The Lumer Phillips Theorem

The Hille-Yosida theorem provides a set of conditions on the operator A that are necessary and sufficient in order for -A to generate a strongly continuous semigroup of contractions. It is often more convenient to have an equivalent set of conditions as described in the following theorem.

Theorem (**Lumer**-**Phillips**) Suppose $A : D_A \to H$ is closed and densely defined. Then the following are equivalent conditions:

1. $-A$ generates a C^0 semigroup of contractions on H
2. A is accretive and $I + A : D_A \to H$ is surjective

Proof- $(2) \Rightarrow (1)$: Suppose A is accretive. Then for $\lambda > 0$

$$\|(\lambda I + A)x\|_H \geq \lambda \|x\|_H \qquad \forall x \in D_A$$

Letting $z = (\lambda I + A)x$ this becomes,

$$\|z\|_H \geq \lambda \|(\lambda I + A)^{-1} z\|_H.$$

In particular, since $(I + A)$ is surjective, this previous estimate implies

$$\|(I + A)^{-1}\|_{L(H)} \leq 1.$$

Now observe that if $0 < \lambda < 2,$ then formally,

$$(\lambda I + A)^{-1} = (I + A)^{-1}[I + (\lambda - 1)(I + A)^{-1}]^{-1}$$
$$= (I + A)^{-1} \sum_{n=0}^{\infty} [(1 - \lambda)(I + A)^{-1}]^n$$

Note that if $0 < \lambda < 2,$ then $(1 - \lambda)(I + A)^{-1} \in L(H)$ and

$$\|(1 - \lambda)(I + A)^{-1}\|_{L(H)} \leq |\lambda - 1| \leq 1.$$

Then the Neumann series for $(1 - \lambda)(I + A)^{-1}$ converges and we get,

$$\left[\, I + (\lambda - 1)(I + A)^{-1} \,\right]^{-1} = \sum_{n=0}^{\infty} [(1 - \lambda)(I + A)^{-1}]^n$$

Now let $B = [I + (\lambda - 1)(I + A)^{-1}]$ and note that

$$(\lambda + A)(I + A)^{-1} B^{-1} = ((\lambda - 1) + (I + A))(I + A)^{-1} B^{-1}$$
$$= [(\lambda - 1)(I + A)^{-1} + I] B^{-1} = BB^{-1} = I$$

and

$$(I + A)^{-1} B^{-1} (\lambda + A) = (I + A)^{-1} B^{-1}((\lambda - 1) + (I + A))$$
$$= (I + A)^{-1} B^{-1}[(\lambda - 1)(I + A)^{-1} + I](I + A)$$
$$= (I + A)^{-1} B^{-1} B (I + A) = I;$$

i.e.,

$$(\lambda + A)^{-1} = (I + A)^{-1} B^{-1}$$

and

$$\|(\lambda + A)^{-1}\|_{L(H)} = \|(I + A)^{-1}\|_{L(H)}\|B^{-1}\|_{L(H)}$$
$$\leq \sum_{n=0}^{\infty}(1-\lambda)^n = 1/\lambda \qquad 0 < \lambda < 2.$$

Now it follows that

$$(\mu I + A)^{-1} = (\lambda I + A)^{-1}[I + (\mu - \lambda)(\lambda I + A)^{-1}]^{-1}$$

and for $\mu > 0$ and $|\mu - \lambda| \|(\lambda I + A)^{-1}\|_{L(H)} < 1$, we have

$(\mu I + A)^{-1}$ exists and $\|(\mu I + A)^{-1}\|_{L(H)} \leq 1/\mu$

Repeating this process we see that $(\lambda I + A)^{-1}$ exists for all $\lambda > 0$, and

$$\|(\lambda I + A)^{-1}\|_{L(H)} \leq 1/\lambda$$

Then the hypotheses of the Hille-Yosida theorem are satisfied and $-A$ generates a C^0 semigroup of contractions on H. This proves (2) implies (1).

Proof that $(1) \Rightarrow (2)$: If $-A$ generates a C^0 semigroup of contractions on H then

$$(S(t)x - x, x)_H = (S(t)x, x)_H - \|x\|_H^2 \leq (\|S(t)x\|_H - \|x\|_H)\|x\|_H \leq 0$$

This implies that for all x in D_A

$$((S(t)x - x)/t, x)_H \to (-Ax, x)_H \leq 0$$

i.e., A is accretive. In addition, it follows from the Hille-Yosida theorem that $(\lambda + A)$ is surjective for all $\lambda > 0$, and for $\lambda = 1$ in particular.■

Examples-

1) Let $H = H^0(R)$, $A = \partial_x$, $D_A = H^1(R)$. Note that $u \in H^1(R)$ implies that $u(x)$ is continuous and tends to zero as $|x|$ tends to infinity.

Then

$$(Au, u)_H = \int_R u'(x)u(x)dx = 1/2\ u(x)^2|_{x=-\infty}^{x=\infty} = 0 \qquad \forall u \in D_A$$

and A is accretive. Note that according to this, $-A$ is also accretive. Next, note that

$$(I \pm A)u = u(x) \pm u'(x) = f(x) \qquad u \in D_A, \quad f \in H$$

implies $(1 \pm i\alpha)U(\alpha) = F(\alpha)$ where U, F denote Fourier transforms of u and f, respectively. Then

$$U(\alpha) = \frac{F(\alpha)}{1 \pm i\alpha} = (1 \mp i\alpha)\frac{F(\alpha)}{1+\alpha^2} = G(\alpha) \mp i\alpha G(\alpha)$$

and

$$u(x) = g(x) \mp g'(x)$$

where

$$g(x) = T_F^{-1}\left[\frac{F(\alpha)}{1+\alpha^2}\right] = \int_R e^{|x-y|}f(y)dy.$$

This shows that $I + A : D_A \to H$ is onto and now it follows from the Lumer-Phillips theorem that $-A$ generates S(t), a $C^0 - s/g$ of contractions, on H. This implies, in turn that for each

$u_0 \in D_A$ there exists a unique $u(t)$ satisfying,

$$u'(t) + Au(t) = \partial_t u(x,t) + \partial_x u(x,t) = 0, \qquad u(0) = u_0.$$

Since it is evident that $u(x,t) = u_0(x-t)$ solves the initial value problem, it follows by uniqueness that $S(t)u_0(x) = u_0(x-t)$.

Since $-A$ is also accretive and $I - A : D \to H$ is onto, $+A$ must also generate a $C^0 - s/g$ of contractions, $Z(t)$. Then the solution of

$$u'(t) - Au(t) = \partial_t u(x,t) - \partial_x u(x,t) = 0, \qquad u(0) = u_0 \in D_A$$

is given by $u(x,t) = Z(t)u_0(x) = u_0(x+t) = S(-t)u_0(x)$. Then uniqueness implies that $Z(t) = S(-t)$. In this case, the operator A generates a C^0 **group** of contractions on H. That is,

$$U(t) = \left\{ \begin{array}{lll} S(t) & \text{if} & t \geq 0 \\ Z(-t) & \text{if} & t < 0 \end{array} \right\}$$

satisfies

$$U(t)U(-t) = S(t)S(-t) = S(0) = I.$$

Then $\{U(t) : t \in R\}$ is a group of contractions on H since for each operator in the collection, the inverse operator is also in the collection.

2) Let $H = H^0(U)$, for $U \subset R^n$, open and bounded. Let $Au = -div(M\nabla u)$, for M a symmetric positive definite matrix, and let $D_A = H_0^1(U) \cap H^2(U)$. Then

$$(Au,u)_H = B[u,u] = \int_U \nabla u \cdot M\nabla u dx \geq 0 \qquad \forall u \in D_A.$$

This shows that A is accretive and, from previous results regarding elliptic operators, it is known that $A + \lambda I$ is surjective for $\lambda \geq 0$. For $\lambda = 1$, in particular, $A + I$ is surjective and the Lumer-Phillips theorem asserts that $-A$ generates a $C^0 - s/g$ of contractions, S(t). Then the IVP

$$u'(t) + Au(t) = f(t), \qquad u(0) = u_0$$

has a unique solution for all $u_0 \in D_A$ and for all $f \in C^1([0.\infty) : H)$. The solution is given by

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)d\tau.$$

This abstract IVP stands for the problem

$$\begin{array}{lll} \partial_t u(x,t) - div(M\nabla u(x,t)) = f(x,t) & & in \ U_T \\ u(x,0) = u_0(x) & & in \ U \\ u(x,t) = 0 & & on \ \Gamma \times (0,T) \end{array}$$

and then

$$u(x,t) = \sum_{n>0} e^{-\lambda_n t}(u_0,\phi_n)_H \phi_n(x) + \sum_{n>0} \int_0^t e^{-\lambda_n(t-\tau)}(f(\cdot,\tau),\phi_n)_H d\tau\, \phi_n(x),$$

where $\{\lambda_n, \phi_n\}$ denote the eigenvalues and normalized eigenfunctions associated with the elliptic operator A. Evidently

$$S(t)u_0 = \sum_{n>0} e^{-\lambda_n t}(u_0,\phi_n)_H \phi_n(x).$$

Groups

A family $\{G(t) : t \in R\} \subset L(H)$ is said to form a **group on H** if:

i $G(t)G(s) = G(t+s) \quad \forall s,t \in R$

ii $G(0) = I$

Note that $G(t)G(-t) = G(0) = I$ so that for each real t, $G(t)^{-1} = G(-t)$. The group G(t) is said to be a C^0-group on H if, $G(t)x \to x$ in H as t tends to 0, for all x in H. As in the case of semigroups, there is a closed, densely defined linear operator associated with G(t),

$$Bx = \lim_{h\to 0} \frac{G(h)-I}{h}x \qquad \text{for all } x \in D_B = \{x \in H : \lim_{h\to 0} \frac{G(h)-I}{h}x \text{ exists}\}$$

and we say this operator B is the generator for the group. Note that if G(t) is generated by B, then B generates a C^0-s/g, $S(t) = G(t)$ for $t \geq 0$, and $-B$ generates a C^0-s/g, $S(-t) = G(t)$ for $t \leq 0$. In this case, the Lumer-Phillips theorem implies that B is accretive and $I+B : D_B \to H$ is surjective and, in addition, $-B$ is accretive and $I-B : D_B \to H$ is surjective. Note that when B and $-B$ are both accretive, then

$$(Bx,x)_H \geq 0 \qquad and \qquad (-Bx,x)_H \geq 0 \qquad \forall x \in D_B$$

i.e.,

$$(Bx,x)_H = 0 \qquad \forall x \in D_B$$

In this case we say the operator B is **conservative**. Note that this implies

$$d/dt\|G(t)x\|_H^2 = 2(G'(t)x, G(t)x)_H = 2(BG(t)x, G(t)x)_H = 0$$

i.e.,

$$\|G(t)x\|_H^2 = \|x\|_H^2 \qquad \text{for all } t \in R$$

Then $\|G(t)\|_{L(H)} = 1$ and we say that $\{G(t) : t \in R\}$ is a **unitary group** on H.

Example- Consider the IVP

$$\begin{aligned}
&\partial_{tt}u(x,t) - \partial_{xx}u(x,t) = 0, && 0 < x < 1, \ t > 0\\
&u(x,0) = f(x), \quad \partial_t u(x,0) = g(x) && t > 0,\\
&u(0,t) = u(1,t) = 0, && 0 < x < 1.
\end{aligned}$$

Let

$$\begin{aligned}
u_1 &= \partial_x u & \partial_t u_1 &= \partial_{xt}u = \partial_x u_2\\
u_2 &= \partial_t u & \partial_t u_2 &= \partial_{tt}u = \partial_x u_1
\end{aligned}$$

Then

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(t=0) = \begin{bmatrix} f'(x) \\ g(x) \end{bmatrix}$$

i.e.,

$$\partial_t U(t) - AU(t) = 0, \qquad U(0) = U_0$$

where

$$H = L^2(0,1)^2, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \partial_x$$

$$D_A = \{U \in H : u_1 \in H^1(0,1),\ u_2 \in H_0^1(0,1)\}$$

Then

$$(AU, U)_H = \int_0^1 (\partial_x u_2 \cdot u_1 + u_2 \cdot \partial_x u_1) dx = \int_0^1 d/dx(u_1 u_2) dx$$

$$= (u_1 u_2)|_{x=0}^{x=1} = 0 \quad \text{(since } u_2 \in H_0^1(0,1))$$

This proves A is conservative. Now for $\lambda \neq 0,\ F \in H,$ consider

$$\lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \lambda u_1 + \partial_x u_2 \\ \lambda u_2 + \partial_x u_1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Then

$$\lambda \partial_x u_1 + \partial_{xx} u_2 = \partial_x F_1 \qquad \textit{and} \qquad \partial_x u_1 = F_2 - \lambda u_2,$$

or

$$\partial_{xx} u_2 - \lambda u_2 = \partial_x F_1 - \lambda F_2$$

Since $\partial_x F_1 - \lambda F_2 \in H^{-1}(0,1),$ this last equation has a unique weak solution $u_2 \in H_0^1(0,1),$ by the previously developed elliptic theory. Then

$$\lambda u_1 = F_1 - \partial_x u_2 \in L^2(0,1), \qquad \partial_x u_1 = F_2 - \lambda u_2 \in L^2(0,1),$$

so $u_1 \in H^1(0,1)$ and $U \in D_A.$ This shows that $\lambda + A : D_A \to H$ is surjective for all $\lambda \neq 0.$ ■

We have the following version of the Hille-Yosida theorem for groups rather than semigroups.

Theorem The following statements are equivalent:

1. $B : D_B \to H$ generates a $C_0 - \textit{unitary group}$ on H

2. a) B is closed and densely defined

b) $\forall \lambda \neq 0 \quad (\lambda - B) : D_A \to H$ is one to one and onto with

$$||\ (\lambda - B)^{-1} ||_{L(H)} \leq \frac{1}{\lambda}$$

Proof- Suppose B generates a group, $\{G(t) : t \in R\}$. Then B generates a contraction semigroup, $\{G(t) : t \geq 0\}$ and $-B$ also generates a contraction semigroup, $\{G(-t) : t \geq 0\}.$ Then Hille-Yosida theorem implies that both B and $-B$ satisfy the necessary conditions for generating a contraction semigroup and this implies 2.

Conversely, suppose the conditions 2 hold. Then, again by the Hille-Yosida theorem, B generates a contraction semigroup $\{S_+(t) : t \geq 0\}$ and $-B$ generates a contraction semigroup $\{S_-(t) : t \geq 0\},$ and these semigroups commute. For $x_0 \in D_B$

$$\frac{d}{dt}(S_+(t)S_-(t)x_0) = 0 \qquad t \geq 0$$

from which it follows that $S_+(t)S_-(t) = I \quad t \geq 0$. Then

$$G(t) = \left\{ \begin{array}{ll} S_+(t) & t \geq 0 \\ S_-(-t) & t \leq 0 \end{array} \right\}$$

satisfies all of the conditions for a group. i.e., (i) and (ii) are evident and

$$1 = \|G(t)\,G(-t)\| \leq \|G(t)\|\|G(-t)\| \leq \|G(t)\| \leq 1.$$

Finally, to see that B generates G(t), note that B generates $\{S_+(t) : t \geq 0\}$ *so for* $h > 0$,

$$\frac{G(h)x - x}{h} = \frac{(S_+(h) - I)x}{h} \to Bx.$$

and $-B$ generates $\{S_-(t) : t \geq 0\}$, so for $h < 0$

$$\frac{G(h)x - x}{h} = -\frac{(S_-(-h) - I)x}{-h} \to -(-Bx). \blacksquare$$

We have also a version of the Lumer-Phillips theorem for groups.

Theorem The following statements are equivalent:

1. . $B : D_B \to H$ generates a $C_0 - \textit{unitary group}$ on H

2. a) B is closed and densely defined

b) B is conservative

c) $\lambda + B$ is onto for some $\lambda > 0$ *and for some* $\lambda < 0$.

Example- Consider the IVP for the wave equation

$$\begin{array}{lll} \partial_{tt}u(x,t) - \partial_{xx}u(x,t) = 0 & & 0 < x < 1, \quad 0 < t < T, \\ u(x,0) = f(x), & \partial_t u(x,0) = g(x), & 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & & 0 < t < T. \end{array}$$

Let $u_1(x,t) = \partial_x u(x,t)$
$u_2(x,t) = \partial_t u(x,t)$

so

$$\begin{array}{ll} \partial_t u_1(x,t) = \partial_{xt}u(x,t) = \partial_x u_2(x,t) & u_1(x,0) = f'(x) \\ \partial_t u_2(x,t) = \partial_{tx}u(x,t) = \partial_x u_1(x,t) & u_2(x,0) = g(x); \end{array}$$

i.e.,

$$\partial_t \vec{U}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \partial_x \vec{U}(t), \qquad \vec{U}(x,0) = \begin{bmatrix} f' \\ g \end{bmatrix}.$$

Let

$$H = L^2(0,1)^2, \qquad D_A = \left\{ \vec{U} \in H : u_1 \in H^1(0,1),\ u_2 \in H_0^1(0,1) \right\}$$

Then

$$\left(A\vec{U},\vec{U}\right)_H = \int_0^1 (u_1\,\partial_x u_2 + u_2\,\partial_x u_1)\,dx = \int_0^1 \frac{d}{dx}(u_1 u_2)dx = (u_1 u_2)|_{x=0}^{x=1} = 0$$

which shows that A is conservative and must therefore generate a group if we can show that condition 2c of the previous theorem is satisfied. But for $\lambda \neq 0$ *and* $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in H$

$$\lambda\vec{U} + A\vec{U} = \vec{F} \quad \Leftrightarrow \begin{bmatrix} \lambda u_1 + \partial_x u_2 \\ \lambda u_2 + \partial_x u_1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

But in this case, $\lambda\partial_x u_1 + \partial_{xx} u_2 = \partial_x F_1$

and $\partial_x u_1 = F_2 - \lambda u_2$

hence

$$\lambda(F_2 - \lambda u_2) + \partial_{xx} u_2 = \partial_x F_1,$$

or

$$\partial_{xx} u_2 - \lambda^2 u_2 = \partial_x F_1 - \lambda F_1.$$

This last equation is uniquely solvable with $u_2 \in H_0^1(0,1)$ since $\partial_x F_1 - \lambda F_1 \in H^{-1}(0,1)$.
Next, $\lambda u_1 + \partial_x u_2 = F_1$ implies $\lambda u_1 = \partial_x F_1 - \partial_x u_2 \in L^2(0,1)$.
In addition, $\partial_x u_1 = F_2 - \lambda u_2 \in L^2(0,1)$ hence $u_1 \in H^1(0,1)$ and

$$\vec{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \begin{bmatrix} H^1(0,1) \\ H_0^1(0,1) \end{bmatrix} = D_A.$$

This shows that $\lambda + A : D_A \to H$ is onto for $\lambda \neq 0$.
Then A generates a group which we know from previous experience is given by

$$G(t)[\vec{U}(x,0)] = \begin{bmatrix} \frac{1}{2}(\tilde{f}'(x+t) + \tilde{f}'(x-t)) + \frac{1}{2}(\tilde{g}(x+t) - \tilde{g}(x-t)) \\ \frac{1}{2}(\tilde{f}'(x+t) - \tilde{f}'(x-t)) + \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) \end{bmatrix}$$

where f̃, g̃ denote the odd 2-periodic extensions of f and g.