Alternative Development of Semigroups

Consider the abstract initial value problem

$$u'(t) + Au(t) = 0, \qquad u(0) = u_0$$

where A denotes an unbounded but closed and densely defined linear operator on a Banach space X. We recall from the previous development that if we approximate this problem by

$$u(t+h) - u(t) \approx -hAu(t+h),$$

then

$$(I + hA)u(t+h) \approx u(t) \quad for \ h \geq 0,$$

or

$$u(t+h) \approx (I + hA)^{-1}u(t)$$

This leads to $u(h) \approx (I + hA)^{-1}u_0$

$$u(2h) \approx (I + hA)^{-1}u(h) \approx (I + hA)^{-2}u_0$$
$$\vdots$$
$$u(nh) \approx (I + hA)^{-n}u_0,$$

or

$$u(t) \approx \left(I + \tfrac{t}{n}A\right)^{-n}u_0 \quad for \quad t \geq 0.$$

This suggests that the solution of the abstract IVP is given by

$$u(t) = Lim_{n\to\infty}\left(I + \tfrac{t}{n}A\right)^{-n}u_0 \quad for \quad t \geq 0, \qquad (1)$$

and raises the question of whether this limit exists under appropriate conditions on A.

Theorem (Hille-Yosida) In order that the limit in (1) exist it is sufficient that

i) A is closed and densely defined
ii) $(\lambda I + A)^{-1}$ exists and is bounded for $\lambda \geq 0$
iii) $||(\lambda I + A)^{-1}||_{L(X)} \leq \frac{1}{\lambda} \quad for \quad \lambda > 0.$

Proof- It follows from iii) that for $\mu \geq 0$,

$$||(I + \mu A)^{-1}||_{L(X)} \leq 1 \qquad (2)$$

Now for $n = 1, 2, \ldots$ define $S_n(t) = \left(I + \tfrac{t}{n}A\right)^{-n}$. Then

$$\|S_n(t)\|_{L(X)} \leq 1 \quad for \quad n = 1,2,\ldots \quad t \geq 0.$$

i.e., the family $\{S_n(t)\}$ is uniformly bounded in the operator norm on $L(X)$. Now write

$$u - (I+\mu A)^{-1}u = \mu(I+\mu A)^{-1}Au \qquad \forall u \in D_A.$$

Then it follows from (2) that

$$\left|\left| u - (I+\mu A)^{-1}u \right|\right|_X \leq \mu\|Au\|_X \qquad \forall u \in D_A.,$$

and this implies that $\forall u \in D_A,$ we have

$$\left|\left| u - (I+\mu A)^{-1}u \right|\right|_X \to 0 \quad as \quad \mu \to 0;$$

i.e., $\quad (I+\mu A)^{-1}u \to u \ in \ X \ as \ \mu \to 0 \quad \forall u \in D_A.$

Since D_A is dense in X and $(I+\mu A)^{-1}$ is uniformly bounded for $\mu \geq 0,$ it follows that $(I+\mu A)^{-1}$ converges to I (in the norm topology on X) as $\mu \to 0$ and $S_n(t)$ converges to I (in the norm topology on X) as $t \to 0$. Now suppose for the moment that we can show

$$\| S_n(t)u - S_m(t)u\|_X \leq \frac{t^2}{2}\left(\tfrac{1}{n} + \tfrac{1}{m}\right)\|A^2u\|_X \quad \forall u \in D_2 = D(A^2). \qquad (3)$$

Then (3) implies $\{S_n(t)u\}$ is a Cauchy sequence in X for every u in D_2. But $D_2 = (\lambda I + A)^{-1}D_A = (\lambda I + A)^{-2}X,$ and since D_A is dense in X, it follows that D_2 is also dense in X. Then, $\{S_n(t)\}$ is a uniformly bounded family of bounded operators on X with $\{S_n(t)u\}$ converging strongly for all u in a dense subset of X. It follows that for all $t \geq 0,$ $S_n(t)$ converges strongly to a bounded linear operator, $S(t),$ as $n \to \infty.$
It remains only to prove (3). Write

$$S_n(t)u - S_m(t)u = \lim_{\varepsilon\to 0}\int_\varepsilon^{t-\varepsilon} \tfrac{d}{ds}(S_m(t-s)\,S_n(s)u)\,ds$$

$$= \lim_{\varepsilon\to 0}\int_\varepsilon^{t-\varepsilon}[S_m(t-s)\,S'_n(s)u - S'_m(t-s)\,S_n(s)u]\,ds.$$

But

$$S'_n(t) = A\left(I + \frac{t}{n}A\right)^{-n-1}$$

hence

$$S_m(t-s)\,S'_n(s)u - S'_m(t-s)\,S_n(s)u =$$

$$= A\left(I + \frac{t-s}{m}A\right)^{-m-1}\left(I + \frac{s}{n}A\right)^{-n} - A\left(I + \frac{t-s}{m}A\right)^{-m}\left(I + \frac{s}{n}A\right)^{-n-1}$$

$$= \left(\frac{t-s}{m} - \frac{s}{n}\right)A^2\left(I + \frac{t-s}{m}A\right)^{-m-1}\left(I + \frac{s}{n}A\right)^{-n-1}$$

Now A commutes with its resolvent and the family $\{S_n(t)\}$ is uniformly bounded which leads to

$$\|S_n(t)u - S_m(t)u\|_X \leq \|A^2u\|_X \int_0^t \left(\frac{t-s}{m} + \frac{s}{n} \right) ds$$

$$\leq \frac{t^2}{2}\left(\frac{1}{n} + \frac{1}{m} \right)\|A^2u\|_X \qquad \forall u \in D_2 = D(A^2) \blacksquare$$

Recall that in the previous approach we began with the assumption that the IVP had a unique solution, which we expressed as $u(t) = S(t)u_0$, and we then proceeded to prove that S(t) had certain properties. In particular, we showed that there was an operator B, called the generator, associated with S(t) and we showed that B was an extension of the operator -A from the IVP. Finally we proved the Hille-Yosida theorem which specified conditions on A sufficient to imply that -A was the generator of a semigroup. Here we have begun with A and showed that under certain conditions on A, the limit (1) must exist. We must now show that the operator S(t) obtained in this limit has properties that are useful for solving the IVP.

Properties of S(t)

Let A denote a linear operator on X satisfying the hypotheses of the theorem and let

$$S(t) = Lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n}.$$

Then

1) $S(t)$ is a strongly continuous function of $t \geq 0;\ i.e.,\ \forall u \in X$

$$\|S(t)u - S(t_0)u\|_X \to 0 \quad as\ t \to t_0 \geq 0.$$

2) $S(0) = I$

3) $\|S(t)\|_{L(X)} \leq 1.$

4) $S(t) : D_A \to D_A \quad and \quad AS(t)u = S(t)Au \quad \forall u \in D_A$

5) $S'(t)u = -AS(t)u = -S(t)Au \quad \forall u \in D_A$

6) $\lim\limits_{t\to 0} \dfrac{S(t) - I}{t}u = -Au \qquad \forall u \in D_A$

To prove 1) we can write

$$\|S(t)u - S(t_0)u\|_X = \|S(t)u - S_m(t)u + S_m(t)u - S_m(t_0)u + S_m(t_0)u - S(t_0)u\|_X$$

and use (3) to write

$$\|S(t)u - S_m(t)u\|_X \leq \frac{t^2}{2m}\,\|A^2u\|_X \quad \forall u \in D_2 = D(A^2).$$

Since $S_m(t)$ is strongly continuous in t for every m,

$$\|S_m(t)u - S_m(t_0)u\|_X \to 0 \quad as\ t \to t_0 \geq 0,$$

and the result follows. The results 2) and 3) are proved similarly. To prove 4) note that since A commutes with its resolvent $(I+\mu A)^{-1}$, it follows that for all $u \in D_A$

$$A\left(I+\tfrac{t}{n}A\right)^{-n-1}u = A\,S_n(t)\left(I+\tfrac{t}{n}A\right)^{-1}u \to A\,S(t)\,Iu = AS(t)u$$

and
$$A\left(I+\tfrac{t}{n}A\right)^{-n-1}u = \left(I+\tfrac{t}{n}A\right)^{-n-1}Au$$
$$= S_n(t)\left(I+\tfrac{t}{n}A\right)^{-1}Au \to S(t)IAu = S(t)Au.$$

This proves 4) and since $S'_n(t) = -\left(I+\tfrac{t}{n}A\right)^{-n-1}A$, we have

$$S_n(t)u - u = -\int_0^t \left(I+\tfrac{\tau}{n}A\right)^{-n-1}Au\,d\tau \qquad \forall u \in D_A.$$

Letting n tend to ∞ we find

7) $S(t)u - u = -\int_0^t S(\tau)Au\,d\tau \qquad \forall u \in D_A.$

Note that passing to the limit under the integral is valid since the convergence

$$\left(I+\tfrac{\tau}{n}A\right)^{-n-1}Au \to S(\tau)Au$$

is uniform on any finite time interval. The result 7) is also of interest. To prove 6), use 7) to write

$$(S(t)-I)u = -\int_0^t S(\tau)Au\,d\tau \qquad \forall u \in D_A.$$

Now the MVT for integrals implies

$$\int_0^t S(\tau)Au\,d\tau = S(t_0)\,tAu \quad for\ \ 0 \leq t_0 \leq t,$$

and

$$\frac{S(t)-I}{t}u = -S(t_0)\,Au \qquad for\ \ 0 \leq t_0 \leq t.$$

Now let t tend to zero and use 2) to get 6).

It is now possible to show that for all $u_0 \in D_A$, the unique solution of

$$u'(t) + Au(t) = 0, \qquad u(0) = u_0$$

is given by

$$u(t) = S(t)u_0.$$

That $u(t) = S(t)u_0$ solves the IVP follows from 5) and 2). To see that the solution is unique, let $v(t)$ satisfy

$$v \in C([0,T] : D_A) \cap C^1((0,T : D_A)$$

$$v'(t) + Av(t) = 0, \qquad v(0) = 0.$$

Let $F(s) = S(t-s)v(s), \qquad 0 \le s \le t, \quad t > 0 \;\; fixed.$

Then

$$F'(s) = -S'(t-s)v(s) + S(t-s)v'(s)$$

$$= AS(t-s)v(s) - S(t-s)Av(s) = 0.$$

i.e., F is constant. But,

$$F(t) = S(0)v(t) = v(t) \;\; and \;\; F(0) = S(t)v(0)$$

hence

$$v(t) = S(t)v(0) = 0 \;\; for\ all \;\; t \ge 0.$$

Since the difference of any two solutions to the IVP must satisfy the IVP with a zero initial state, the uniqueness follows.

Finally consider $F(s) = S(t-s)v(s)$ in the special case that $v(s) = S(s)u_0$. Then

$$F(s) = S(t-s)S(s)u_0 \qquad and \qquad F(0) = S(t)u_0.$$

But F is constant even in this case so

$$F(s) = S(t-s)S(s)u_0 = S(t)u_0 = F(0). \; \forall u_0 \in D_A.$$

Since D_A is dense in X we have

$$.S(t-s)S(s) = S(t) \;\; for \;\; 0 \le s \le t$$

or

8) $S(\tau + s) = S(\tau)S(s) \;\; for \;\; s, \tau \ge 0.$

Note that if we let $J(t)u = e^{-\lambda t}S(t)u \;\; for\ \lambda, t \ge 0 \; and \;\; u \in X,$ then

$$J'(t)u = -\lambda e^{-\lambda t}S(t)u + e^{-\lambda t}S'(t)u = -e^{-\lambda t}S(t)(\lambda I + A)u \;\; \forall u \in D_A$$

i.e.,

$$J'(t)u = -J(t)(\lambda I + A)u \qquad \forall u \in D_A$$

and we see that $J(t)$ is a contraction semigroup generated by $(\lambda I + A)$. Let $v = (\lambda I + A)u$. Then

$$J(t)u - u = -\int_0^t J(\tau)(\lambda I + A)u\,d\tau \qquad \forall u \in D_A$$

and if we apply $(\lambda I + A)$ to both sides of this equation, then we get

$$J(t)v - v = -(\lambda I + A)\int_0^t J(\tau)v\,d\tau \qquad \forall v \in X.$$

Let t tend to infinity and use the fact that

$$\|J(t)u\|_X = \|e^{-\lambda t}S(t)u\|_X \le e^{-\lambda t}\|u\|_X$$

to conclude that

$$u = \int_0^\infty e^{-\lambda\tau}S(\tau)(\lambda I + A)u\,d\tau \qquad \forall u \in D_A.$$

Note that this implies $(\lambda I + A)$ is one to one. Moreover,

$$v = (\lambda I + A)\int_0^\infty e^{-\lambda\tau}S(\tau)v\,d\tau \qquad \forall v \in X.$$

implies that $(\lambda I + A)$ is onto and

$$(\lambda I + A)^{-1}v = \int_0^\infty e^{-\lambda\tau}S(\tau)v\,d\tau \qquad \forall v \in X,$$

with

$$\left|\left|(\lambda I + A)^{-1}v\right|\right|_X \le \frac{1}{\lambda}\|v\|_X. \;\; \forall \lambda > 0.$$

Together, these properties of S(t) show that the Hille Yosida conditions on A are not only sufficient for the generation of S(t), they are also necessary.