\{u_N\}$ converges strongly to u in $L_2[0,T:V]$. To see this, we write,

$$\int_0^T (\partial_t u, u(t))_H dt + \int_0^T B[u,u(t),t]dt = \int_0^T (f,u(t))_H dt$$

and

$$\int_0^T (\partial_t u, u(t))_H dt = \frac{1}{2}\int_0^T \frac{d}{dt}\|u(t)\|_H^2 dt = \frac{1}{2}\|u(T)\|_H^2 - \frac{1}{2}\|u(0)\|_H^2$$

Then

$$\int_0^T (f,u(t))_H dt - \int_0^T B[u(t),u(t),t]dt = \frac{1}{2}\|u(T)\|_H^2 - \frac{1}{2}\|u(0)\|_H^2 \qquad (4)$$

Next, we observe that

$$a_0\|u_N - u\|_{L_2[0,T:V]}^2 \le \int_0^T B[u_N - u, u_N - u(t), t]dt$$
$$\le \int_0^T B[u_N,u_N,t]dt - \int_0^T B[u_N,u,t]dt - \int_0^T B[u,u_N-u,t]dt$$

But,

$$\int_0^T B[u_N,u_N,t]dt = \int_0^T (f,u_N(t))_H dt - \int_0^T (\partial_t u_N, u_N)_H dt$$
$$= \int_0^T (f,u_N(t))_H dt - \frac{1}{2}\left(\|u_N(T)\|_H^2 - \|u_N(0)\|_H^2\right)$$

and this leads to,

$$a_0\|u_N - u\|_{L_2[0,T:V]}^2 \le \int_0^T \{(f,u_N(t))_H - B[u_N,u,t]\}dt$$
$$- \int_0^T B[u,u_N-u,t]dt - \frac{1}{2}\left(\|u_N(T)\|_H^2 - \|u_N(0)\|_H^2\right)$$

Now, the weak convergence results imply we can allow N to tend to infinity on the right side of this last expression and make use of (4) to obtain

$$RHS \to \int_0^T (f, u(t))_H dt - \int_0^T B[u(t), u(t), t] dt - \frac{1}{2}\| u(T) \|_H^2 + \frac{1}{2}\| u(0) \|_H^2 = 0$$

from which it follows that $\| u_N - u \|_{L_2[0,T:V]}^2 \to 0$ as $N \to \infty$.

Note it now follows that the energy inequalities b) and c) apply to the weak solution u and this implies that the solution depends continuously on the data. That is,

$$\{u_0, f\} \in H \times L_2[0,T;V'] \to u \in W[0,T] \qquad \textit{is continuous},$$

where we recall that

$$\| u \|_{W[0,T]}^2 = \int_0^T \left\{ \| u(t) \|_V^2 + \| \partial_t u(t) \|_{V'}^2 \right\} dt.$$