Applications of Semigroups to Nonlinear IVP's

1.The Abstract IVP

Consider the following nonlinear initial value problem

$$u'(t) + Au(t) = F(u(t)) \qquad 0 < t < T, \qquad u(0) = u_0 \qquad (1.1)$$

where $-A : D_A \to H$ generates a $C^0 - s/g$ of contractions on H. Of course this includes the special case that the semigroup generated by $-A$ is analytic. A strong solution of (1) on [0,T] is a function $u(t) \in C^0([0,T) : H) \cap C^1((0,T) : H)$ which solves the equation and we will define a funtion $u(t)$ to be a mild solution of (1) if $u(t) \in C^0([0,T) : H)$ satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds \qquad 0 < t < T. \qquad (1.2)$$

The simplest existence proofs for problems like this make the assumption that $F : V \to H$ is locally Lipschitz; i.e., V denotes a closed subspace of H ($V = H$ is allowed) and for some $R > 0,$ there exists $C_R > 0$ such that

$$\|F(u) - F(v)\|_H \leq C_R\|u - v\|_V \qquad \forall u,v \in B_R(0) \subset V \qquad (1.3)$$

For some nonlinearities it will suffice to take V=H, while for others it will be necessary to choose V to be an appropriate proper closed subspace of H. In these cases we will suppose that S(t) maps H into $V \subset D_\infty$ with $\|S(t)x\|_V \leq C_V\|x\|_H,$ and, for convenience we will assume $C_V = 1.$

To show that (1) has a mild solution under the assumption (3), let

$$\Phi(u) = \int_0^t S(t-s)F(u(s))ds \qquad and \qquad v(t) = S(t)u_0,$$

also

$$R = 2\|u_0\|_V \qquad and \qquad K_R = RC_R + \|F(0)\|_H.$$

Then

$$\|F(u)\|_H - \|F(0)\|_H \leq \|F(u) - F(0)\|_H \leq C_R\|u\|_V \leq RC_R$$

and

$$\|F(u)\|_H \leq K_R \qquad \forall u \in B_R(0) \subset V.$$

This bound on $\|F(u)\|_H$ implies

$$\|\Phi(u(t))\|_V \leq T\max_{0\leq s\leq t\leq T} \|S(t-s)F(u(s))\|_V \leq TK_R$$

if $u(t) \in B_R(0)$ for $0 \leq t \leq T.$ Now if we let

$$M_R = \left\{u \in C([0,T] : H) : \|u(t)\|_V \leq R, \ \ 0 \leq t \leq T\right\}$$

Then for $u \in M_R$ and $0 < T < R/(2K_R)$ we have

$$\|\Phi(u(t))\|_V \leq TK_R < R/2 = \|u_0\|_V$$

i.e.,

$$\Phi : M_R \to M_R \qquad for \qquad 0 \leq t \leq T < \frac{R}{2K_R}$$

In addition, for $0 \le t \le T$,

$$\|\Phi(u(t)) - \Phi(w(t))\|_H \le C_R t \|u(t) - w(t)\|_V \qquad \forall u,v \in M_R$$

hence, for $t < 1/C_R$, Φ is a strict contraction on M_R. Now let

$$T_0 = \min[1/C_R, R/2K_R]$$

Then for $u \in M_R$ and $0 \le t \le T_0$,

$$\|v(t) + \Phi(u(t))\|_V \le \|u_0\|_V + \|\Phi(u(t))\|_V \le 2\|u_0\|_V = R$$

and it follows that $M_R \ni u \rightsquigarrow v + \Phi(u) \in M_R$ is a strict contraction. Then there is a unique fixed point, $\hat{u} \in M_R$ such that

$$\hat{u}(t) = v(t) + \Phi(\hat{u}(t)), \qquad 0 \le t \le T_0$$

i.e., $\hat{u}$ is a mild solution of the IVP. In order to prove that $\hat{u}$ is, in fact, a strong solution to the IVP, additional hypotheses on A or on F are needed. For example, if A generates an analytic semigroup, then $\hat{u}$ would have the additional smoothness required of a strong solution. Also if additional smoothness on F were assumed, we may be able to show the mild solution is strong.

Since the solution has only been shown to exist for $0 \le t \le T_0$, it is referred to as a local solution. In an effort to extend the solution to larger time, suppose we use $u_1 = \hat{u}(T_0)$as the initial condition for a new IVP and follow the same procedure to obtain a new mild solution on an interval $[T_0, T_1]$ for some $T_1 > T_0$. Repeating this procedure N times leads to solutions on $[0,T_0] \cup [T_0,T_1] \cup [T_1,T_2] \cup \ldots \cup [T_{N-1},T_N] = [0,T_N]$. In general, the length $|[T_j, T_{j+1}]|$ tends to zero with increasing j due to the fact that R, C_R, K_R grow as T increases. However, if it is known, say from some a-priori estimate of the solution, that any solution of the IVP must satisfy $\|u(t)\|_V \le C$ for $0 \le t \le T$, then we may take $R = \max[2\|u_0\|_H, C]$ in the procedure just described. Then we can divide [0,T] into subintervals $[T_j, T_{j+1}]$ of uniform length and in this way, obtain a solution for the interval [0,T]; i.e., a uniform bound on solutions implies a global solution.

The nonlinear operator $\Xi[u(t)] = v(t) + \Phi[u(t)]: H \to H$ may be interpreted as the continuous flow on H associated with the IVP.

2. A Nonlinear Diffusion Equation on R^n

Consider the problem

$$\begin{aligned} \partial_t u(x,t) &= \nabla^2 u(x,t) + f(u(x,t)) \qquad x \in R^n,\ \ t > 0 \qquad\qquad (2.1) \\ u(x,0) &= u_0(x) \qquad\qquad x \in R^n. \end{aligned}$$

In this problem we take, instead of a Hilbert space H, the Banach space of functions which are defined and continuous on R^n and have a finite max. This linear space of functions $X = C_b(R^n)$ is a Banach space for the sup norm. We assume also that the nonlinearity, $f : R \to R$ satisfies,

$$|f(u) - f(v)| \le C_R |u - v| \qquad \forall\ |u|,|v| \le R \qquad\qquad (2.2)$$

Note that $f(u) = u^2$ satisfies condition (2.2) for $C_R = 2R$. Then (2.2) implies that $F(u) = f(u(x,t))$ satisfies the condition (1.3) with $H = V = X$, and, since the composition of continuous functions is continuous, that $F(u) = f(u(x,t))$ maps X to itself.

Since the operator $A = -\nabla^2$ on $D_A = \{u \in X : Au \in X\} = C^2(R^n) \cap C_b(R^n)$ can be shown to generate a C^0 semigroup of contractions on X, it follows from the result of the previous section that the initial value problem has a unique mild solution, û(x,t) which satisfies,

$$\hat{u}(t) = S(t)u_0 + \int_0^t S(t-s)F(\hat{u}(s))ds \qquad 0 < t < T_0$$

i.e.,

$$\hat{u}(x,t) = \int_{R^n} K(x-y,t)u_0(y)dy. + \int_0^t \int_{R^n} K(x-y,t-s)f(\hat{u}(y,s))dyds. \qquad (2.3)$$

where

$$K(x,t) = 1/\sqrt{4\pi t}\ e^{-x^2/4t}, \qquad t > 0.$$

Since the semigroup generated by $-A = \nabla^2$ is, in fact, analytic, we can show that the mild solution to the IVP is actually a strong solution. This follows from the fact that when the semigroup is analytic, the abstract IVP has a strong solution when the inhomogeneous term $f(t)$ is only Lipschitz continuous in t. The condition (2.2) is sufficient to imply that $f(t) = f(u(x,t))$ is Lipschitz in t for any $u(x,t) \in X$.

In addition, for this problem it is possible to use monotonicity methods to establish uniform bounds on the solution under appropriate conditions on f. When f is such that such bounds can be established, the solution can be shown to be global in t.

3. An IBVP in 1-dimension

Consider the problem

$$\begin{aligned} &\partial_t u(x,t) - \partial_{xx} u(x,t) = f(u(x,t)) && 0 < x < 1,\ \ t > 0 \\ &u(x,0) = u_0(x) && 0 < x < 1, \\ &u(0,t) = u(1,t) = 0 && t > 0, \end{aligned}$$

where we suppose $f \in C^1(R)$.

Let $H = L^2(0,1)$ and $V = H_0^1(0,1)$. Then we can show that

$$V \subset C^{0,\alpha}(0,1) \qquad for \qquad 0 < \alpha \leq 1/2.$$

i.e., for $u \in V$, and $0 \leq x,y \leq 1$,

$$|u(x) - u(y)| = \left|\int_y^x u'(s)ds\right| \leq \left(\int_y^x 1^2 ds\right)^{1/2}\left(\int_y^x u'(s)^2 ds\right)^{1/2}$$

$$\leq |x-y|^{1/2}\left(\int_0^1 u'(s)^2 ds\right)^{1/2} \leq ||u||_V |x-y|^{1/2}$$

Then it follows that for $0 \leq x \leq 1$, $|u(x)| \leq ||u||_V$; i.e., $||u||_\infty \leq ||u||_V$. In particular then for $u \in V$, $f(u) \in H$ so $F = f(u)$ maps V to H. Now, for $u,v \in B_R(0) \subset V$,

$$\begin{aligned} ||f(u) - f(v)||_H^2 &= \int_0^1 |f(u(x)) - f(v(x))|^2 dx \\ &\leq (\max_{|s| \leq R}\left|f'(s)\right|)^2 \int_0^1 |u(x) - v(x)|^2 dx \\ &\leq C_R ||u-v||_H^2 \leq C_R ||u-v||_V^2 \end{aligned}$$

and we see that $f : V \mapsto H$ is locally Lipschitz. It follows from the results of section 1 that the abstract IVP has a unique mild solution, $\hat{u} \in C([0,T] : H)$ for $T > 0$, sufficiently small. However, since the semigroup generated by $-A$ is, in fact, an analytic semigroup, the Lipschitz smoothness of f is sufficient to imply that the mild solution is actually strong.

Note that we used that $V \subset C^{0,1/2}([0,1]) \subset H$ in order to assert that $f(u) \in H$ for $u \in V$ and that

$$u,v \in B_R(0) \subset V \text{ implies } \|u\|_\infty \leq R, \text{ and } \|v\|_\infty \leq R$$

which leads then to the result, $|f(u) - f(v)| \leq \max_{|s|\leq R}|f'(s)|\,|u - v|$. i.e., this is a case where we have to take V to be an appropriate closed subspace of H in order to get the behavior we need for f.

4. A Semilinear IBVP on $\mathbf{R}^1$

Consider the semilinear problem

$$\begin{aligned} \partial_t u(x,t) - \partial_{xx}u(x,t) + u(x,t)\partial_x u(x,t) &= f(u(x,t)) \qquad 0 < x < 1,\ t > 0 \\ u(x,0) &= u_0(x) \qquad 0 < x < 1, \\ u(0,t) = u(1,t) &= 0 \qquad t > 0, \end{aligned} \tag{4.1}$$

where we suppose $f \in C^1(R)$. Let

$$F(u) = f(u) - u\partial_x u$$

$$H = L^2(0,1) \qquad V = H_0^1(0,1) \subset C^{0,1/2}([0,1])$$

Then $\quad f : V \mapsto H$

and $\quad \|u\partial_x u\|_H \leq \|u\|_\infty \|\partial_x u\|_H \leq \|u\|_V^2$

so we have $\quad F : V \mapsto H.$ Moreover, for all $u,v \in B_R(0) \subset V,$

$$\begin{aligned} \|u\partial_x u - v\partial_x v\|_H &\leq \|u(\partial_x u - \partial_x v)\|_H + \|(u-v)\partial_x v\|_H \\ &\leq \|u\|_\infty \|u - v\|_V + \|u - v\|_\infty \|v\|_V \\ &\leq (\|u\|_V + \|v\|_V)\|u - v\|_V \leq 2R\|u - v\|_V \end{aligned}$$

and this implies F is locally Lipschitz on V. It follows then that the abstract IVP has a unique mild solution which can again be seen to be a strong solution due to the fact that $-A$ generates an analytic semigroup on H. The strong solution is only local in t unless some a-priori bound on the solution can be established.

5. A Semilinear IBVP on $\mathbf{R}^n$, n=2,3

The previous two examples were set in one space dimension where it happens that $V \subset C^{0,\alpha}(0,1)$ *for* $0 < \alpha \leq 1/2$. For $n \geq 2$, the Sobolev embedding theorem changes the situation and we have to deal more carefully with the function spaces in order to get the Lipschitz behavior for the nonlinearity.

For U a bounded open set in R^n, $n \geq 2$ and for $\alpha \geq 0$, define

$$H_\alpha(U) = \left\{ u \in H^0(U) : \sum_{j\geq 1} |\lambda_j|^{2\alpha} |(u,\varphi_j)_H|^2 < \infty \right\}$$

where $\{\varphi_j\}_{j\geq 1}$ denote the orthonormal family of eigenfunctions for $A = -\nabla^2$ on

$V = H_0^1(U); i.e.,$

$$H = H^0(U) \ni u = \sum_{j\geq 1}(u,\varphi_j)_H\varphi_j \qquad \|u\|_H^2 = \sum_{j\geq 1}|(u,\varphi_j)_H|^2$$

$$H_1 = D_A = \left\{u \in H : Au = \sum_{j\geq 1}\lambda_j(u,\varphi_j)_H\varphi_j \in H\right\}$$

i.e., $u \in D_A$ iff $\|Au\|_H^2 = \sum_{j\geq 1}|\lambda_j|^2|(u,\varphi_j)_H|^2 < \infty$

for $u \in H_\alpha, \qquad A^\alpha u = \sum_{j\geq 1}\lambda_j^\alpha(u,\varphi_j)_H\varphi_j \qquad 0 \leq \alpha \leq 1,$

$$\|u\|_\alpha^2 = \|A^\alpha u\|_H^2 = \sum_{j\geq 1}|\lambda_j|^{2\alpha}|(u,\varphi_j)_H|^2$$

This defines a sequence of linear spaces,

$$D_A = H_1 \subset H_\alpha \subset H_0 = H^0(U), \qquad 0 < \alpha < 1.$$

Evidently, H_α is a Hilbert space for

$$(u,v)_\alpha = (u,v)_H + (A^{\alpha/2}u, A^{\alpha/2}v)_H = \sum_{j\geq 1}(1+|\lambda_j|^{2\alpha})|u_j v_j|$$

i.e.,

$$\|u\|_\alpha^2 = \|u\|_H^2 + \|A^\alpha u\|_H^2$$

And since this can be seen to be the graph norm on D_A, it follows from the closed graph theorem that H_α is a Banach space for this norm. Of course the norm then supports this inner product and H_α becomes a Hilbert space. In particular, $H_{1/2} = H_0^1(U)$.

Embedding Results

We state now some results regarding the embedding of the H_α spaces.

If $\quad H_0^1(U) \cap H^2(U) \subset D_A = H_1 \subset H_\alpha \subset H_0 = H^0(U), \qquad 0 < \alpha < 1.$

then we can show that

H_α is continuously embedded in $W^{p,q}(U)$ if $\left\{ \begin{array}{c} 2\alpha > p \\ 2\alpha - n/2 > p - n/q \end{array} \right\}$

H_α is continuously embedded in $C^m(\bar{U})$ if $2\alpha - n/2 > m$

Now consider

$$\begin{aligned} \partial_t u(x,t) - \nabla^2 u(x,t) &= f(u(x,t)) && x \in U \subset R^n,\ t > 0 \\ u(x,0) &= u_0(x) && x \in U \\ u(x,t) &= 0 && x \in \Gamma,\ t > 0, \end{aligned}$$

where $f \in C^1(R)$. Then $F(u) = f(u(x,t)) \ : H_\alpha \to H$ provided $H_\alpha \hookrightarrow C^0(U)$; i.e., for $\alpha > n/4$. In addition, F is locally Lipschitz if

$$u,v \in B_R(0) \subset H_\alpha \text{ implies } \|u\|_\infty, \|v\|_\infty \leq R$$

Again, we need H_α continuously embedded in $C^0(\bar{U})$ which means that $\alpha > n/4$. It follows that for $u_0 \in H_\alpha$ with $\alpha > n/4$ there is a unique mild solution for the IBVP, $\hat{u}(t) \in C([0,T] : H)$ for sufficiently small $T > 0$. Since the semigroup generated by $-A$ is analytic here, the solution is actually a strong solution belonging to $C^0([0,T] : H)$

$\cap C^1((0,T):H)$. Note that for $n \geq 2$ it is not sufficient to choose $H_{1/2} = H_0^1(U)$ as the closed subspace of H which leads to Lipschitz behavior for F.

Now let us consider the IBVP in the more difficult case where $n = 3$ and the nonlinearity $F(u) = f(u(x,t))$ is given by

$$f(u) = \sum_{i=1}^{3} u(\partial u/\partial x_i)$$

This nonlinearity is more difficult to deal with than the previous $f \in C^1(R)$ and we need some lemmas before trying to prove existence of the solution to the IBVP.

Lemma 1 There exists a constant $C > 0$, such that for all $u \in H_1 = D_A$,

$$|u(x) - u(y)| \leq C\,\|Au\|_H\,|x-y|_{R^3}^{1/2} \qquad \forall x,y \in R^3$$

Proof- For $\varphi \in C_0^\infty(U)$ we have the classical representation for a solution of Poisson's equation in terms of a fundamental solution, (cf sec 2.2.1 in the Evans text)

$$\varphi(x) = C\int_U \frac{\nabla^2\varphi(y)}{|x-y|}dy$$

for C an appropriate constant. Applying the C-S inequality to this expression leads to

$$|\varphi(x) - \varphi(z)|^2 \leq C^2\left(\int_U \nabla^2\varphi(y)\left\{\frac{1}{|x-y|} - \frac{1}{|z-y|}\right\}dy\right)^2$$

$$\leq C^2\int_U |\nabla^2\varphi(y)|^2 dy \bullet \int_U \left\{\frac{1}{|x-y|} - \frac{1}{|z-y|}\right\}^2 dy$$

But $$\int_U \left\{\frac{1}{|x-y|} - \frac{1}{|z-y|}\right\}^2 dy \leq C_U|x-z|$$

for $C_U > 0$ depending only on U. Then it follows that

$$|\varphi(x) - \varphi(z)| \leq C\|A\varphi\|_H|x-z|^{1/2}$$

Since $C_0^\infty(U)$ is dense in $D_A = H_1 \subset C^0(\bar{U})$, we can approximate any $u \in D_A$ by $\{\varphi_n\} \subset C_0^\infty(U)$ and pass to the limit to get the result.■

Lemma 2 There exists a constant C $>$ 0, such that for all $u \in H_1 = D_A$,

$$\|u\|_\infty^4 \leq C\,\|Au\|_H^3\,\|u\|_H$$

Proof- The embedding results imply $D_A = H_1 \subset C^0(\bar{U})$ and, assuming the boundary Γ is smooth, we have that $u|_\Gamma = 0$, since $H_0^1(U) \cap H^2(U)$ is dense in $D_A = H_1$. Now if u is identically zero, the result is trivial so suppose $\|u\|_\infty = ess - \sup_U |u(x)| = L > 0$.

We have from the previous lemma

$$|u(x) - u(y)| \leq K\,|x-y|_{R^3}^{1/2} \qquad \text{for } K = C\|Au\|_H$$

and WOLG we may suppose $L = |u(0)|$. Let $R = (L/K)^2$ and consider the open ball, $B_R(0) \subset R^3$. For $x \in B_R(0)$

$$|u(x)| > |u(0)| - |u(0) - u(x)| \geq L - K|x|^{1/2} > L - (K/L) = 0$$

Since $u|_{\Gamma} = 0$ this last estimate implies $B_R(0) \subset U$ and for $x \in B_R(0), \quad |u(x)| \geq L - K|x|^{1/2}$. Now the result follows from,

$$\|u\|_H^2 \geq \int_{B_R(0)} |u(x)|^2 dx \geq \int_{B_R(0)} \left|L - K|x|^{1/2}\right|^2 dx$$

$$\geq 4\pi L^2 R^3 \int_0^1 (1 - z^{1/2})^2 z^2 dz = CL^2R^3 = CL^8K^{-6}$$

i.e.,

$$L^4 \leq CK^3 \|u\|_H. \blacksquare$$

Lemma 3 For $1 \geq \alpha > 3/4,$ and $\forall u, v \in D_A$

1. $f : H_\alpha \to H$ with $\|f(u)\|_H \leq C\|A^\alpha u\|_H \|A^{1/2}u\|_H$

2. $\|f(u) - f(v)\|_H \leq C(\|A^\alpha u\|_H \|A^{1/2}u - A^{1/2}v\|_H + \|A^{1/2}v\|_H \|A^\alpha u - A^\alpha v\|_H)$

Proof- Note that the embedding result asserts that for $1 \geq \alpha > 3/4,$ H_α is continuously embedded in $C(\bar{U})$. This implies that there exists a constant $C > 0,$ depending on U and α such that for all $u \in D_A, \quad \|u\|_\infty \leq C\,\|A^\alpha u\|_H$. Then for $u \in D_A, \quad u \in L^\infty(U)$ and $\partial u/\partial x_i \in L^2(U) = H$ so $f(u) \in H.$ Moreover

$$\|f(u)\|_H \leq \|u\|_\infty \|\nabla u\|_H \leq C\,\|A^\alpha u\|_H \|\nabla u\|_H \leq C\|A^\alpha u\|_H \|A^{1/2}u\|_H \quad .$$

This proves 1). Now note that

$$\|f(u) - f(v)\|_H \leq \|u\nabla u - v\nabla v\|_H = \|u\nabla(u - v) - (u - v)\nabla v\|_H$$

$$\leq \|u\|_\infty \|\nabla(u - v)\|_H + \|u - v\|_\infty \|\nabla v\|_H$$

$$\leq C\left(\|A^\alpha u\|_H \|A^{1/2}u - A^{1/2}v\|_H + \|A^{1/2}v\|_H \|A^\alpha u - A^\alpha v\|_H\right).$$

This proves 2).$\blacksquare$

Now we can show the results needed to establish existence for the solution of the IBVP. Since $D_A = H_1 \subset H_\alpha \subset H_0 = H^0(U), \quad 0 < \alpha < 1,$ it follows that the mapping f can be extended from H_1 to H_α for $1 \geq \alpha > 3/4.$ Moreover, $H_{3/4} \subset H_{1/2}$ and

$$\|A^{1/2}u - A^{1/2}v\|_H \leq \|A^{3/4}u - A^{3/4}v\|_H$$

It follows that f satisfies, for $1 \geq \alpha > 3/4,$

$$\|f(u) - f(v)\|_H \leq C(\|A^\alpha u\|_H + \|A^\alpha v\|_H)\,\|A^\alpha u - A^\alpha v\|_H$$

i.e., $f : H_\alpha \to H$ is locally Lipschitz for $1 \geq \alpha > 3/4.$ Then the IBVP has a unique mild solution for every $u_0 \in H_\alpha,\ 1 \geq \alpha > 3/4.$ Since the semigroup S(t), generated by $-A$ is analytic, this is also a strong solution.