

Hyperbolic Partial Differential Equations

Evolution equations associated with irreversible physical processes like diffusion and heat conduction lead to parabolic partial differential equations. When the equation is a model for a reversible physical process like propagation of acoustic or electromagnetic waves, then the evolution equation is generally hyperbolic.

The mathematical models usually begin with a conservation statement that is some version of Newton's second law

$$\frac{d}{dt} (m\vec{V}) = \vec{F}$$

which, when applied to an arbitrary ball B inside the domain $U \subset R^n$ where the process is occurring, often takes the form

$$\underbrace{\partial_t \int_B \rho \partial_t u \, dx}_{\text{momentum}} = \underbrace{\int_{\partial B} \vec{F}(x,t) \cdot \vec{n} \, dS}_{\text{surface tractions}} + \underbrace{\int_B f \, dx}_{\text{body forces}}$$

In the usual way, this leads to

$$\int_B [\partial_t(\rho \partial_t u) - \text{div} \vec{F} - f] \, dx = 0 \quad \forall B \subset U$$

and, since B is arbitrary,

$$\partial_t(\rho \partial_t u(x,t)) - \text{div} \vec{F}(x,t) = f(x,t) \quad \text{for } (x,t) \in U \times (0,T)$$

If u here represents, say, the deflection from the equilibrium state for an elastic membrane, then we will have as a constitutive relation,

$$\vec{F}(x,t) = A \nabla u(x,t)$$

and then

$$\rho \partial_{tt} u(x,t) = \nabla(A \nabla u(x,t)) + f(x,t) \quad \text{for } (x,t) \in U \times (0,T).$$

To complete the specifications for a well posed problem, we would add two initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad \partial_t u(x,0) = u_1(x) \quad \text{for } x \in U,$$

and boundary conditions of some sort, say

$$u(x,t) = 0 \quad \text{for } x \in \partial U, t \in (0,T).$$

We will represent this problem, in general, in the form

$$\begin{aligned} \partial_{tt} u(x,t) + L[u(x,t)] &= f(x,t) \quad \text{in } U_T & (1) \\ u(x,0) &= u_0(x) \quad \text{in } U \\ \partial_t u(x,0) &= u_1(x) \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \times (0,T) \end{aligned}$$

where L denotes a second order operator which is uniformly elliptic on U_T with coefficients which we will suppose to be in $C^1[U_T]$. In this case, the initial boundary value problem is said to be "hyperbolic". Now we are going to try to devise a framework in which this problem can be viewed as if it behaves in more or less the same way as an ordinary differential equation

for a real, vector valued function of t .

Because of what we already know about the elliptic operator L , if our solution u were to belong to $L_2[0, T : V]$, where $V = H^1(U)$, (and $H = H^0(U)$, $V' = H^{-1}(U)$) then $L[u(x, t)] \in L_2[0, T : V']$ and if $f \in L_2[0, T : H] = L_2(U_T) \subset L_2[0, T : V']$, then this imposes the condition $\partial_{tt}u \in L_2[0, T : V']$. Then this creates the following situation with regard to the spaces where u and its time derivatives are found,

$$u \in L_2[0, T : V] \quad \partial_{tt}u \in L_2[0, T : V']$$

which suggests $\partial_t u \in L_2[0, T : H]$;

$$\text{i.e.,} \quad \langle \partial_{tt}u, u \rangle_{V' \times V} = (\partial_{tt}u, u)_H = -(\partial_t u, \partial_t u)_H = -\|\partial_t u\|_H^2.$$

Then, since $L_2[0, T : V] \subset L_2[0, T : H] \subset L_2[0, T : V']$, we have

$$\begin{aligned} u, \partial_t u &\in L_2[0, T : H] & \text{hence} & \quad u \in C[0, T : H] \\ \partial_t u, \partial_{tt}u &\in L_2[0, T : V'] & \text{hence} & \quad \partial_t u \in C[0, T : V'] \end{aligned}$$

Here we made use of the fact that if a function and its first derivative are both square integrable on $[0, T]$ then the function must, in fact, be Holder continuous on $[0, T]$ (of order $\frac{1}{2}$). This permits the following interpretation of the initial conditions,

$$\begin{aligned} u(\cdot, t) &\rightarrow u_0 \in H & \text{as } t &\rightarrow 0 \\ \partial_t u(\cdot, t) &\rightarrow u_1 \in V' & \text{as } t &\rightarrow 0. \end{aligned}$$

Then a reasonable weak formulation of (1) might be the following:

$$\text{given } f \in L_2[0, T : H], u_0 \in H, \text{ and } u_1 \in V'$$

$$\text{find } u \in L_2[0, T : V], \text{ with } \partial_t u \in L_2[0, T : H] \text{ and } \partial_{tt}u \in L_2[0, T : V']$$

$$\text{such that } \langle \partial_{tt}u, v \rangle_{V' \times V} + B[u, v, t] = \langle f, v \rangle_{V' \times V} \quad \forall v \in V \text{ a.e. in } (0, T)$$

$$u(0) = u_0 \quad \partial_t u(0) = u_1$$

In fact, we are going to assume the initial data is such that $u_0 \in V$, and $u_1 \in H$ and then it remains to be seen if we can show that this weak problem has a unique solution for all admissible data.

Theorem For all data $f \in L_2[0, T : H]$ and $u_0 \in V$, $u_1 \in H$, there exists a unique $u \in L_2[0, T : V]$, with $\partial_t u \in L_2[0, T : H]$ and $\partial_{tt}u \in L_2[0, T : V']$ such that

$$\begin{aligned} \text{i)} \quad &\langle \partial_{tt}u, v \rangle_{V' \times V} + B[u, v, t] = (f, v)_H \quad \forall v \in V \text{ a.e. in } (0, T) & (2) \\ \text{ii)} \quad &u(0) = u_0 \quad \partial_t u(0) = u_1 \end{aligned}$$

Moreover, the solution mapping

$$L_2[0, T : H] \times H \times V' : \{f, u_0, u_1\} \rightarrow \{u, \partial_t u\} \in L_2[0, T : V] \times L_2[0, T : H]$$

is continuous.

Proof- (uniqueness) We will first prove that the solution is unique if it exists. Let u denote a

solution of the weak problem (2) corresponding to data, $u_0 = u_1 = f = 0$. We would like to proceed by writing

$$\int_0^T \{ \langle u'', u' \rangle_{V' \times V} + B[u, u', t] \} dt = 0.$$

However, since u' belongs to $L_2[0, T : H]$, and not to $L_2[0, T : V]$, neither of the terms $\langle u'', u' \rangle_{V' \times V}$, $B[u, u', t]$ is defined. Instead, we must define for a fixed $s \in (0, T]$,

$$v(t) = \begin{cases} \int_t^s u(\tau) d\tau & \text{if } 0 \leq t \leq s \\ 0 & \text{if } s \leq t \leq T \end{cases}$$

Then $v \in V$ for all t , $0 \leq s \leq T$, and

$$\int_0^s \{ \langle \partial_{tt} u, v \rangle_{V' \times V} + B[u, v, t] \} dt = 0$$

Now $u'(0) = 0$, and $v(s) = 0$ so,

$$\int_0^s \{ - \langle u', v' \rangle_{V' \times V} + B[u, v, t] \} dt = 0$$

Next, we observe that $v'(t) = -u(t)$ for $0 \leq t \leq s$, and hence

$$\int_0^s \{ \langle u', u \rangle_{V' \times V} - B[v', v, t] \} dt = 0$$

This implies that

$$\begin{aligned} \int_0^s \frac{1}{2} \frac{d}{dt} \{ \|u(t)\|_H^2 - B[v, v, t] \} dt &= \\ = - \int_0^s \{ C[u, v, t] + D[v, v, t] \} dt \end{aligned}$$

where

$$\begin{aligned} C[u, v, t] &= \int_U [u \vec{b} \cdot \nabla v + \frac{1}{2} uv \nabla \cdot \vec{b}] dx \\ D[u, v, t] &= \frac{1}{2} \int_U [\nabla v \cdot \partial_t A \nabla u + v \partial_t \vec{b} \cdot \nabla u + uv \partial_t c] dx. \end{aligned}$$

This leads to

$$\frac{1}{2} \{ \|u(s)\|_H^2 + B[v(0), v(0), t] \} = - \int_0^s \{ C[u, v, t] + D[v, v, t] \} dt$$

and $\|u(s)\|_H^2 + \|v(0)\|_V^2 \leq C \int_0^s \{ \|u(t)\|_H^2 + \|v(t)\|_V^2 \} dt$

Now write $v(0) = \int_0^t u(\tau) d\tau := w(t)$

and express this last estimate in terms of w ,

$$\|u(s)\|_H^2 + \|w(s)\|_V^2 \leq C \int_0^s \{ \|u(t)\|_H^2 + \|w(t) - w(s)\|_V^2 \} dt$$

But

$$\|w(t) - w(s)\|_V^2 \leq 2\|w(t)\|_V^2 + 2\|w(s)\|_V^2$$

hence

$$\|u(s)\|_H^2 + (1 - 2sC)\|w(s)\|_V^2 \leq C \int_0^s \{\|u(t)\|_H^2 + \|w(t)\|_V^2\} dt$$

If we choose $T_1 > 0$ sufficiently small that $1 - 2T_1C \geq \frac{1}{2}$, then we will have

$$\|u(s)\|_H^2 + \|w(s)\|_V^2 \leq C \int_0^s \{\|u(t)\|_H^2 + \|w(t)\|_V^2\} dt \quad \text{for } 0 \leq s \leq T_1$$

If we let $U(s) = \int_0^s \{\|u(t)\|_H^2 + \|w(t)\|_V^2\} dt$, this estimate asserts

$$U'(s) \leq C U(s);$$

i.e.,

$$U(s) \leq U(0) e^{Cs}.$$

But $U(0) = 0$ so $U(s) = 0$ for $0 \leq s \leq T_1$ and this implies $u(s) = 0$ for $0 \leq s \leq T_1$. We can repeat this argument on $[T_1, 2T_1], [2T_1, 3T_1], \dots, [(n-1)T_1, nT_1]$ for $nT_1 > T$ in order to eventually conclude that $u(t) = 0$ for $0 \leq t \leq T$. This proves the uniqueness of the weak solution.

Observe that weakening the notion of the solution to the IBVP enlarges the class of admissible solutions so that proving existence becomes easier in general. At the same time, however, proving uniqueness in a larger class usually becomes more difficult, as this proof illustrates.

The existence proof, like the proof for the existence of a solution to the parabolic problem will proceed in a series of steps.

1) Existence of Approximate Solutions-

Let $\{w_k\}$ denote an orthonormal basis for H that is, simultaneously, an orthogonal basis for V . Then for each positive integer N , define

$$u_N(t) = \sum_{j=1}^N C_{j,N}(t) w_j$$

where the $C_{j,N}$ are required to satisfy, for each k , $1 \leq k \leq N$,

$$\begin{aligned} \text{i)} \quad & (u_N''(t), w_k)_H + B[u_N, w_k, t] = (f, w_k)_H \\ \text{ii)} \quad & C_{k,N}(0) = (u_0, w_k)_H \quad C'_{k,N}(0) = (u_1, w_k)_H \end{aligned} \quad (3)$$

Here we are using the fact that $\langle u, v \rangle_{V' \times V} = (u, v)_H$. Now (3) is equivalent to,

$$C''_{k,N}(t) + \sum_{j=1}^N B[w_j, w_k, t] C_{j,N}(t) = f_k(t)$$

$$C_{k,N}(0) = (u_0, w_k)_H \quad C'_{k,N}(0) = (u_1, w_k)_H$$

which is a system of second order linear ODE's of the form,

$$\begin{aligned} \vec{C}''_N(t) + [B_{jk}(t)] \vec{C}_N(t) &= \vec{f}(t), \\ \vec{C}_N(0) &= \vec{U}_0, \quad \vec{C}'_N(0) = \vec{U}_1, \end{aligned}$$

where the coefficient matrix, $[B_{jk}(t)]$ is uniformly positive definite on $[0, T]$. It is well known that such a system has a unique global solution, $\vec{C}_N(t)$, for each N , hence there exists for each N , a unique approximate solution $u_N(t)$ for the weak boundary problem.

2) Energy Estimates

$$\text{a) } \max_{0 \leq t \leq T} \left(\|u_N(t)\|_V^2 + \|u'_N(t)\|_H^2 \right) \leq C_1 \left\{ \|u_0\|_V + \|u_1\|_H + \|f\|_{L_2[0,T]} \right\} \quad (4)$$

$$\text{b) } \|u''_N\|_{L_2[0,T;V']} \leq C_2 \left\{ \|u_0\|_V + \|u_1\|_H + \|f\|_{L_2[0,T]} \right\}$$

It follows from (3) that

$$(u_N''(t), u'_N)_H + B[u_N, u'_N, t] = (f, u'_N)_H$$

and

$$\begin{aligned} (u_N''(t), u'_N)_H &= \frac{1}{2} \frac{d}{dt} (u'_N(t), u'_N)_H = \frac{1}{2} \frac{d}{dt} \|u'_N(t)\|_H^2 \\ B[u_N, u'_N, t] &= \int_U \nabla u'_N A \nabla u_N dx + \int_U [u'_N \vec{b} \cdot \nabla u_N + c u'_N u_N] dx \\ &=: B_1[u_N, u'_N, t] + B_2[u_N, u'_N, t] \end{aligned}$$

We can assume WLOG that the matrix A is symmetric, which leads to

$$\begin{aligned} B_1[u_N, u'_N, t] &= \int_U \nabla u'_N A \nabla u_N dx \\ &= \frac{1}{2} \frac{d}{dt} \int_U \nabla u_N A \nabla u_N dx - \frac{1}{2} \int_U \nabla u_N A' \nabla u_N dx \end{aligned}$$

and

$$B_1[u_N, u'_N, t] \geq \frac{1}{2} \frac{d}{dt} \int_U \nabla u_N A \nabla u_N dx - C \|u_N\|_V^2$$

In addition,

$$B_2[u_N, u'_N, t] \leq C \left(\|u_N\|_V^2 + \|u'_N\|_H^2 \right)$$

and, combining these leads to

$$\frac{1}{2} \frac{d}{dt} \left(\|u'_N(t)\|_H^2 + \int_U \nabla u_N A \nabla u_N dx \right) \leq C \left(\|u'_N\|_H^2 + \|u_N\|_V^2 + \|f\|_H^2 \right)$$

Note here that in order to estimate $|(f, u'_N)_H|$ in terms of $\|u'_N\|_H$, we need f in $L_2[0, T : H]$ and not in $L_2[0, T : V']$ where we might have thought it should be. Next we observe that by the ellipticity assumption, we have

$$\int_U \nabla u_N A \nabla u_N dx \geq a_0 \|u_N\|_V^2$$

which implies that

$$\frac{1}{2} \frac{d}{dt} \left(\|u'_N(t)\|_H^2 + \int_U \nabla u_N A \nabla u_N dx \right) \leq C' \left(\|u'_N\|_H^2 + \int_U \nabla u_N A \nabla u_N dx + \|f\|_H^2 \right).$$

If we let,

$$U(t) := \|u'_N(t)\|_H^2 + \int_U \nabla u_N A \nabla u_N dx$$

then

$$\frac{d}{dt} U(t) - C U(t) = \frac{d}{dt} (e^{-Ct} U(t)) e^{Ct} \leq C'' \|f\|_H^2.$$

and it follows that $U(t)$ satisfies,

$$U(t) \leq \left(U(0) + C'' \int_0^t \|f(s)\|_H^2 ds \right) e^{Ct}$$

But,

$$\begin{aligned} U(0) &= \|u'_N(0)\|_H^2 + \int_U \nabla u_N(0) A \nabla u_N(0) dx \\ &\leq C \left(\|u_1\|_H^2 + \|u_0\|_V^2 \right) \end{aligned}$$

and this leads to

$$\|u'_N(t)\|_H^2 + \int_U \nabla u_N(t) A \nabla u_N(t) dx \leq C \left(\|u_1\|_H^2 + \|u_0\|_V^2 + \|f\|_{L_2[U_T]}^2 \right).$$

Finally, we can make use of the ellipticity assumption together with the Poincare inequality to arrive at the conclusion

$$\|u'_N(t)\|_H^2 + \|u_N(t)\|_V^2 \leq C \left(\|u_1\|_H^2 + \|u_0\|_V^2 + \|f\|_{L_2[U_T]}^2 \right) \quad (5)$$

since this last estimate holds for all $t \in (0, T)$,

$$\max_{0 \leq t \leq T} \left[\|u'_N(t)\|_H^2 + \|u_N(t)\|_V^2 \right] \leq C \left(\|u_1\|_H^2 + \|u_0\|_V^2 + \|f\|_{L_2[U_T]}^2 \right).$$

This proves (4a). Alternatively, we could integrate (5) to obtain

$$\|u'_N\|_{L_2[0,T,H]}^2 + \|u_N\|_{L_2[0,T,V]}^2 \leq C(T) \left(\|u_1\|_H^2 + \|u_0\|_V^2 + \|f\|_{L_2[U_T]}^2 \right). \quad (6)$$

Next, fix $v \in V$ with $\|v\|_V \leq 1$, and write $v = v_1 + v_2$ where $v_1 \in \text{span}[w_1, \dots, w_N] = M_N$ and $v_2 \perp M_N$. Then $\|v_1\|_V \leq \|v\|_V \leq 1$ and

$$\langle u''_N, v \rangle_{V' \times V} = (u''_N, v_1)_H = (f, v_1)_H - B[u_N, v_1].$$

It follows then, using the Cauchy-Schwartz inequality and the boundedness of B , that

$$|\langle u''_N, v \rangle_{V' \times V}| \leq C(\|f(t)\|_H + \|u_N(t)\|_V)$$

hence

$$\|u''_N(t)\|_{V'} = \sup_{\|v\|_V \leq 1} |\langle u''_N, v \rangle_{V' \times V}| \leq C(\|f(t)\|_H + \|u_N(t)\|_V)$$

and

$$\begin{aligned} \int_0^T \|u''_N(t)\|_{V'}^2 dt &\leq C \int_0^T (\|f(t)\|_H^2 + \|u_N(t)\|_V^2) dt \\ &\leq C \left(\|f\|_{L_2(U_T)}^2 + \|u_1\|_H^2 + \|u_0\|_V^2 \right) \end{aligned}$$

and this is equivalent to (4b).

3. Existence of Weak Solutions

The energy estimates (4) imply that

$$\begin{aligned} \{u_N(t)\} &\text{ is bounded in } L_2[0, T : V] \\ \{u'_N(t)\} &\text{ is bounded in } L_2[0, T : H] \\ \{u''_N(t)\} &\text{ is bounded in } L_2[0, T : V'] \end{aligned}$$

and it follows that there exists a subsequence $\{u_n(t)\} \subset \{u_N(t)\}$ such that

$$u_n(t) \text{ converges weakly to } u \text{ in } L_2[0, T : V]$$

$$u'_n(t) \text{ converges weakly to } v \text{ in } L_2[0, T : H]$$

$$u''_n(t) \text{ converges weakly to } w \text{ in } L_2[0, T : V']$$

In the usual way, making use of the fact that

$$L_2[0, T : V] \subset L_2[0, T : H] \subset L_2[0, T : V'] \subset D'[0, T : V']$$

we get that $u' = v$, $u'' = v' = w$. It remains now to show that this weak limit point is a weak solution, i.e., that it satisfies (2). Let

$$V_m = \left\{ v(t) = \sum_{j=1}^m d_j(t) w_j : d_j(t) \in C^2[0, T] \ 1 \leq j \leq m \right\}.$$

For $m \leq n$,

$$\int_0^T \{(u''_n(t), v)_H + B[u_n, v, t]\} dt = \int_0^T (f, v)_H dt \quad \text{for all } v \in V_m$$

For $m = n$, let n tend to infinity and use the weak convergence results to get

$$\int_0^T \{(u''(t), v)_H + B[u, v, t]\} dt = \int_0^T (f, v)_H dt \quad \text{for all } v \in \bigcup_{m>0} V_m$$

Since $\{w_k\}$ is a basis for V , it follows that $\bigcup_{m>0} V_m = L_2[0, T : V]$ and therefore,

$$\int_0^T \{(u''(t), v)_H + B[u, v, t]\} dt = \int_0^T (f, v)_H dt \quad \text{for all } v \in L_2[0, T : V].$$

In addition, $u \in C[0, T : H]$, and $u' \in C[0, T : V']$ hence $u(t) \rightarrow u(0)$ in H and $u'(t) \rightarrow u'(0)$ in V' as $t \rightarrow 0^+$.

Finally, for any $v \in C^2[0, T : V]$ such that $v(T) = v'(T) = 0$, we have

$$\int_0^T \{(u, v'')_H + B[u, v, t] - (f, v)_H\} dt = \langle u'(0), v(0) \rangle - \langle u(0), v'(0) \rangle$$

and

$$\int_0^T \{(u_n, v'')_H + B[u_n, v, t] - (f, v)_H\} dt = \langle u'_n(0), v(0) \rangle - \langle u_n(0), v'(0) \rangle$$

Then

$$\int_0^T \{(u - u_n, v'')_H + B[u - u_n, v, t]\} dt = \langle u'(0) - u'_n(0), v(0) \rangle - \langle u(0) - u_n(0), v'(0) \rangle$$

and the weak convergence of the subsequence $\{u_n\}$ implies that the left side of this equation tends to zero as n tends to infinity. On the right side, recalling (3ii), we have $u_n(0)$ converges in V to u_0 and $u'_n(0)$ converges in H to u_1 , which implies

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u_1.$$

Then it follows that u is a weak solution of the IBVP. But then every subsequence of the sequence of approximate solutions, $\{u_N\}$, must converge to a weak solution. Since the weak solution has been shown to be unique, it follows that all subsequences have the same weak

limit. But in this case, the sequence $\{u_N\}$, must itself converge, weakly, to the weak solution.

Note that the estimate (5) applies to $u = \lim u_N$ which shows that the mapping

$$L_2[0, T : H] \times H \times V' : \{f, u_0, u_1\} \rightarrow \{u, \partial_t u\} \in L_2[0, T : V] \times L_2[0, T : H]$$

is continuous.

Finally notice that if the initial conditions in (1), $u(x, 0) = u_0(x)$ and $\partial_t u(x, 0) = u_1(x)$, were replaced by final conditions, $u(x, T) = u_0(x)$ and $\partial_t u(x, T) = u_1(x)$, then the new problem still admits a unique weak solution which depends continuously on the data. This is in contrast to the parabolic problem where the final value problem is not well posed.