The Hille-Yosida Theorem

We have seen that when the abstract IVP is uniquely solvable then the solution operator defines a semigroup of bounded operators. We have not yet discussed the conditions under which the IVP is uniquely solvable. However, it is clear that S(t) is some sort of generalized version of $\exp\{tA\}$ where A is an unbounded operator. To see this connection between S(t) and the exponential of A, consider the scalar equation

$$u'(t) = -a\,u(t), \qquad u(0) = u_0.$$

Then $$u(t) = S(t)[u_0] = e^{-at}u_0.$$

In this simple situation, we have also

$$\hat{u}(s) = L[u(t)] = \int_0^\infty e^{-st}e^{-at}u_0 dt = \frac{u_0}{s+a};$$

i.e.,

$$|\hat{u}(s)| = |\int_0^\infty e^{-st}S(t)u_0\,dt| = \left|(s+a)^{-1}u_0\right| \le \frac{1}{s}|u_0|.$$

We will see this result appear again in a more general setting. An example more general than this is provided by the system of linear ODE's

$$\vec{X}'(t) = -[A]\vec{X}(t) \qquad \vec{X}(0) = \vec{X}_0,$$

where A denotes an n by n symmetric matrix of constants. Then the solution to this system is given by

$$\vec{X}(t) = \sum_{k=1}^{n}\left(\vec{X}_0,\vec{V}_k\right)e^{-\lambda_k t}\vec{V}_k$$

where $\left\{\lambda_k,\vec{V}_k\right\}$ denote the eigenvalues and normalized eigenvectors of A. Then

$$\begin{aligned}\vec{X}(t) &= \sum_{k=1}^{n}\left(\vec{X}_0,\vec{V}_k\right)\sum_{m=0}^{\infty}\frac{(-\lambda_k t)^m}{m!}\vec{V}_k = \sum_{k=1}^{n}\left(\vec{X}_0,\vec{V}_k\right)\sum_{m=0}^{\infty}\frac{t^m}{m!}(-\lambda_k)^m\vec{V}_k \\ &= \sum_{k=1}^{n}\left(\vec{X}_0,\vec{V}_k\right)\sum_{m=0}^{\infty}\frac{t^m}{m!}(-A)^m\vec{V}_k = \sum_{k=1}^{n}\left(\vec{X}_0,\vec{V}_k\right)\sum_{m=0}^{\infty}\frac{(-At)^m}{m!}\vec{V}_k \\ &= \sum_{k=1}^{n}\left(\vec{X}_0,\vec{V}_k\right)[e^{-At}]\vec{V}_k = [e^{-At}]\sum_{k=1}^{n}\left(\vec{X}_0,\vec{V}_k\right)\vec{V}_k = [e^{-At}]\vec{X}_0.\end{aligned}$$

We observe that $[e^{-At}] = \sum_{m=0}^{\infty}\frac{(-At)^m}{m!}$ and, in fact, when A is a bounded linear operator on a Hilbert space H (as is the case in this example for $H = R^1$) then we can expect that

$$S(t) = \lim_{M\to\infty}\sum_{m=0}^{M}\frac{(-At)^m}{m!} .= \lim_{M\to\infty}S_M(t).$$

For any fixed value of t, we have

$$\|S_M(t) - S_N(t)\|_{L(H)} \le \sum_{m=N}^{M}\frac{|t|^m}{m!}\|A\|_{L(H)}^m \to 0 \quad as\ M,N\to\infty$$

so the meaning of $S(t) = \lim_{M\to\infty}S_M(t)$ is to be understood as a limit in the complete normed linear space, $L(H)$. Clearly the limit $S(t)$ satisfies

$$S(0) = I \qquad and \quad S(t+\tau) = S(t)S(\tau)$$

since these equalities hold for every $S_M(t)$. Also, an easy calculation with the series shows that

$$\|S(t) - I\|_{L(H)} \le |t|\,\|A\|_{L(H)} e^{-|t|\,\|A\|_{L(h)}}$$

and

$$\left\| \frac{S(t)-I}{t} + A \right\|_{L(H)} \le \|A\|_{L(H)} \|S(t) - I\|_{L(H)}.$$

We say, in this case, that the semigroup $S(t)$ is **uniformly continuous** on H. Of course, all this is true under the assumption that A is bounded.

On the other hand when A is unbounded, since $D[A^{n+1}] \subset D[A^n]$, it may be that the domain in the infinite sum shrinks to zero. We could make use of the fact that $S(t)[u_0]$ solves

$$u'(t) = -Au(t), \qquad u(0) = u_0,$$

to write $$u(t+h) \approx u(t) - hAu(t) = (I - hA)u(t)$$

from which we find $$u(t) = \left(I - \tfrac{t}{n}A\right)^n u_0 \qquad \left(t = nh,\ n = 1,2,\ldots\right).$$

Since we are again iterating an unbounded operator, the difficulty of the shrinking domain has not disappeared. However, if we write

$$u(t+h) \approx u(t) - hAu(t+h) \quad or \qquad (I + hA)u(t+h) = u(t),$$

then

$$u(t+h) \approx (I+hA)^{-1}u(t) \qquad and \qquad u(t) \approx \left(I + \tfrac{t}{n}A\right)^{-n} u_0$$

so in this case we are iterating a bounded operator, $(I+hA)^{-1}$. The point of this discussion is just that it is plausible that we will be able to find a meaningful definition for e^{At} even when A is unbounded but we must be careful how we do it. Of course, there must also be some restrictions on A. We will now state and prove a theorem giving a set of necessary and sufficient conditions on A in order that $-A$ generates a C_0 –semigroup.

Theorem (**Hille**-**Yosida**) The following statements are equivalent:

1. $-A : D_A \to H$ generates a C_0 –semigroup of contractions on H

2. a) A is closed and densely defined

b) $\forall \lambda > 0 \quad (\lambda + A) : D_A \to H$ is one to one and onto with

$$\| (\lambda + A)^{-1} \|_{L(H)} \le \frac{1}{\lambda}$$

Proof- We already showed in a previous lemma that if $B = -A$, generates a C_0 –semigroup, then $B = -A$ is closed and densely defined. We also found that

$$(i) \qquad S(t)u_0 - u_0 = \int_0^t S(\tau)\,Bu_0\,d\tau \qquad \forall u_0 \in D_A = D_B$$

$$(ii) \qquad S(t)x - x = \int_0^t B\,S(\tau)\,x\,d\tau \qquad \forall x \in H.$$

Now, for any $\lambda > 0,\ T(t) = e^{-\lambda t}S(t)$ is a C_0 –semigroup of contractions that is generated by $B - \lambda I,\ D_{B-\lambda I} = D_B$. Apply the results $(i),(ii)$ to $T(t)$ to get

$$T(t)u_0 - u_0 = e^{-\lambda t}S(t)u_0 - u_0 = \int_0^t e^{-\lambda\tau}S(\tau)\,(B - \lambda I)u_0\,d\tau \qquad \forall u_0 \in D_A = D_B$$

and
$$T(t)x - x = e^{-\lambda t}S(t)x - x = \int_0^t e^{-\lambda\tau}(B - \lambda I)\,S(\tau)\,x\,d\tau \qquad \forall x \in H.$$

Now let $t \to \infty$ and use the fact that $-A = B$ is closed to conclude

$$u_0 = \int_0^\infty e^{-\lambda\tau}S(\tau)\,(\lambda I - B)u_0\,d\tau \qquad \forall u_0 \in D_A = D_B$$

$$x = \int_0^\infty (\lambda I - B)\,e^{-\lambda\tau}\,S(\tau)\,x\,d\tau = (\lambda I - B)\int_0^\infty e^{-\lambda\tau}\,S(\tau)\,x\,d\tau \qquad \forall x \in H.$$

That is, $\forall u_0 \in D_A \qquad u_0 = \int_0^\infty e^{-\lambda\tau}S(\tau)\,(\lambda I + A)u_0\,d\tau \quad i.e.,\ (\lambda I + A)$ is 1-1

$$\forall x \in H \qquad x = (\lambda I + A)\int_0^\infty e^{-\lambda\tau}\,S(\tau)\,x\,d\tau$$

$$= (\lambda I + A)\,z, \qquad z \in D_A$$ $i.e.,\ (\lambda I + A)$ is onto

Finally, $$(\lambda I + A)^{-1}x = \int_0^\infty e^{-\lambda\tau}\,S(\tau)\,x\,d\tau$$
implies

$$\|(\lambda I + A)^{-1}x\|_H \le \int_0^\infty e^{-\lambda\tau}\,d\tau\|S(\tau)\,x\|_H \le \int_0^\infty e^{-\lambda\tau}\,d\tau\ \|x\|_H \le \frac{1}{\lambda}\,\|x\|_H$$

This proves that 2)a),b) are necessary conditions if $-A$ is to generate a C_0-semigroup of contractions. Note that

$$(\lambda I + A)^{-1}x = \int_0^\infty e^{-\lambda\tau}\,S(\tau)\,x\,d\tau = L[S(t)],$$

and

$$\|(\lambda I + A)^{-1}x\|_H \le \frac{1}{\lambda}\,\|x\|_H$$

which generalizes the analogous result observed earlier for the scalar equation.

Now we suppose A satisfies 2)a),b) and we will show $-A$ generates a C_0-semigroup of contractions, S(t). We will accomplish this by approximating A by a bounded linear operator A_λ and showing

$$A_\lambda \in L(H) \qquad A_\lambda x \ \longrightarrow\ Ax \qquad in\ H\ as\ \lambda \to \infty \qquad \forall x \in D_A$$
$$\Downarrow$$
$$e^{-tA_\lambda}x \longrightarrow S(t)\,x \quad in\ L(H)\ as\ \lambda \to \infty \quad \forall x \in H.$$

Now if A satisfies 2a) and 2b) then $(\lambda I + A) : D_A \to H$ is bijective for $\lambda > 0$ so that $(\lambda I + A)^{-1}$ exists. Then we may define $A_\lambda := \lambda A(\lambda I + A)^{-1}$ a bounded operator which will be shown to be an approximation to A.

Lemma 1 Under the conditions 2a) and 2b), the operator $A_\lambda \in L(H),\ \lambda > 0\ \ and$

a. $A_\lambda = \lambda I - \lambda^2(\lambda I + A)^{-1}$

b. $\|A_\lambda x\|_H \le \|Ax\|_H \qquad \forall x \in D_A$

c. $A_\lambda x \longrightarrow Ax \quad in\ H \quad as \quad \lambda \to \infty \qquad \forall x \in D_A$

Proof of lemma- For $x \in D_A \quad write$

$$\big(\lambda A(\lambda I + A)^{-1} - \lambda I\big)(\lambda I + A)x = \lambda Ax - \lambda^2 x - \lambda Ax = -\lambda^2 x.$$

i.e., $\quad (A_\lambda - \lambda I)\, z = -\lambda^2(\lambda I + A)^{-1} z \qquad \forall z \in H$

or $\qquad A_\lambda z = \lambda z - \lambda^2(\lambda I + A)^{-1} z \qquad \forall z \in H.$

For $x \in D_A$, 2b) implies

$$\|A_\lambda x\|_H = \big|\big|\, \lambda A(\lambda I + A)^{-1} x \big|\big|_H \le \|Ax\|_H.$$

If we combine these two results, we get

$$\big|\big|\lambda(\lambda I + A)^{-1} z - z\big|\big|_H = \frac{1}{\lambda}\|A_\lambda z\|_H \le \frac{1}{\lambda}\|Az\|_H \qquad \forall z \in D_A$$

from which it follows that $\lambda(\lambda I + A)^{-1} z \longrightarrow z\ in\ H\ as\ \lambda \to \infty\ \forall z \in D_A$. But D_A is dense in H and $\lambda(\lambda I + A)^{-1}$ is uniformly bounded on H by 2b) hence, by continuity,

$$\lambda(\lambda I + A)^{-1} z \longrightarrow z \quad in\ H \quad as\ \lambda \to \infty \qquad \forall z \in H.$$

Then $A_\lambda x = \lambda(\lambda I + A)^{-1} Ax \longrightarrow Ax\ in\ H\ as\ \lambda \to \infty\ \forall x \in D_B.$ ■

Since $A_\lambda \in L(H),\ \lambda > 0$ we can define

$$S_\lambda(t) = e^{-tA_\lambda} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} A_\lambda^n \qquad for\ t \ge 0,\ \lambda > 0.$$

Lemma 2 Under the conditions 2a) and 2b), for each $\lambda > 0$, $\{S_\lambda(t) : t \ge 0\}$ is a strongly continuous semigroup of contractions on H with generator equal to $-A_\lambda$. For each $x \in D_A$ $\{S_\lambda(t)[x]\}$ converges in H as $\lambda \to \infty$. Moreover, the convergence is uniform in t on all bounded intervals $[0, T]$.

Proof of lemma- Lemma 1a) implies $S_\lambda(t) = e^{-tA_\lambda} = e^{-t\lambda} e^{\lambda^2 t(\lambda + A)^{-1}}$ and then we have as a result of hypothesis 2b)

$$\|S_\lambda(t)\|_{L(H)} \le e^{-t\lambda} \| e^{\lambda^2 t(\lambda + A)^{-1}} \|_{L(H)} \le e^{-t\lambda} e^{t\lambda} = 1.$$

Thus $S_\lambda(t)$ is a strongly continuous semigroup of contractions on H. Also

$$\tfrac{d}{dt} S_\lambda(t) = -A_\lambda\ S_\lambda(t)$$

and

$$S_\lambda(t) - S_\mu(t) = \int_0^t \tfrac{d}{ds}(S_\mu(t-s)S_\lambda(s))\, ds = \int_0^t S_\mu(t-s) S_\lambda(s)(A_\mu - A_\lambda)\, ds.$$

Then

$$\|S_\lambda(t)x - S_\mu(t)x\|_H \le t\|A_\mu x - A_\lambda x\|_H \qquad \forall t \ge 0, \quad \lambda, \mu > 0,\ x \in D_A.$$

i.e., $\{S_\lambda(t)[x]\}$ is a Cauchy sequence in H, uniformly in t$\in [0,T]$.■

To complete the proof of the theorem, observe that each $S_\lambda(t)$ is a contraction and D_A is dense in H, so the limit

$$S_\lambda(t)[x] \longrightarrow S(t)[x] \ \text{in } H \text{ as } \lambda \to \infty, \qquad t \geq 0$$

extends to all x in H, and holds uniformly in $t \in [0,T]$. Since every $S_\lambda(t)$ is a strongly continuous contraction, clearly $S(t) \in L(H)$ is a contraction and in addition, since the convergence is uniform in t on bounded intervals, $[0,T]$, it follows that $S(t)$ is strongly continuous. The semigroup identity also holds since $S_\lambda(t)S_\lambda(\tau) \longrightarrow S(t)S(\tau)$ etc. Then $S(t)$ is a strongly continuous semigroup of contractions on H. Now for $x \in D_A$ and $h > 0$,

$$S_\lambda(h)x - x = \int_0^h S_\lambda(t)(-A_\lambda x)dt \qquad \forall \lambda > 0,$$

and as $\lambda \to \infty$, this becomes

$$S(h)x - x = \int_0^h S(t)(-Ax)dt$$

and this implies that the generator B of S(t) is an extension of $-A$. But we have assumed that $(I+A)$ is onto and since B is the generator of S(t), $I-B$ is one to one. Then $(I-B)D_A = (I+A)D_A = H$, which is to say, $(I-B)^{-1}H = D_A$ or $B = -A$.■

Now we can prove:

Theorem- Existence and Uniqueness for the IVP

Suppose $-A$ generates $S(t)$, a strongly continuous semigroup of contractions on H. Then for any $u_0 \in D_A$, $u(t) = S(t)[u_0]$ satisfies

i $u(t) \in C^0([0,\infty);H) \cap C^1((0,\infty);H)$,

ii $u'(t) + Au(t) = 0, \qquad t > 0,$

iii $u(0) = u_0$

and $u(t)$ is unique.

Proof- We have already shown that for $S(t)$, a strongly continuous semigroup and $u_0 \in D_A$, $S(t)[u_0]$ satisfies i) and iii), and moreover,

$$\tfrac{d}{dt}(S(t)[u_0]) = B(S(t)[u_0]) = -A(S(t)[u_0]) \qquad \forall u_0 \in D_A$$

The Hille-Yosida theorem shows that if $-A$ generates a strongly continuous semigroup, then

$$\left|\left|(\lambda I + A)^{-1}x\right|\right|_H \leq \frac{1}{\lambda}\|x\|_H \qquad \lambda > 0,\ x \in H,$$

or

$$\lambda^2\|z\|_H^2 \leq \|(\lambda I + A)z\|_H^2 \qquad \lambda > 0,\ z \in D_A.$$

This implies

$$2(Az,z)_H \geq -\frac{\|Az\|_H^2}{\lambda} \qquad \lambda > 0, \quad z \in D_A$$

or

$$(Az,z)_H \geq 0 \qquad \forall z \in D_A.$$

Then A is accretive and $u(t) = S(t)[u_0]$ is unique.■

Corollary- Suppose $-A$ generates $S(t)$, a strongly continuous semigroup of contractions on H. Then for any $u_0 \in D_A$, and every $f \in C^1([0,T];H)$, the unique solution of

$$u'(t) + Au(t) = f(t), \quad 0 < t < T, \qquad u(0) = u_0,$$

is given by $$u(t) = S(t)[u_0] + \int_0^t S(t-s)[f(s)]\,ds.$$

Proof- Let $$g(t) = \int_0^t S(t-s)[f(s)]\,ds.$$

Then $g(0) = 0$ and

$$\begin{aligned} g(t+h) - g(t) &= \int_0^{t+h} S(t+h-s)[f(s)]ds - \int_0^t S(t-s)[f(s)]\,ds \\ &= \int_0^{t+h} S(\sigma)[f(t+h-\sigma)]\,ds - \int_0^t S(\sigma)[f(t-\sigma)]\,ds \\ &= \int_0^t S(\sigma)[f(t+h-\sigma) - f(t-\sigma)]\,ds + \int_t^{t+h} S(\sigma)[f(t+h-\sigma)]\,ds \end{aligned}$$

Dividing by $h > 0$ and letting $h \to 0$, we find

$$g'(t) = \int_0^t S(\sigma)[f'(t-\sigma)]\,ds + S(t)[f(0)],$$

which shows that g is differentiable.

But, $$\begin{aligned} \frac{g(t+h)-g(t)}{h} &= \frac{1}{h}\left[\int_0^{t+h} S(t+h-s)[f(s)]ds - \int_0^t S(t-s)[f(s)]\,ds\right] \\ &= \frac{S(h)-I}{h}\int_0^t S(t-s)[f(s)]\,ds + \frac{1}{h}\int_t^{t+h} S(t+h-s)[f(s)]ds \\ &= \frac{S(h)-I}{h}\,g(t) + \frac{1}{h}\int_t^{t+h} S(t+h-s)[f(s)]ds \end{aligned}$$

and, as $h \to 0$,

$$\frac{g(t+h)-g(t)}{h} \longrightarrow g'(t) \quad and \quad \frac{1}{h}\int_t^{t+h} S(t+h-s)[f(s)]ds \longrightarrow S(0)f(t) = f(t).$$

Since these limits exist, it follows that the limit

$$\lim_{h\to 0}\frac{S(h)-I}{h}\,g(t)$$

exists and equals $-Ag(t)$. Then we conclude $g'(t) - f(t) = -Ag(t), \;\; and \;\; g(t)$ satifies

$$g'(t) + Ag(t) = f(t), \quad t \in (0,T) \quad and \quad g(0) = 0.$$

Evidently, $u(t) = S(t)[u_0] + g(t)$ solves the inhomogeneous IVP. This solution is unique since if there are two such solutions, their difference satisfies the IVP with $f(t) = 0,\; u_0 = 0,$ and then this difference is zero since A is accretive.■

Examples-

1. Consider the Banach space
$X = L^1(0,\infty)$ *with* $A = -\frac{d}{dx}$, $D_A = \{u \in X : Au \in X\}$. This corresponds to solving the following initial value problem,

$$u'(t) + Au(t) = \partial_t u(x,t) - \partial_x u(x,t) = 0,$$
$$u(x,0) = u_0(x) \in D_A$$

Note that if $u(x)$ *and* $-u'(x) = Au(x)$ both belong to X, then $u(x) \to 0$ *as* $x \to \infty$. Then it follows that for $\lambda \geq 0$, *and* $u \in D_A$

$$(\lambda I + A)u(x) = \lambda u(x) - u'(x) = 0 \quad \text{implies} \quad u(x) = 0;$$

i.e., $u(x) = Ce^{\lambda x}$ and $u(x) \to 0$ *as* $x \to \infty$ if and only if $C = 0$.

Then $(\lambda I + A)^{-1}$ exists. In fact,

$$(\lambda I + A)u(x) = v(x) \qquad \Leftrightarrow \qquad u(x) = Ce^{\lambda x} - \int_0^x e^{\lambda(x-y)}v(y)\,dy,$$

where $u \in D_A$ *if* $C = \int_0^\infty e^{-\lambda y}v(y)\,dy$. Then we have,

$$u(x) = \int_x^\infty e^{\lambda(x-y)}v(y)\,dy = \int_0^\infty e^{-\lambda z}v(x+z)\,dz = (\lambda I + A)^{-1}v(x).$$

Note that

$$\begin{aligned} ||(\lambda I + A)^{-1}v||_X &= \int_0^\infty \left| \int_x^\infty e^{\lambda(x-y)}v(y)\,dy \right| dx \\ &\leq \int_0^\infty \int_x^\infty e^{\lambda(x-y)}|v(y)|\,dydx \leq \int_0^\infty \int_0^y e^{\lambda(x-y)}|v(y)|\,dx\,dy \\ &\leq \frac{1}{\lambda}\int_0^\infty (1 - e^{-\lambda y})|v(y)|\,dy \leq \frac{1}{\lambda}\|v\|_X \end{aligned}$$

If we take A to be a closed extension of the indicated operator then we have an operator that satisfies the hypotheses of the Hille Yosida theorem and it follows that -A generates a C_0 –semigroup of contractions on X. Note that the equation at the bottom of page 3 implies

$$(\lambda I + A)^{-1}v(x) = \int_0^\infty e^{-\lambda\tau}v(x+\tau)\,d\tau = \int_0^\infty e^{-\lambda\tau}S(\tau)v(x)\,d\tau,$$

from which it follows that for $x, \tau \geq 0$, $S(\tau)v(x) = v(x+\tau)$ *for* $v \in X$. Then the solution of

$$u'(t) + Au(t) = \partial_t u(x,t) - \partial_x u(x,t) = 0, \qquad u(x,0) = u_0(x) \in D_A$$

is given by $u(x,t) = S(t)[u_0(x)] = u_0(x+t)$. This is a wave travelling from **right to left** with speed one.

2. Consider the Banach space
$X = L^1(0,\infty)$ *with* $A = \frac{d}{dx}$, $D_A = \{u \in X : Au \in X,\ u(0) = 0\}$. This corresponds to solving the following initial-boundary value problem,

$$u'(t) + Au(t) = \partial_t u(x,t) + \partial_x u(x,t) = 0,$$
$$u(x,0) = u_0(x) \in D_A$$
$$u(0,t) = 0, \qquad t > 0,$$

Then it follows that for $\lambda \geq 0$, *and* $u \in D_A$

$$(\lambda I + A)u(x) = \lambda u(x) + u'(x) = 0 \quad \text{implies} \quad u(x) = 0$$

since

$$u(x) = Ce^{-\lambda x} \ and \ \ u(0) = 0 \ \ implies \ \ C = 0.$$

Then $(\lambda I + A)^{-1}$ exists. In fact,

$$(\lambda I + A)u(x) = v(x), \quad u(0) = 0 \iff \quad u(x) = \int_0^x e^{-\lambda(x-y)} v(y)\, dy = (\lambda I + A)^{-1} v(x).$$

Then $\quad || (\lambda I + A)^{-1} v(x) ||_X = \int_0^\infty \left| \int_0^x e^{-\lambda(x-y)} v(y)\, dy \right| dx \leq \frac{1}{\lambda} \|v\|_X$

and

$$(\lambda I + A)^{-1} v(x) = \int_0^x e^{-\lambda(x-y)} v(y)\, dy = \int_0^\infty e^{-\lambda\tau} S(\tau) v(x)\, d\tau.$$

But

$$\int_0^x e^{-\lambda(x-y)} v(y)\, dy = \int_0^x e^{-\lambda\tau} v(x-\tau)\, d\tau$$

hence

$$\int_0^x e^{-\lambda\tau} v(x-\tau)\, d\tau = \int_0^\infty e^{-\lambda\tau} S(\tau) v(x)\, d\tau.$$

Then $\quad S(\tau)v(x) = \left\{ \begin{array}{ccc} v(x-\tau) & if & 0 \leq \tau \leq x \\ 0 & if & \tau > x \end{array} \right\},$

and the solution of the initial boundary value problem

is given by $\quad u(x,t) = S(t)[u_0(x)] = \left\{ \begin{array}{ccc} u_0(x-t) & if & 0 \leq t \leq x \\ 0 & if & t > x \end{array} \right\}$

This is a wave travelling from **left to right** with speed one.