# Introduction

Partial differential equations arise mathematical models based on a continuum view of nature. Since there are generally more than a single independent variable, the derivatives in the equations are partial derivatives, hence the name. Problems rarely call for the solution of an equation or system of equations without side conditions and one of the difficult issues that must be adressed is the question of what is the right number and type of side conditions to impose on the solution of a given partial differential equation in order to produce a problem which has a unique solution.

Avoiding that issue for the moment, we remark that in elementary courses on partial differential equations, proofs of existence of solutions tend to be direct. That is, we prove the solution exists by constructing it and proving that it works by substituting into the equation and side conditions. In the case that the equations are not explicitly solvable, as is often the case if the equation has variable coefficients or if the domain is not sufficiently simple, the direct approach cannot be used. In such cases we employ an indirect approach in which we cast the PDE in abstract setting where the partial differential operator in the equation is viewed as a mapping acting on function spaces. We then try to show that the mapping has properties which imply that for all reasonable data in the problem there is a function in the solution space that gets mapped by the operator onto the given data function. This procedure consists of three major steps:

1. Build a function space that is suitable for the problem at hand
2. Show how the PDE problem (including side conditions) can be abstractly formulated in the function space
3. Prove that the mapping associated with the PDE is one to one and onto its range.

There are several guidlines to follow in building a function space. Choosing a large function space requires weakening the notion of what it means to be a solution for the PDE. This usually makes existence easier to prove but makes it more difficult to show the solution is unique. If the notion of solution is too weak, then the solutions may have no physical meaning or they may be very difficult to compute. On the other hand, choosing a small function space makes it harder to prove existence but then it is usually easier to show that the solution is unique if it exists. In a very small function space solutions may be hard to compute and they may exist only under practically unrealistic assumptions on the data. In some problems it may happen that existence can be proved in one function space setting and uniqueness is proved in another. This is a very unsatisfactory situation and suggests that some mistake may have been made in modeling the physical system.

Properties that a function space must have include:

1. it must be a linear space (i.e., closed under forming linear combinations)
2. there must be a notion of convergence and the space must be complete for this definition of convergence
3. there must be a suitable way to define the derivative of functions in the space (often this requires a weakening of the meaning of derivative)

The ultimate aim is to translate the classical formulation of a problem involving a PDE with side conditions into an abstract statement about a (linear) mapping L on a function space X with values in a function space Y, in such a way that all the information about the PDE and the side conditions is encoded in the definition of L, X and Y. Questions of existence and uniqueness are then answered using the tools of functional analysis and usually the answers follow quite simply. The reason the results follow simply is that all the hard work gets done in the set up where the function spaces are defined and their properties discovered.