

Injections, Embeddings and the Trace Theorem

We want to derive a number of properties for the spaces $H^k(U)$ for U a bounded open set in R^n . It turns out that the most efficient way to do this is to consider first the much simpler case where $U = R^n$.

1. Fourier Characterization of the Hilbert-Sobolev Spaces

For $\phi \in C_c^\infty(R^n)$ let the Fourier transform be defined by,

$$T_F[\phi] = (2\pi)^{-n/2} \int_{R^n} \phi(x) e^{-ix \cdot z} dx = \hat{\phi}(z)$$

We have showed previously that the Fourier transform is an isometric isomorphism on $L^2 = C_c^\infty(R^n) \cap L^2(R^n)$. In particular, we have

$$(a) \quad \forall u \in L^2 \quad \|u\|_0 = \|\hat{u}\|_0 \quad (\text{Parseval relation})$$

$$(b) \quad \forall u, v \in L^2, \quad (u, v)_0 = (\hat{u}, \hat{v})_0 \quad (\text{Plancherel relation})$$

The results (a) and (b) hold for every test function, and since the test functions are dense in $L^2(R^n)$, they extend to $L^2(R^n)$ by continuity. Then the Fourier transform can be extended to $L^2(R^n)$ by continuity as well. In addition,

$$(i) \quad \text{if } u, D^\alpha u \in L^2 \quad \text{then} \quad T_F[D^\alpha u] = (iz)^\alpha \hat{u}(z) \in L^2$$

$$(ii) \quad \text{if } u, x^\alpha u(x) \in L^2 \quad \text{then} \quad T_F[x^\alpha u(x)] = (iD_z)^\alpha \hat{u}(z) \in L^2.$$

i.e.,

$$\begin{aligned} T_F[D^\alpha u] &= \int_{R^n} D^\alpha u(x) e^{-ix \cdot z} d\hat{x} = (-1)^{|\alpha|} \int_{R^n} u(x) D_x^\alpha e^{-ix \cdot z} d\hat{x} \\ &= (-1)^{|\alpha|} \int_{R^n} u(x) (-iz)^\alpha e^{-ix \cdot z} d\hat{x} = (iz)^\alpha \hat{u}(z); \end{aligned}$$

$$T_F[x^\alpha u(x)] = i^\alpha \int_{R^n} u(x) (-ix)^\alpha e^{-ix \cdot z} d\hat{x} = i^\alpha \int_{R^n} u(x) D_z^\alpha (e^{-ix \cdot z}) d\hat{x} = (iD_z)^\alpha \hat{u}(z).$$

The result (i) asserts that when u is smooth, \hat{u} decays rapidly at infinity and result (ii) asserts the converse. This suggested the definition,

$$\text{for } s \geq 0, \quad H^s = \left\{ u \in L^2 : (1 + |z|^2)^{s/2} \hat{u}(z) \in L^2 \right\}$$

$$\text{with} \quad \|u\|_s^2 = \int_{R^n} (1 + |z|^2)^s \hat{u}(z)^2 dz,$$

$$\text{and} \quad (u, v)_s = \int_{R^n} (1 + |z|^2)^s \hat{u}(z) \hat{v}(z) dz.$$

Then $H^0 = L^2$, and $s \geq t \geq 0$ implies $H^s \subset H^t \subset H^0$.

To see how partial differential operators act on these spaces, let

$$P_m(z) = \sum_{|\alpha| \leq m} c_\alpha z^\alpha$$

and define $P(D_x)u(x) = T_F^{-1}[P_m(iz) \hat{u}(z)]$ for $u \in H^m(R^n) \subset H^0 = L^2$.

Then

$$P(D_x)u(x) = \int_{R^n} e^{ix \cdot z} \sum_{|\alpha| \leq m} c_\alpha (iz)^\alpha \hat{u}(z) dz$$

$$\begin{aligned}
&= \sum_{|\alpha| \leq m} c_\alpha \int_{R^n} e^{ix \cdot z} (iz)^\alpha \hat{u}(z) d\hat{z} \\
&= \sum_{|\alpha| \leq m} c_\alpha D_x^\alpha u(x)
\end{aligned}$$

where the last step follows from (i). Then $P(D_x)$ is a differential operator of order m . For $s > m$ we have

$$\begin{aligned}
\|P(D_x)u(x)\|_{s-m} &= \int_{R^n} (1 + |z|^2)^{s-m} P_m(iz)^2 \hat{u}(z)^2 dz, \\
&= \int_{R^n} (1 + |z|^2)^s \left[\frac{P_m(iz)^2}{(1 + |z|^2)^m} \right] \hat{u}(z)^2 dz \leq C_m \|u\|_s^2,
\end{aligned}$$

where

$$C_m = \max_z \frac{|P_m(iz)|^2}{(1 + |z|^2)^m} < \infty.$$

This implies that any differential operator of order m is a bounded linear mapping from H^s into H^{s-m} for $s > m$.

Recall that the definition of the norm in $H^m(R^n) = \{u \in L^2 : D^\alpha u \in L^2, |\alpha| \leq m\}$ was defined as

$$\|u\|_m^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|_0^2$$

But

$$\|D^\alpha u\|_0^2 = \|z^\alpha \hat{u}(z)\|_0^2 = \int_{R^n} |z|^{2\alpha} \hat{u}(z)^2 dz$$

hence

$$\begin{aligned}
\|u\|_m^2 &= \sum_{|\alpha| \leq m} \int_{R^n} |z|^{2\alpha} \hat{u}(z)^2 dz \\
&= \int_{R^n} \sum_{|\alpha| \leq m} \frac{|z|^{2\alpha}}{(1 + |z|^2)^m} (1 + |z|^2)^m \hat{u}(z)^2 dz \leq C_m \|u\|_m^2
\end{aligned}$$

where

$$(1 + |z|^2)^m \leq \sum_{|\alpha| \leq m} |z|^{2\alpha} \leq C_m (1 + |z|^2)^m$$

This shows that the Fourier norm is equivalent to the original Sobolev norm on H^m , and the two definitions of the space are therefore also equivalent. We will now prove the first of the important embedding results for these spaces.

2. Inclusions and Embeddings

We prove first a result that says that if $u \in H^s$ for s sufficiently large, then u is equivalent (i.e., equal almost everywhere to) a function that is continuous.

Theorem 2.1 (Sobolev embedding theorem, primitive version) If $s > m + \frac{n}{2}$ then H^s is continuously injected into C^m

Proof- We will first prove the result for $m = 0$. If $u \in H^s$ then

$\hat{v}(z) = (1 + |z|^2)^s \hat{u}(z) \in H^0 = L^2$. Then note that

$$\|\hat{u}(z)\|_{L^1} = \int_{R^n} |\hat{u}(z)| dz = \int_{R^n} (1 + |z|^2)^{-s/2} (1 + |z|^2)^{s/2} |\hat{u}(z)| dz = \int_{R^n} (1 + |z|^2)^{-s/2} |\hat{v}(z)| dz$$

and $\|\hat{u}(z)\|_{L^1} \leq \left\| (1 + |z|^2)^{-s/2} \right\|_0 \|\hat{v}\|_0$.

Now $\left\| (1 + |z|^2)^{-s/2} \right\|_0^2 = \int_{R^n} (1 + |z|^2)^{-s} dz = \int_{\Omega} \int_0^{\infty} (1 + r^2)^{-s} r^{n-1} dr d\omega$,

and $\int_0^{\infty} (1 + r^2)^{-s} r^{n-1} dr \sim \int_1^{\infty} r^{n-1-2s} dr \sim r^{n-2s}|_1^{\infty}$.

Then the integral is finite if $n - 2s < 0$; i.e., if $s > \frac{n}{2}$. We have proven that $u \in H^s$, $s > \frac{n}{2}$ implies $\hat{u} \in L^1$. But $\hat{u} \in L^1$ implies, in turn, that $\tilde{u} = T_F^{-1}[\hat{u}]$ is in the space, C_0 , of continuous functions which tend to zero at infinity. That is to say, after possibly modifying u on a set of measure zero, it is a continuous function.

For $m > 0$, and $u \in H^s$ for $s > m + \frac{n}{2}$, we have, from the result about differential operators of order m , that $D^\alpha u \in H^{s-m}$ for $|\alpha| \leq m$, and hence, if $s - m > \frac{n}{2}$ then $D^\alpha u \in C_0$ for $|\alpha| \leq m$. ■

We interpret this result to mean that for $s > m + \frac{n}{2}$, every $u \in H^s$ can be identified with a unique function $\tilde{u} \in C^m$ and that this injection is continuous. More precisely, if $u \in H^s$ then there exists a $\tilde{u} \in C^m$ such that $\|u - \tilde{u}\|_{H^s} = 0$ and $\|\tilde{u}\|_{C^m} \leq \|u\|_{H^s}$.

The next result we are going to prove has to do with the embedding of $H_0^1(U)$ into $H^0(U)$. For U a bounded open set in R^n , let $\{u_m\} \in H_0^1(U)$ be such that $\|u_m\|_1 \leq M$, $\forall m$. Since $H_0^1(U)$ is a Hilbert space, it follows from a classical property of Hilbert spaces that $\{u_m\}$ must contain a weakly convergent subsequence. Moreover, since $H_0^1(U)$ is continuously injected into $H^0(U)$, the subsequence must also converge weakly in $H^0(U)$. The somewhat surprising fact is that the subsequence is actually strongly convergent in $H^0(U)$; i.e., the continuous injection of $H_0^1(U)$ into $H^0(U)$ is a compact embedding.

To prove the compact embedding result, we first define, for $u \in H_0^1(U)$,

$$Zu(x) = \begin{cases} u(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases} \quad (\text{extension by zero})$$

Since the support of u is compactly contained in U , it follows that

$$\|Zu\|_{H^1(R^n)} = \|u\|_{H^1(U)} \quad \text{and} \quad Zu \in H^1(R^n).$$

More generally,

$$u \in H_0^m(U) = \text{completion of } C_c^\infty(U) \text{ in the norm } \|\cdot\|_m \text{ implies } Zu \in H^s \quad s \leq m.$$

For $u \in H^m(U)$, it is not generally the case that the extension by zero, Zu , belongs to $H^m(R^n)$ since extending by zero introduces a sharp change in u at the boundary of the set U . For example, $u(x) = 1$, $x \in (0, 1)$ belongs to $H^1(0, 1)$ but $Zu(x)$ is just the indicator function of $(0, 1)$. Since this is a discontinuous function, it is not in $H^1(R)$.

Theorem 2.2 (Rellich's lemma, primitive version) Suppose U is open and bounded in R^n . Then the natural injection

$$i : H_0^1(U) \rightarrow H^0(U) \quad \text{is a compact mapping.}$$

i.e., every sequence that is bounded in $H_0^1(U)$ contains a subsequence that is strongly

convergent in $H^0(U)$.

Proof- Suppose $\{u_m\} \in H_0^1(U)$ is such that $\|u_m\|_1 \leq M, \forall m$. Since $\{u_m\} \in H_0^1(U)$ it is evident that $\{Zu_m\} \in H^1 \subset L^2$ and Zu_m has a Fourier transform, $\hat{u}_m \in L^2$. Then

$$\begin{aligned} Zu_m(x) &= T_F^{-1}[\hat{u}_m] = \int_{\mathbb{R}^n} \hat{u}_m(z) e^{ix \cdot z} dz = \int_{|z| < A} \hat{u}_m(z) e^{ix \cdot z} dz + \int_{|z| > A} \hat{u}_m(z) e^{ix \cdot z} dz, \\ &= T_F^{-1}[\hat{u}_m(z)I_A(z)] + T_F^{-1}[\hat{u}_m(z)(1 - I_A(z))]. \end{aligned}$$

Now $\hat{u}_m(z)I_A(z)$ has compact support which implies that $T_F^{-1}[\hat{u}_m(z)I_A(z)]$ is smooth. Moreover, we will use the hypothesis $\|u_m\|_1 \leq M, \forall m$, to show that the second term here can be made arbitrarily small by choosing A large. In fact,

$$\begin{aligned} \|T_F^{-1}[\hat{u}_m(z)(1 - I_A(z))]\|_0^2 &= \int_{|z| > A} \hat{u}_m(z)^2 dz = \int_{|z| > A} (1 + |z|^2)^{-1} (1 + |z|^2) \hat{u}_m(z)^2 dz, \\ &\leq \max_{|z| > A} (1 + |z|^2)^{-1} \|u_m\|_1^2 \leq \frac{M^2}{A^2}, \quad \forall m. \end{aligned}$$

Then, for any $\varepsilon > 0$, choose $A > \frac{M}{\varepsilon}$, to obtain $\|T_F^{-1}[\hat{u}_m(z)(1 - I_A(z))]\|_0 \leq \varepsilon$.

Now the smooth functions $\{v_m(x) = T_F^{-1}[\hat{u}_m(z)I_A(z)]\}$ can be shown to be a uniformly bounded and equicontinuous family of continuous functions. Postponing the proof of this fact for the moment, we note that this implies the existence (via Ascoli-Arzelà) of a subsequence $\{v_{m'}(x)\}$ which converges uniformly on U to a limit $v \in C(\bar{U})$. Since uniform convergence on U implies convergence in $L^2(U)$, it follows that this subsequence converges strongly to v in $L^2(U)$. In particular then, it must be true of the subsequence that

$$\|v_{m'} - v_{k'}\|_{H^0(U)} < \varepsilon \quad \text{for } m', k' > N_\varepsilon$$

and

$$\begin{aligned} \|u_{m'} - u_{k'}\|_{H^0(U)} &\leq \|v_{m'} - v_{k'}\|_0 + \|T_F^{-1}[(\hat{u}_{m'}(z) - \hat{u}_{k'}(z))(1 - I_A(z))]\|_0 \\ &\leq \varepsilon + 2\varepsilon \quad \text{for } m', k' > N_\varepsilon. \end{aligned}$$

Repeat this argument now for the subsequence $\{v_{m'} = T_F^{-1}[\hat{u}_{m'}(z)I_A(z)]\}$ replacing ε by $\varepsilon/2$. Iterate the procedure in order to generate a sequence of subsequences to which we can apply a diagonal procedure to get, finally, a subsequence $\{u_{m''}\}$ of $\{u_m\}$ that is Cauchy in $H^0(U)$.

To see that $\{v_m(x) = T_F^{-1}[\hat{u}_m(z)I_A(z)]\}$ is uniformly bounded and equicontinuous, note first that for any x and every m ,

$$|v_m(x)| = \left| \int_{|z| < A} \hat{u}_m(z) e^{ix \cdot z} dz \right| \leq \int_{|z| < A} |\hat{u}_m(z)| dz \leq C_A \|\hat{u}_m\|_0 = C_M \|u_m\|_0 \leq C_M \|u_m\|_1 \leq M \cdot C_M.$$

Thus the family is uniformly bounded. Next,

$$\begin{aligned} |v_m(x) - v_m(y)| &= \left| \int_{|z| < A} \hat{u}_m(z) (e^{ix \cdot z} - e^{iy \cdot z}) dz \right| \leq \int_{|z| < A} |\hat{u}_m(z) e^{ix \cdot z} (1 - e^{i(y-x) \cdot z})| dz \\ &\leq \sup_{|z| \leq A} |1 - e^{i(y-x) \cdot z}| C_A M \rightarrow 0 \text{ as } y \rightarrow x \text{ uniformly in } m. \end{aligned}$$

Since $\sup_{|z| \leq A} |1 - e^{i(y-x) \cdot z}| \rightarrow 0$ as $y \rightarrow x$ uniformly in m , it follows that the family is equicontinuous. ■

Corollary- For U open and bounded in R^n , and $m > 1$, $H_0^m(U)$ is compactly embedded in $H^{m-1}(U)$.

Proof- Suppose $\{u_m\} \subset H_0^m(U)$ with $\|u_m\|_m \leq M \quad \forall m$. Then for each α , $|\alpha| \leq m-1$, we have $\|D^\alpha u_m\|_1 \leq M$ and it follows from the theorem that there is a subsequence $\{u_{m'}\} \subset \{u_m\}$ such that $\{D^\alpha u_{m'}\}$ is Cauchy in $H^0(U)$. Then a diagonal argument implies the existence of a single subsequence $\{u_{m''}\} \subset \{u_m\}$ such that $\{D^\alpha u_{m''}\}$ is Cauchy in $H^0(U)$ for every α , $|\alpha| \leq m-1$. Then this subsequence is Cauchy in $H^{m-1}(U)$. ■

We will think of this result in the following way. Any sequence $\{u_m\} \subset H_0^m(U)$ with $\|u_m\|_m \leq M \quad \forall m$ contains a weakly convergent (in $H_0^m(U)$) subsequence just by virtue of the fact that $H_0^m(U)$ is a Hilbert space. Then we use the embedding lemma to conclude that this subsequence is, in fact, strongly convergent in $H^{m-1}(U)$. So, a bounded sequence in $H_0^m(U)$ contains a subsequence that is strongly convergent in $H^{m-1}(U)$.

3. The Trace Theorem

Since functions in $H^0(U)$, are, in fact, equivalence classes of functions they are defined only up to sets of measure zero. Then restriction of such functions to the boundary of U (a set of measure zero) has no meaning. On the other hand, for $u \in H^m(U)$ for m sufficiently large relative to n , it will be possible to give meaning to the restrictions to ∂U not just for the functions, but for some derivatives of the functions as well. We will begin with the simplest possible case of a set with a boundary, the set

$$U = R_+^n = \{(x', x_n) : x' \in R^{n-1}, 0 < x_n < \infty\} = R^{n-1} \times R_+.$$

In this case, $\partial U = R^{n-1} = \{(x', x_n) : x' \in R^{n-1}, x_n = 0\}$.

For $u \in C^\infty(R_+^n)$ define

$$T_j u(x') = \lim_{x_n \rightarrow 0^+} \partial_n^j u(x', x_n), \quad 0 \leq j \leq m-1.$$

We will show that this operator, called the **trace operator** of order j , can be extended to $H^m(R_+^n)$.

Theorem 3.1 (Trace theorem, primitive version)

1. $\|T_j u\|_{H^{m-j-1/2}(R^{n-1})} \leq C \|u\|_{H^m(R_+^n)} \quad \forall u \in C^\infty(R_+^n)$ hence T_j extends as a bounded linear mapping $T_j : H^m(R_+^n) \rightarrow H^{m-j-1/2}(R^{n-1})$
2. T_j maps $H^m(R_+^n)$ onto $H^{m-j-1/2}(R^{n-1})$
3. $T_j u = 0$ if and only if $u \in H_0^m(R_+^n)$

idea of the proof- In order to use the Fourier transform techniques, the functions must belong to $H^m(R^n)$ rather than $H^m(R_+^n)$. Thus for $u \in H^m(R_+^n)$ we define

$$Zu(x) = \left\{ \begin{array}{ll} u(x', x_n) & \text{if } x_n > 0 \\ 0 & \text{if } x_n < 0 \end{array} \right\}.$$

In general, $u \in H^m(\mathbb{R}_+^n)$ does not imply $Zu \in H^m(\mathbb{R}^n)$. Thus we define an operator, $E : H^m(\mathbb{R}_+^n) \rightarrow H^m(\mathbb{R}^n)$,

$$Eu(x) = \left\{ \begin{array}{ll} u(x', x_n) & \text{if } x_n > 0 \\ \sum_{k=1}^m a_k u(x', -kx_n) & \text{if } x_n < 0 \end{array} \right\}$$

Here the a'_k s are required to satisfy

$$\sum_{k=1}^m a_k (-k)^j = 1 \quad \text{for } j = 0, 1, \dots, m-1.$$

This (unique) choice of the a'_k s ensures

$$\lim_{x_n \rightarrow 0^-} \partial_n^j Eu(x', x_n) = \lim_{x_n \rightarrow 0^+} \partial_n^j u(x', x_n) \quad \text{for } j = 0, 1, \dots, m-1.$$

Then $u \in H^m(\mathbb{R}_+^n)$ implies $Eu \in H^m(\mathbb{R}^n)$ and, moreover

$$\|Eu\|_{H^m(\mathbb{R}^n)} \leq C_0 \|u\|_{H^m(\mathbb{R}_+^n)}$$

For future reference we also define

$$E_1 u(x', x_n) = \left\{ \begin{array}{ll} u(x', x_n) & \text{if } x_n > 0 \\ a(x_n) Eu(x', x_n) & \text{if } x_n < 0 \end{array} \right\}$$

where

$$a(x_n) \in C^\infty(\mathbb{R}^1), \quad a(x) = \left\{ \begin{array}{ll} 1 & \text{if } x_n > 0 \\ 0 & \text{if } x_n < -1 \end{array} \right\} = a \text{ smooth cutoff function}$$

Then $E_1 u \in H^m(\mathbb{R}_{-1}^n)$ for all $u \in H^m(\mathbb{R}_+^n)$ where $\mathbb{R}_{-1}^n = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, -1 < x_n < \infty\}$.

This modified extension operator smoothly extends the function $u \in H^m(\mathbb{R}_+^n)$ to a neighborhood of the boundary of \mathbb{R}_+^n .

For $v \in H^m(\mathbb{R}^n)$ let

$$\begin{aligned} \hat{v}(z) &= \hat{v}(z', z_n) = \int_{\mathbb{R}^n} e^{-ix \cdot z} v(x) d\hat{x} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} e^{-ix' \cdot z'} e^{-ix_n \cdot z_n} v(x', x_n) d\hat{x}' dx_n \\ &= \int_{\mathbb{R}^{n-1}} e^{-ix' \cdot z'} \hat{v}(x', z_n) d\hat{x}'. \end{aligned}$$

where $\hat{v}(x', z_n)$ denotes the Fourier transform of $v(x', x_n)$ with respect to the variable x_n only; i.e., $\hat{v}(x', z_n) := T_F[v(x', x_n)]_n$

This splitting off of the x_n to z_n part of the Fourier transform now allows us to define the action of the mapping T_j in a convenient way for purposes of telling to which space $H^s(\mathbb{R}^{n-1})$ the function $T_j v$ belongs. We write,

$$v(x', 0) = \int_{\mathbb{R}} e^{-ix_n \cdot z_n} |_{x_n=0} \hat{v}(x', z_n) dz_n = \int_{\mathbb{R}} \hat{v}(x', z_n) dz_n$$

and

$$T_j v(x') = \partial_n^j v(x', 0) = \int_{\mathbb{R}} (iz_n)^j \hat{v}(x', z_n) dz_n, \quad j = 0, 1, \dots, m-1.$$

This leads to

$$\|T_j v\|_{H^s(\mathbb{R}^{n-1})}^2 = \int_{\mathbb{R}^{n-1}} (1 + |z'|^2)^s T_F[T_j v(z')]^2_{n-1} dz', \quad T_F[T_j v(z')]_{n-1} = \text{transform in } x' \text{ only}$$

and since

$$\begin{aligned} T_F[T_j v(z')]_{n-1} &= T_F \left[\int_{\mathbb{R}} (iz_n)^j \hat{v}(x', z_n) dz_n \right]_{n-1} \\ &= \int_{\mathbb{R}} (iz_n)^j T_F[\hat{v}(x', z_n)]_{n-1} dz_n, \end{aligned}$$

we have

$$\|T_j v\|_{H^s(\mathbb{R}^{n-1})}^2 = \int_{\mathbb{R}^{n-1}} (1 + |z'|^2)^s \left(\int_{\mathbb{R}} (iz_n)^j \hat{v}(z) dz \right)^2 dz_n dz'$$

Then, using several clever tricks, we can show

$$\|T_j v\|_{H^s(\mathbb{R}^{n-1})}^2 \leq C_1 \|v\|_{H^m(\mathbb{R}^n)}^2 \text{ for } v \in H^m(\mathbb{R}^n), \quad s \leq m - j - 1/2.$$

and

$$E : H^m(\mathbb{R}_+^n) \rightarrow H^m(\mathbb{R}^n) \quad T_j : H^m(\mathbb{R}^n) \rightarrow H^{m-j-1/2}(\mathbb{R}^{n-1})$$

so the composition $T_j \circ E : H^m(\mathbb{R}_+^n) \rightarrow H^{m-j-1/2}(\mathbb{R}^{n-1})$ is linear and bounded.

To show that $T_j \circ E$ is onto we will construct a continuous right inverse for T_j . Define, for $v \in H^s(\mathbb{R}^{n-1})$

$$K[v](x', x_n) = T_F^{-1} \left[e^{-x_n} (1 + |z'|^2)^{1/2} \hat{v}(z') \right] = \int_{\mathbb{R}^{n-1}} e^{ix' \cdot z'} e^{-x_n} (1 + |z'|^2)^{1/2} \hat{v}(z') dz'.$$

Then $K[v](x', 0) = v(x')$ which implies $T_0[K[v]] = v \quad \forall v \in H^s(\mathbb{R}^{n-1})$

Note that this definition of K implies that $K[v](x)$ satisfies

$$\begin{aligned} (1 - \nabla^2)K[v](x) &= 0 \text{ in } \mathbb{R}_+^n \\ T_0[K[v]] &= v \text{ on } \partial\mathbb{R}_+^n \end{aligned}$$

Then for every $v \in H^s(\mathbb{R}^{n-1})$ this boundary value problem is uniquely solvable; i.e., there exists a $u = K[v]$ and we can show that

$$\|K[v]\|_{H^m(\mathbb{R}_+^n)} \leq C \|v\|_{H^{m-1/2}(\mathbb{R}^{n-1})}$$

This is the beginning of the idea of the proof that each T_j has a continuous right inverse; i.e., $T_j \circ K = id$, and T_j maps $H^m(\mathbb{R}_+^n)$ onto $H^{m-j-1/2}(\mathbb{R}^{n-1})$ continuously. The choice of the operator K is not unique.

Finally, it is easy to see that $T_j u = 0$ for all $u \in H^m(\mathbb{R}_+^n)$. That the converse is also true is more difficult to show and we will omit this proof. ■