

## Evolution Equations

For  $T > 0$ , fixed, let  $U_T = U \times (0, T]$  where  $U$  denotes a bounded open set in  $R^n$ . Suppose that  $U$  is filled with a material in which a contaminant is being transported by various means including diffusion and convection. Let the concentration of contaminant at  $(x, t)$  in  $U_T$  be denoted by  $u(x, t)$  and let the vector field  $\vec{F}(x, t)$  denote the flux field for the contaminant. Then we can state a conservation principle for the contaminant as follows. For an arbitrary ball  $B$  in  $U$  and an arbitrary time interval  $(t_1, t_2) \subset (0, T)$  we can assert that

$$\underbrace{\int_B u(x, t_2) dx}_A = \underbrace{\int_B u(x, t_1) dx}_B - \underbrace{\int_{t_1}^{t_2} \int_{\partial B} \vec{F}(x, t) \cdot \vec{n} dS dt}_C + \underbrace{\int_{t_1}^{t_2} \int_B f(x, t) dx dt}_D$$

where the integrals A, B represent the amount of contaminant inside  $B$  at time  $t_2, t_1$ , respectively, integral C represents the amount of contaminant flowing out of  $B$  during the time interval  $(t_1, t_2)$ , (i.e.,  $\vec{n}$  here denotes the outward unit normal to the boundary of  $B$ ) and the integral D represents the amount of contaminant that is created or destroyed inside  $B$  during the time interval  $(t_1, t_2)$  as a result of a source density given by  $f(x, t)$ . Using the divergence theorem to convert integral C to an integral over  $B$ , and using the fundamental theorem of calculus to write the difference A minus B as

$$\int_B u(x, t_2) dx - \int_B u(x, t_1) dx = \int_{t_1}^{t_2} \int_B \partial_t u(x, t) dx dt$$

we get 
$$\int_{t_1}^{t_2} \int_B [\partial_t u(x, t) + \text{div} \vec{F} - f] dx dt = 0$$

Since this holds for all  $B$  in  $U$  and every time interval  $(t_1, t_2) \subset (0, T)$ , we can conclude that

$$\partial_t u(x, t) + \text{div} \vec{F}(x, t) - f(x, t) = 0 \quad \forall (x, t) \in U_T.$$

Next, we observe that if the contaminant is being transported through the medium due to a combination of diffusion and convection by a velocity field  $\vec{V}(x, t)$ , then the flux field is related to the contaminant concentration by,

$$\vec{F}(x, t) = -a(x, t) \nabla u(x, t) + u(x, t) \vec{V}(x, t),$$

where  $a(x, t)$  denotes a material dependent property called "diffusivity". The diffusivity may be a tensor (n by n matrix) in anisotropic materials, a scalar in materials which are isotropic but not homogeneous, and for isotropic and homogeneous materials, the diffusivity is a constant. Then the conservation statement becomes,

$$\partial_t u(x, t) - \text{div} [a(x, t) \nabla u] + \vec{V} \cdot \nabla u + \text{div} \vec{V} u = f(x, t) \quad \forall (x, t) \in U_T.$$

which means that the concentration satisfies an equation of the form

$$\partial_t u(x, t) + L[u(x, t)] = f(x, t) \quad \text{in } U_T$$

where 
$$L[u(x, t)] = -\text{div}(a(x, t) \nabla u) + \vec{V}(x, t) \cdot \nabla u + \text{div} \vec{V}(x, t) u.$$

The diffusivity is only physically meaningful if it is nonnegative, and we will assume further that it is strictly positive at every point in  $U_T$ . Then the operator  $L$  is said to be elliptic and the operator  $\partial_t + L$  is said to be parabolic. Evidently, parabolic operators arise in modelling the time evolution of physical processes that behave like diffusion-convection situation described here.

We are going to consider the problem

$$\begin{aligned} \partial_t u(x, t) + L[u(x, t)] &= f(x, t) && \text{in } U_T \\ u(x, t) &= g(x, t) && \text{on } \Gamma \times (0, T) \\ u(x, 0) &= u_0(x) && \text{on } U \times (t = 0). \end{aligned}$$

Such a problem is called an initial value problem for an evolution equation, (IVP for short). We shall show that a modification of the approach we used for the elliptic BVP's can be employed to solve these problems. First it will be necessary to describe the function spaces in which the variable  $t$  is viewed differently from the space variables,  $x$ .

### H Valued Functions of t

In dealing with evolution equations we will be faced with functions of  $(x, t)$  where  $x$  is in  $R^n$  and  $t$  is in  $R^1$  or in  $R^1_+$ . It will turn out to be useful to view such functions as functions of  $t$ , taking their values in an appropriate space of functions of the variable  $x$ ; i.e.,

$$(0, T) \ni t \rightsquigarrow \underbrace{[U \ni x \rightsquigarrow u(x, t) \in H]}_{u(\cdot, t) = \text{function of } t \text{ with values in } H}$$

The function  $u(x, t)$  will be written as  $u(t)$  where this is interpreted to mean that the value at each  $t$  is  $u(\cdot, t)$ , a function which is an element of the function space  $H$ . For us, most of the function spaces will be Hilbert spaces.

### H Valued Distributions: $D'[0, T; H]$

For  $H$  a Hilbert space, a linear mapping,

$$J : C_c^\infty(0, T) \rightarrow H$$

defines a distribution in  $t$  with values in  $H$  if and only if

$$\phi_n \rightarrow 0 \text{ in } C_c^\infty(0, T) \quad \text{implies} \quad J(\phi_n) \rightarrow 0 \text{ in } H$$

We will use the notation  $D[0, T]$  to denote  $C_c^\infty(0, T)$ .

For  $f : (0, T) \rightarrow H$ , locally integrable, define

$$J_f(\phi) = \int_0^T f(t)\phi(t)dt \quad \forall \phi \in D[0, T].$$

Then  $J_f$  is easily seen to be a distribution in  $t$  with values in  $H$ . The notation for this will be  $J_f \in D'[0, T : H]$ , and we shall use the notation  $L^1_{loc}[0, T : H]$  to denote locally integrable functions of  $t$ , with values in  $H$ . These are the regular  $H$ -valued distributions.

For  $J(t) \in D'[0, T : H]$ , the derivative,  $J'(t) \in D'[0, T : H]$ , is defined by

$$J'(\phi) = -T(\phi') \quad \text{for all } \phi \in D[0, T].$$

More generally,  $T^{(n)}(\phi) = (-1)^n T(\phi^{(n)})$  for all  $\phi \in D[0, T]$ .

Now define  $L^2[0, T : H]$  to be the set of all measurable functions of  $t$ ,  $0 < t < T$ , with values in  $H$  such that

$$\|u\|_{L^2[0,T;H]}^2 = \int_0^T \|u(t)\|_H^2 dt < \infty.$$

**Theorem 1-** The natural inclusion,  $L^2[0, T : H] \hookrightarrow D'[0, T : H]$ , is a continuous injection; i.e.,

$$f_n \rightarrow f \text{ in } L^2[0, T : H] \text{ implies } J_{f_n}(\phi) \rightarrow T_f(\phi) \text{ in } H \quad \forall \phi \in D[0, T]$$

and

$$T_f(\phi) = 0 \text{ in } H \quad \forall \phi \in D[0, T] \text{ implies } f = 0 \text{ in } L^2[0, T : H]$$

If we define linear spaces with norms,

- $C[0, T : H]$  = continuous functions of  $t$ ,  $0 \leq t \leq T$ , with values in  $H$

$$\|u\|_{C[0,T;H]} = \max_{0 \leq t \leq T} \|u(t)\|_H$$

- $H^1[0, T : H] = \{u \in L^2[0, T : H] : u' \in L^2[0, T : H]\}$

$$\|u\|_{H^1[0,T;H]}^2 = \int_0^T \{\|u(t)\|_H^2 + \|u'(t)\|_H^2\} dt$$

then it is shown on p 286 in the text, that for any  $u$  in  $H^1[0, T : H]$ ,

1. there exists  $\hat{u}$  in  $C[0, T : H]$  such that  $u = \hat{u}$  a.e., in  $(0, T)$
2. ii) for all  $(s, t) \subset [0, T]$ ,  $u(t) = u(s) + \int_0^t u'(z) dz$
3. iii)  $\|u\|_{C[0,T;H]} \leq C \|u\|_{H^1[0,T;H]}$

The arguments are very close to those that show that functions in  $H^1(a, b)$  are continuous.

For Hilbert spaces,  $H_1$  and  $H_2$  with  $H_1$  continuously included in  $H_2$  it is easy to show that

$$D'[0, T : H_1] \hookrightarrow D'[0, T : H_2]$$

In particular, for the situation  $V \hookrightarrow H \hookrightarrow V'$

$$f \in L^2[0, T : V] \hookrightarrow D'[0, T : V] \text{ implies } f' \in D'[0, T : V] \hookrightarrow D'[0, T : V']$$

In the evolution equations we are going to study, the elliptic operator  $L$  is a bounded linear map from  $V$  onto its dual,  $V'$ . Then a natural choice of solution space for the evolution equations is

$$W[0, T] = \{u \in L^2[0, T : V] : \partial_t u \in L^2[0, T : V']\}$$

i.e., if  $u$  is in  $W[0, T]$  then  $\partial_t u$  and  $Lu$  both belong to  $L^2[0, T : V']$ .

**Theorem 2-**  $W[0, T]$  is a Hilbert space for the inner product

$$(f, g)_{W[0,T]} = \int_0^T \{(f(t), g(t))_V + (f'(t), g'(t))_{V'}\}$$

**proof-** Suppose  $\{f_n\}$  is a Cauchy sequence in  $W[0, T]$ . Then the definition of the inner product implies

$$f_n \rightarrow F_0 \text{ in } L^2[0, T : V] : \quad \text{and} \quad f'_n \rightarrow G_0 \text{ in } L^2[0, T : V']$$

Now  $L^2[0, T : V] \hookrightarrow D'[0, T : V]$  implies

$$f_n \rightarrow F_0 \text{ in } D'[0, T : V] : \quad \text{and} \quad f'_n \rightarrow F'_0 \text{ in } D'[0, T : V]$$

But  $D'[0, T : V] \hookrightarrow D'[0, T : V']$ ,  
hence  $f'_n \rightarrow F'_0 \text{ in } D'[0, T : V']$ .

However,

$$f'_n \rightarrow G_0 \text{ in } L^2[0, T : V'] \hookrightarrow D'[0, T : V'],$$

and it follows that  $F'_0 = G_0$  and  $F_0 \in W[0, T]$ . ■

Since we are studying equations of the form  $\partial_t u + Lu$  for  $L$  an elliptic operator of order 2, we will be particularly interested in the case of  $W[0, T]$  when  $V = H^1_0(U)$  (and  $H = H^0(U)$ ,  $V' = H^{-1}(U)$ ). It is not hard to show, using mollifiers, that in this case  $C^\infty[0, T : H^1_0(U)]$  is dense in  $W[0, T]$ . Then we have the following result,

**Theorem 3-** The natural injection

$$C^\infty[0, T : H^1_0(U)] \hookrightarrow C[0, T : H^0(U)]$$

can be extended as a continuous injection of

$$W[0, T] \hookrightarrow C[0, T : H^0(U)]$$

Moreover, if  $u \in W[0, T]$ , then

- i)  $\|u(\cdot, t)\|_H^2$  is absolutely continuous on  $[0, T]$
- ii)  $\frac{d}{dt} \|u(\cdot, t)\|_H^2 = 2 \langle u'(t), u(t) \rangle_{V' \times V}$   
and  $\frac{d}{dt} (u(t), v(t))_H = \langle u'(t), v(t) \rangle_{V' \times V} + \langle v'(t), u(t) \rangle_{V' \times V}$
- iii) there is a constant  $C > 0$  such that  $\|u\|_{C[0, T; H]} \leq C \|u\|_{W[0, T]}$

**proof-**Heuristically, this result is nothing more than the observation that for  $u \in W[0, T]$ ,

$$\|u(\cdot, t)\|_H^2 = (u(\cdot, t), u(\cdot, t))_H$$

$$\text{hence} \quad \frac{d}{dt} \|u(\cdot, t)\|_H^2 = 2(u'(\cdot, t), u(\cdot, t))_H = 2 \langle u'(t), u(t) \rangle_{V' \times V}$$

where we have noted that  $u' \in L^2[0, T : V']$  and that the duality pairing for  $V \times V'$  is just the  $H$ -inner product. This can all be made rigorous as follows.

Let  $u \in W[0, T] = \{u \in L^2[0, T : H^1_0(U)] : \partial_t u \in L^2[0, T : H^{-1}(U)]\}$  Extend  $u(t)$  to  $[-T, T]$  as

$$\tilde{u}(t) = \begin{cases} u(t) & t > 0 \\ u(-t) & t < 0 \end{cases}$$

Clearly this defines a continuous injection of  $W[0, T]$  into  $W[-T, T]$ . Now let  $a(t)$  denote a

smooth cut-off function; i.e.,

$$\begin{aligned} a(t) &\in C^\infty(\mathbb{R}) \\ a(t) &= 1 \quad \text{for } t \geq 0, \quad a(t) = 0 \quad \text{for } t \leq -T, \\ \text{and } 0 &\leq a(t) \leq 1 \quad \text{for } -T < t < 0. \end{aligned}$$

Now if  $u \in C^\infty[0, T : H_0^1(U)]$ , then

$$\begin{aligned} d/dt \|a(t)\tilde{u}(\cdot, t)\|_H^2 &= d/dt (a\tilde{u}(\cdot, t), a\tilde{u}(\cdot, t))_H \\ &= 2 \langle (a\tilde{u})', (a\tilde{u}) \rangle_{V' \times V} \end{aligned}$$

But 
$$\int_{-T}^t d/dt \|a(t)\tilde{u}(\cdot, s)\|_H^2 ds = \|a(t)\tilde{u}(\cdot, t)\|_H^2 - 0$$

from which it follows that

$$\begin{aligned} \|a(t)\tilde{u}(\cdot, t)\|_H^2 &= 2 \int_{-T}^t \langle (a\tilde{u}(\cdot, s))', (a\tilde{u}(\cdot, s)) \rangle_{V' \times V} ds \\ &\leq 2 \int_{-T}^t \|(a\tilde{u}(\cdot, s))'\|_{H^{-1}(U)} \|(a\tilde{u}(\cdot, s))\|_{H_0^1(U)} ds \\ &\leq \int_{-T}^t \{ \|(a\tilde{u}(\cdot, s))'\|_{H^{-1}(U)}^2 + \|(a\tilde{u}(\cdot, s))\|_{H_0^1(U)}^2 \} ds \\ &\leq 2C_a \int_0^T \{ \|u'(\cdot, s)\|_{H^{-1}(U)}^2 + \|u(\cdot, s)\|_{H_0^1(U)}^2 \} ds \\ &\leq 2C_a \|u\|_{W[0, T]}^2 \end{aligned}$$

where  $C_a$  denotes a constant depending on the cut-off function,  $a(t)$ . Now it follows that

$$\max_{0 \leq t \leq T} \|u(\cdot, t)\|_H^2 \leq 2C_a \|u\|_{W[0, T]}^2$$

and, since  $C^\infty[0, T : H_0^1(U)]$  is dense in  $W[0, T]$ , this holds not just for  $u$  in  $C^\infty[0, T : H_0^1(U)]$ , but for all  $u$  in  $W[0, T]$ . ■

The meaning of this result is that for every  $u$  in  $W[0, T]$  there exists a  $\hat{u} \in C[0, T : H^0(U)]$  such that  $u(t) = \hat{u}(t)$  a.e.,  $0 < t < T$ . In particular, for  $u$  in  $W[0, T]$ , a solution of an IVP, it means that as a continuous function of  $t$  with values in  $H^0(U)$ , as  $t$  tends to zero,  $u(t)$  tends to  $u(0)$  in  $H^0(U)$ . This gives meaning to the initial condition,  $u(0) = u_0 \in H^0(U)$ .

With regard to boundary conditions associated with an IVP, any  $u$  such that,

$$u \in W[0, T] = \{u \in L^2[0, T : H_0^1(U)] : \partial_t u \in L^2[0, T : H^{-1}(U)]\}$$

automatically satisfies the homogeneous boundary condition,  $T_0 u = u|_\Gamma = 0$ .

An inhomogeneous boundary condition of the form,  $T_0 u = u|_\Gamma = g$ , for a function,  $g(\cdot, t) \in L^2[0, T : H^{1/2}(\Gamma)]$  can be formulated as follows. Let  $G(x, t)$  be such that

$$G \in L^2[0, T : H^1(U)], \quad \partial_t G \in L^2[0, T : H^{-1}(U)]$$

$$G(\cdot, t) \rightarrow G(\cdot, 0) \text{ in } H^0(U) \text{ as } t \rightarrow 0+,$$

$$T_0 G = g, \text{ a.e., } 0 < t < T.$$

Then  $u \in L^2[0, T : H^1(U)]$  satisfies the boundary condition  $T_0 u = g$ , if

$u(\cdot, t) - G(\cdot, t) \in H_0^1(U)$ , a.e.,  $0 < t < T$ ; i.e.,

$$u - G \in \{u \in L^2[0, T : H_0^1(U)] : \partial_t u \in L^2[0, T : H^{-1}(U)]\}.$$

It is worth noting that there is no condition of compatibility required between  $u_0$  and  $G(\cdot, 0)$ . While the conditions

$$u(\cdot, t) - G(\cdot, t) \in H_0^1(U), \text{ a.e., } 0 < t < T;$$

$$u(\cdot, t) \rightarrow u_0, \text{ in } H^0(U) \text{ as } t \rightarrow 0$$

seem to require that  $u_0 - G(\cdot, 0) \in H_0^1(U)$ , it is clear from the fact that  $u_0 \in H^0(U)$  that this condition has no meaning. In fact, if  $h$  is any function such that

$$h \in \{u \in L^2[0, T : H_0^1(U)] : \partial_t u \in L^2[0, T : H^{-1}(U)]\}$$

and

$$h(\cdot, 0) = G(\cdot, 0),$$

then replacing  $G$  in the IVP with  $G - h$  would have no effect on the boundary condition; i.e., the value of  $G$  at  $t = 0$  is not relevant to the formulation of the IVP. For example, consider the IVP in 1-d,

$$\begin{aligned} \partial_t u(x, t) - \partial_{xx} u(x, t) &= 0, & 0 < x < 1, & 0 < t < T, \\ u(x, 0) &= 1 & 0 < x < 1, \\ u(0, t) = u(1, t) &= 0, & 0 < t < T. \end{aligned}$$

Although the incompatibility between the initial and boundary conditions prevents the existence of a solution which is continuous on the closure of  $U_T$ , it does not preclude the existence of a solution in  $W[0, T]$ . This solution is then continuous as a function of  $t$  with values in  $H^0(U)$ , which means that as  $t$  tends to zero,  $u(x, t)$  tends to  $u_0(x) = 1$ , in the norm of  $H^0(U)$ . At the same time, for each positive  $t$ ,  $u(x, t)$  is valued in  $L^2[0, T : H_0^1(U)]$ , which means that  $T_0 u(\cdot, t) = 0$ ; i.e., the boundary condition is satisfied. That this implies no contradiction at  $(0, 0)$  and  $(1, 0)$  is a result of the fact that the functions involved are not pointwise defined but are defined only up to sets of measure zero.