The Trace and Embedding Theorems for a General Bounded Open Set

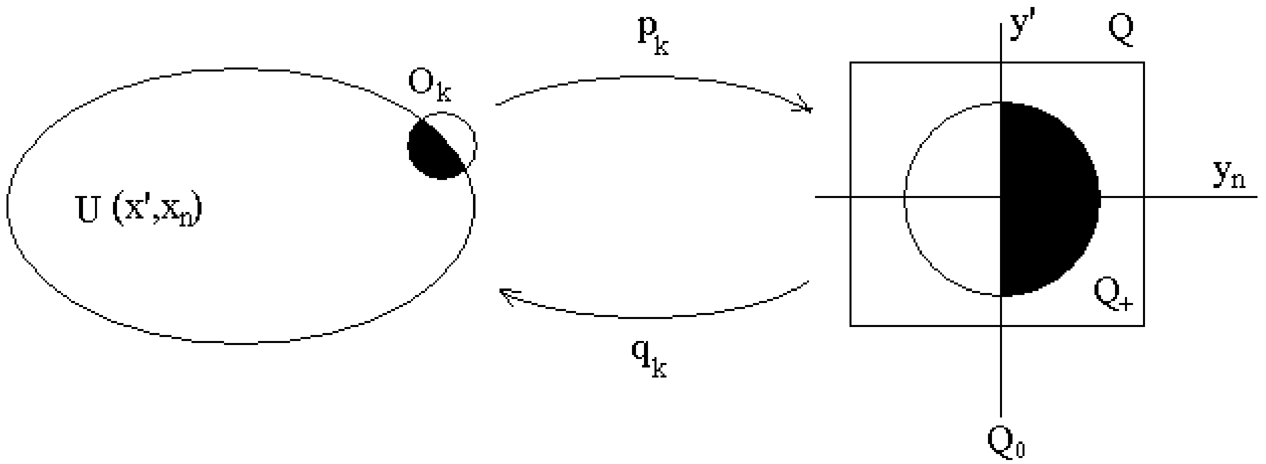
Now we show how the primitive versions of the results we have proved (i.e., when $U = R_+^n$) can be used to deduce analogous results when U is a more general open set. We will describe now the special properties U must have if this extension of results is to work.

1. Flattening the Boundary

Suppose U is a bounded open set in R^n. Then U is said to be ***regular*** if:

the boundary ∂U can be covered by open sets $O_k,\ k = 1,\ldots,M$ such that for each k,

- $p_k \in C^m(R^n)$ maps O_k onto open set Q
- $q_k = p_k^{-1} \in C^m(R^n)$ maps Q onto O_k
- $p_k : O_k \cap U \to Q^+ = Q \cap \{y_n > 0\}$
- $p_k : O_k \cap \partial U \to Q_0 = Q \cap \{y_n = 0\}$



Then $\partial U \subset \bigcup_{k=1}^{M} O_k$ and we suppose also that $\bar{U} \subset \bigcup_{k=0}^{M} O_k,$ where O_0 denotes an open set in the interior of $U.$ This property of having a boundary that "looks like" R^{n-1} near each of its points is what will allow us to extend our primitive versions of results to $U.$ We need one more devise to make this argument work.

We define a **partition of unity** subordinate to the open covering $\{O_k : 0 \le k \le M\}$. This is a set of functions $a_k(x) \in C_c^\infty(R^n)$ such that

a. $supp\, a_k \subset O_k \quad \left(then \quad a_0 \in C_c^\infty(U)\right)$

b. $a_k \geq 0 \quad on \quad O_k$

c. $\sum_{k=0}^{M} a_k(x) = 1 \quad \forall x \in U$

Then for $f \in H^m(U),$ we can write

$$f(x) = a_0(x)f(x) + \sum_{k=1}^{M} a_k(x)f(x)$$

$$= \sqrt{a_0(x)}\,(f\sqrt{a_0}\,)(x) + \sum_{k=1}^{M} \sqrt{a_k(x)}\; q_k^{\#}(p_k^{\#}(f\sqrt{a_k}\,))(x)$$

where

$$H^m(U) \ni \; f \to p^{\#}f = f(q(y)) \in H^m(Q_+)$$

$$H^m(Q_+) \ni \; g \to q^{\#}g = g(p(x)) \in H^m(U).$$

Then

$$f\sqrt{a_0} \in H_0^m(U) \qquad and \qquad Z(f\sqrt{a_0}\,) \in H^m(R^n),$$

$$p_k^{\#}(f\sqrt{a_k}\,) \in H^m(R_+^n) \quad with \quad \text{supp } p_k^{\#}(f\sqrt{a_k}\,) \subset Q_+.$$

Now define $A : H^m(U) \to H^m(R^n) \times H^m(R_+^n)^M$
as

$$Af = \{f\sqrt{a_0}\,, p_1^{\#}(f\sqrt{a_1}\,), \ldots, p_M^{\#}(f\sqrt{a_M}\,)\}$$

and $B : H^m(R^n) \times H^m(R_+^n)^M \to H^m(U)$

as

$$B[v_0, \ldots, v_M] = v_0\sqrt{a_0} + \sum_{k=1}^{M} \sqrt{a_k}\; q_k^{\#}(v_k)\,.$$

Then $B[Af] = f \qquad \forall f \in H^m(U).$

Evidently, A decomposes $f \in H^m(U)$ into $M+1$ pieces, one of which lives on O_0 and M others living on the sets $O_k \cap U,\ 1 \le k \le M.$ The mapping B reassembles these pieces into the original function, f.

Similarly, define

$$A' : H^m(\partial U) \to H^m(R^{n-1})^M \quad and \quad B' : H^m(R_+^n)^M \to H^m(U)$$

by

$$A'f = \{p_1^{\#}(f\sqrt{a_1}\,), \ldots, p_M^{\#}(f\sqrt{a_M}\,)\}$$

$$B'[v_1, \ldots, v_M] = \sum_{k=1}^{M} \sqrt{a_k}\; q_k^{\#}(v_k)\,.$$

These two mappings deal only with functions living on the boundary of U so A' decomposes $f \in H^m(\partial U)$ into M pieces, living on the sets $O_k \cap U,\ 1 \le k \le M.$ The mapping B' reassembles these pieces into the original function, f.

2. Basic Extension Lemma

Lemma 2.1 (Basic Extension lemma) For U a bounded, open and regular set in $R^n,$ every $u \in H^m(U)$ can be extended to $\tilde{u} \in H_0^m(V)$ *for* $U \subset\subset V.$

Proof- Recall the definition, for $u \in H^m(R_+^n),$

$$E_1 u(x', x_n) = \left\{ \begin{array}{lll} u(x', x_n) & if & x_n > 0 \\ a(x_n)\, Eu(x', x_n) & if & x_n < 0 \end{array} \right\}$$

where

$$a(x_n) \in C^\infty(R^1), \qquad a(x) = \left\{ \begin{array}{ll} 1 & if \;\; x_x > 0 \\ 0 & if \;\; x_n < -1 \end{array} \right\} = \text{ a smooth cutoff function}$$

Then $E_1 u \in H^m(R^n_{-1})$ for all $u \in H^m(R^n_+)$ where $R^n_{-1} = \left\{(x', x_n) : x' \in R^{n-1}, \; -1 < x_n < \infty\right\}$. This modified extension operator smoothly extends the function $u \in H^m(R^n_+)$ to a neighborhood of the boundary of R^n_+. Now for $U \subset\subset V$ we have

$$\begin{array}{ll} H^m(U) \;\; ---A--> & H^m(R^n) \times H^m(R^n_+)^M \\ & \quad\;\; \downarrow id \qquad \downarrow E_1 \\ H^m_0(V) <--B--- & H^m(R^n) \times H^m(R^n_{-1})^M \end{array}$$

i.e., E_1 extends each function in $H^m(R^n_+)$ smoothly to a function in $H^m(R^n_{-1})$. Since $R^n_+ \subset\subset R^n_{-1}$, applying the mapping B produces a smooth function with support in an open neighborhood of U.■

3. Trace and Embedding Theorems for a General Open Set

***Theorem 3.1* Sobolev Embedding Theorem**

For U a bounded, open and regular set in R^n, every $u \in H^m(U)$ can be identified with $\tilde{u} \in C^k(\bar{U}),\; for \;\; m > k + \frac{n}{2};$

i.e., $e : H^m(U) \hookrightarrow C^k(\bar{U}),\; for \;\; m > k + \frac{n}{2}$ is a continuous injection.

Proof-

$$\begin{array}{lll} & H^m(U) \;\; ---A--> & H^m(R^n) \times H^m(R^n_+)^M \\ & & \quad\;\; \downarrow id \qquad \downarrow E_1 \\ U \subset\subset V & H^m_0(V) <--B--- & H^m(R^n) \times H^m(R^n_{-1})^M \\ & \quad \downarrow Z & \\ & H^m(R^n) ----> C^k(R^n) ----> C^k(\bar{U}) & \end{array}$$

where $H^m(R^n) ----> C^k(R^n)$ denotes the continuous injection of theorem 2.1

and $C^k(R^n) ----> C^k(\bar{U})$ denotes the restriction from R^n to U.

***Theorem 3.2* Rellich Embedding Theorem**

For U a bounded, open and regular set in R^n, the embedding of $H^m(U)$ into $H^{m-1}(U)$ is compact; i.e. any sequence that is bounded in the norm of $H^m(U)$ contains a subsequence that is convergent in the norm of $H^{m-1}(U)$.

Proof -

$$\begin{array}{lll} & H^m(U) \;\; ---A--> & H^m(R^n) \times H^m(R^n_+)^M \\ & & \quad\;\; \downarrow id \qquad \downarrow E_1 \\ U \subset\subset V & H^m_0(V) <--B--- & H^m(R^n) \times H^m(R^n_{-1})^M \\ & \quad \downarrow e & \\ & H^{m-1}(V) ----> H^{m-1}(U) \; , & \end{array}$$

where $e : \; H^m_0(V) ---> \; H^{m-1}(V)$

denotes the compact embedding of the corollary to theorem 2.2 and

$$H^{m-1}(V) - - - - > H^{m-1}(U)$$

denotes the restriction from $V\ to\ U$,

Theorem 3.3 **The Smooth Approximation Theorem**
For U a bounded, open and regular set in R^n, $C^\infty(\bar{U})$ is dense in $H^m(U)$.

Proof-

$$\begin{array}{lccc} & C^\infty(\bar{U}) & ---A--> & H^m(R^n)\times H^m(R^n_+)^M \\ & & & \downarrow id \quad \downarrow E_1 \\ U \subset\subset V & C^m_0(V) & <--B--- & H^m(R^n)\times H^m(R^n_{-1})^M \\ & \downarrow i & & \\ & H^m_0(V) & ----> & H^m(U)\ , \end{array}$$

where

$$C^\infty_0(V) -- > \ C^m_0(V) -- > H^m_0(V)$$

is an injection with a dense image since $C^\infty_0(V)$ is dense in both $C^m_0(V)$ and $H^m_0(V)$ so it follows that $C^m_0(V)$ is dense in $H^m_0(V)$.
and

$$H^m_0(V) - - - - > H^m(U)$$

denotes restriction from V to U■

Theorem 3.4 **The Trace Theorem**.
For U a bounded, open and regular set in R^n,

$$T_j : H^m(U) - - - > H^{m-j-1/2}(\partial U) \qquad 0 \le j \le m-1,$$

is a continuous linear surjection and $T_j u = 0$ if and only if $u \in H^m_0(U)$.

Proof

$$\begin{array}{ccc} H^m(U) & ---A--> & H^m(R^n)\times H^m(R^n_+)^M \\ & & \downarrow 0 \quad \downarrow T_j = \text{Primitive Trace Operator} \\ H^{m-j-1/2}(\partial U) & <--B--- & 0 \ \times \ H^{m-j-1/2}(R^{n-1})^M \end{array}$$

Note $$T_j(Au) = (T_j[u\sqrt{a_0}\,], T_j[p^\#_1(u\sqrt{a_1}\,)], \ldots, T_j[\,p^\#_M(u\sqrt{a_M}\,)])$$

$$= \left(0, \partial^j_n[p^\#_1(u\sqrt{a_1}\,)], \ldots, \partial^j_n[\,p^\#_M(u\sqrt{a_M}\,)]\right)$$

and

$$\partial^j_n[p^\#_k(u\sqrt{a_k}\,)] \in \ H^{m-j-1/2}(R^{n-1})\ for\ k = 1,2,\ldots,M$$

Since the components of A, B and the primitive trace maps are all continuous, the general trace map is continuous as well (i.e., the composition of continuous mappings is continuous).■

Often we will wish to extend a function defined only on the boundary of a set, into the interior of the set and be able to say that the extended function belongs to some Sobolev space on the large set. Here is a theorem that allows us to do this.

Theorem 3.5 **Extension from the Boundary to the Interior**

Suppose U is a bounded, open and regular set in R^n. Then for any $f \in H^m(\partial U)$ there exists $\tilde{f} \in H^m(U)$ such that $T_0\tilde{f} = f$

Proof-

$$
\begin{array}{ccc}
H^m(\partial U) ----A'--> & [f_1^{\#}, \ldots, f_M^{\#}] \in H^m(R^{n-1})^M \\
 & \downarrow \; K \\
H^m(U) <---B'--- & [Kf_1^{\#}, \ldots, Kf_M^{\#}] \in H^m(R_+^n)^M
\end{array}
$$

Here, K denotes the continuous right inverse of the trace operator T_0.