# Weak Formulation of Elliptic BVP's

There are a large number of problems of physical interest that can be formulated in the abstract setting in which the Lax-Milgram lemma is applied to an equation expressed in terms of an appropriate bilinear form to prove existence of a unique solution. The selection of the solution space and the bilinear form controls the PDE and boundary conditions that the solution will satisfy. We will give some examples to illustrate the interplay between the bilinear form, the solution space and the formal problem that the corresponding solution satisfies.

**The Abstract Green's Formula**

In each of the problems, the linkage between the formal problem and the weak problem is provided by the *Green's formula.* For $U \subset R^n$, bounded open and connected, let

$$L[u(x)] \;=-div[A(x)\nabla u(x)] \;+b(x)\cdot\nabla u(x)+c(x)u(x) \qquad (1)$$

where $A=[a_{ij}(x)]$ denotes an n by n symmetric matrix which is uniformly positive definite on U. Suppose also that $a_{ij},b_j,c \in L^\infty(U).$ Then we have, for all $\phi,\psi$ in $C^\infty(\bar{U})$,

$$\int_U \psi(x)L[\phi(x)]\,dx = B[\phi,\psi] - \int_\Gamma \psi(x)\{n(x)\cdot A(x)\nabla\phi(x)\}dS$$

where

$$B[\phi,\psi] = \int_U [\nabla\psi\cdot A(x)\nabla\phi(x)+\psi(x)b(x)\cdot\nabla\phi(x)+c(x)\psi(x)\phi(x)]. \qquad (2)$$

Since $C^\infty(\bar{U})$ is dense in $H^1(U),$ it follows that for every $u,v\in H^1(U),$ there exist sequences $\{\phi_m\},\{\psi_m\}$ in $C^\infty(\bar{U})$such that

$$\phi_m \to u, \qquad \psi_m \to v \qquad in\, H^1(U)$$

$$B[\phi_m,\psi_m] \to B[u,v] \qquad in\, R,$$

$$\int_\Gamma \psi_m(x)\{n(x)\cdot A(x)\nabla\phi_m(x)\}dS \to \int_\Gamma T_0v\cdot T_1u\; dS$$

Here $T_0$ is just the previously defined zero order trace operator that amounts to restriction to the boundary of $U.$In view of the trace theorem, the fact that $\|\psi_m-\psi_n\|_1\to 0$ implies that $\|T_0\psi_m - T_0\psi_n\|_{H^{1/2}(\Gamma)}\to 0,$hence $T_0\psi_m \to T_0u\in .H^{1/2}(\Gamma).$Similarly, $T_1\phi_m = n(x)\cdot A(x)\nabla\phi_m(x)$ converges in $H^{-1/2}(\Gamma)$ to a limit we denote by $T_1u\;.H^{-1/2}(\Gamma).$Then we are tempted to write the generalized Green's formula as follows,

$$(L[u],v)_0 = B[u,v]-(T_0v,T_1u)_{H^0(\Gamma)} \qquad for\;\; u,v\in H^1(U),$$

where $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_{H^0(\Gamma)}$denote the inner products on $H^0(U)$ and $H^0(\Gamma),$ respectively. Of course, what this really means is that

$$\langle L[u],v\rangle_{V'\times V} = B[u,v]-\langle T_0v,T_1u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \qquad for\;\; u,v\in H^1(U). \qquad (3)$$

That is, $(L[u],v)_0$must be interpreted as $< L[\phi],\psi >_{V'\times V}$ , the action of the linear functional, $L[\phi]\in V'$ on the element $\psi\in V=H^1(U).$ and $\langle T_0\psi,T_1\phi\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$ indicates the analogous duality pairing on $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma);$this duality pairing is just the extension of the $H^0(\Gamma)$ inner product to $H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma).$ Note that the operator $T_1$ is not just the usual normal derivative obtained by projecting the gradient on the outward normal of the boundary. In fact, $T_1 u = n(x) \cdot A(x)\nabla u(x)$ and it is possible to show that this is continuous from $H^1(U)$ *onto* $H^{-1/2}(\Gamma)$

We are going to be using the abstract Green's formula for Hilbert spaces $V, H$ with $C^\infty(\bar{U})$ dense in $V$ and $V$ dense in $H$. Then, taking $H' = H$, we have $H = H'$ is contained in $V'$, so $V \subset H = H' \subset V'$. This is the case when $V$ is any space such that $H_0^1(U) \subset V \subset H^1(U)$ and $H = H^0(U)$. That is, the smallest $V$ we will encounter is $V = H_0^1(U)$, and the largest will be $V = H^1(U)$. Spaces in between these two extremes are composed of subspaces of functions from $H^1(U)$ that "vanish on part of the boundary". With this set up, known as a Gelfand triple of spaces, then the abstract Green's formula (3) makes sense, and since all the spaces $V$ are contained in $H^1(U)$, there is a trace theorem asserting that $T_j : V \to H^{1/2-j}(\Gamma)$ is a bounded linear mapping whose kernel is $H_0^1(U)$.

The abstract Green's formula is the basis for the weak formulation of all the BVP's we are going to consider. We will now illustrate how the weak formulation is obtained from the strong problem in several examples. Many of the remarks are repetitions of results that have already been developed but they are repeated in order to emphasize their importance and enhance their understanding.

## 1. Dirichlet Problem

Consider the Dirichlet problem with homogeneous boundary data,

$$L_0[u(x)] = F \quad \text{in } U, \quad \text{and} \quad u = 0 \ \text{ on } \Gamma \qquad (1.1)$$

in the case that the operator is symmetric; i.e.,

$$L_0[u(x)] = -div[A(x)\nabla u(x)], \qquad x \in U \subset R^n.$$

We assume that $A(x)$ is a symmetric n by n matrix of functions, $a_{ij}(x) \in L^\infty(U)$, such that

$$\vec{z} \cdot A(x)\vec{z} \geq a_0|z|^2 \qquad \forall z \in R^n \ \text{and} \ \forall x \in \bar{U}$$

Then $L_0$ is said to be uniformly elliptic on $U$ and for all $\phi, \psi$ in $C_c^\infty(U)$, we have

$$(L_0\phi, \psi)_0 = \int_U \psi(x) L\phi(x)dx = B_0[\phi, \psi]$$

$$B_0[\phi, \psi] = \int_U \nabla\psi \cdot A(x)\nabla\phi(x)dx.$$

The Poincare inequality then implies

$$B_0[\phi, \phi] \geq a_0\|\nabla\phi\|_0^2 \geq \tfrac{1}{2}a_0\|\nabla\phi\|_0^2 + \tfrac{1}{2C}a_0\|\phi\|_0^2 \geq C\|\phi\|_1^2 \quad \text{for all } \phi \text{ in } C_C^\infty(U),$$

and this result extends to all of $H_0^1(U)$ by continuity. Thus $B_0$ is coercive (or V-elliptic) on $V = H_0^1(U)$ which implies then, that $L_0$ is an isomorphism from $H_0^1(U)$ *onto* $H^{-1}(U)$. That is, for every continuous linear functional $F$ acting on $H_0^1(U)$, there exists a unique $u \in H_0^1(U)$ *such that*

$$B_0[u, v] = F(v) \quad \text{for all } v \in H_0^1(U).$$

This $u \in H_0^1(U)$ is then the weak solution of $(1)$, the Dirichlet problem with homogeneous boundary data.

Note that for $F$ given by $\qquad F(v) = (f, v)_0$

$$f = f_0 + \sum_{j=1}^n \partial_j f_j \quad \text{with} \quad f_j \in H^0(U) \ \ 1 \leq j \leq n,$$

we have

$$(f,\phi)_0 = (f_0,\phi)_0 + \sum_{j=1}^n (\partial_j f_j,\phi)_0 \;= (f_0,\phi)_0 - \sum_{j=1}^n (f_j,\partial_j\phi)_0$$

and

$$|(f,\phi)_0| \leq \|f_0\|_0\|\phi_0\|_0 + \sum_{j=1}^n \|f_j\|_0\|\partial_j\phi\|_0 \leq \left(\sum_{j=0}^n \|f_j\|_0^2\right)^{1/2}\|\phi\|_1.$$

Then $F(v) = (f,v)_0$ defines a bounded linear functional on $H_0^1(U)$ repeating the already known fact that the space

$$H^{-1}(U) = \left\{f = f_0 + \sum_{j=1}^n \partial_j f_j \quad with \quad f_j \in H^0(U)\ 1 \leq j \leq n\right\}$$

is a realization for the dual of $H_0^1(U)$. We recall that since the test functions $C_c^\infty(U)$ are dense in $H_0^1(U)$, it follows that the dual of $H_0^1(U)$, is continuously embedded in the dual of the test functions, that is $H^{-1}(U)$ is continuously embedded in the space of distributions, $D'(U)$. Then

$$B_0[u,\phi] = F(\phi) \quad for\ all \qquad \phi \in C_c^\infty(U)$$

is equivalent to

$$(L_0[u] - F,\phi)_0 = 0 \qquad for\ all \qquad \phi \in C_c^\infty(U)$$

which is just the statement that $L_0[u] - F = 0$ in the sense of distributions. Then the weak solution of $(1.1)$ can be said to satisfy the partial differential equation in sense of distributions, at least. It may follow from arguments based on additional information about the ingredients of the problem that the equation is satisfied in some stronger sense, possibly even in the classical pointwise sense. In the abscence of such arguments, we can only conclude the equation is satisfied in the distributional sense. Since the weak solution is in $H_0^1(U)$, we know that $u$ vanishes on the boundary of $U$ in the trace sense. This is also weaker than being pointwise equal to zero (except in the case that $U$ is 1-dimensional). Roughly speaking, to say that $u = 0$ on the boundary in the trace sense means that $u$ has average value zero over any neighborhood of a point on the boundary of $U$.

Now consider the Dirichlet problem with homogeneous boundary data for the modified operator,

$$L[u(x)] = L_0[u(x)] + \sum_{j=1}^n b_j(x)\,\partial_j u(x) + c(x)\,u(x).$$

The associated bilinear form

$$B[u,v] = B_0[u,v] + \int_U \left[\sum_{j=1}^n b_j(x)\,\partial_j u(x) + c(x)\,u(x)\right] v(x)\,dx,$$

is not,symmetric and in general, is not V-elliptic but is only V-H coercive, which is to say

$$|B[u,u]| \geq \alpha\|u\|_1^2 - \mu\|u\|_0^2 \qquad \forall u \in H_0^1(U).$$

This estimate is not sufficient to imply that L is an isomorphism from $H_0^1(U)$ *onto* $H^{-1}(U)$, which would imply existence of a weak solution to the Dirichlet problem. However, the equation

$$B[u,v] = F(v) \quad \forall v \in H_0^1(U)$$

is equivalent to

$$B_\lambda[u,v] = F(v) + \lambda(u,v)_0 \quad \forall v \in H_0^1(U),$$

and the bilinear form $B_\lambda[u,v]$ is V-elliptic for $\lambda > \mu$. Then $L_\lambda$ is an isomorphism for $\lambda > \mu$ and for every $f \in H^{-1}(U)$ there exists a unique $u \in H_0^1(U)$ satisfying $u = L_\lambda^{-1}(f + \lambda u)$, or, equivalently, satisfying $(I - K)u = L_\lambda^{-1} f$ where $K = \lambda L_\lambda^{-1}$ is compact so that $(I - K)$ is a Fredholm type operator. Then, we can apply the classical Fredholm alternative theorem to the weak Dirichlet problem,

given $f \in H^{-1}(U)$, find $u \in H_0^1(U)$ such that

$$B[u,v] = F(v) = (f,v)_0 \quad \forall v \in H_0^1(U) \qquad (1.2)$$

in order to assert that exactly one of the following alternatives must hold,

i) $B^*[u,v] = 0 \quad \forall v \in H_0^1(U)$ if and only if $u = 0$ and there exists a unique $u \in H_0^1(U)$ which solves $(1.2)$

ii) $$N^* = \left\{u \in H_0^1(U) : B^*[u,v] = 0 \quad \forall v \in H_0^1(U)\right\} = span[w_1, \ldots, w_p] \quad p < \infty,$$

$$N = \left\{u \in H_0^1(U) : B[u,v] = 0 \quad \forall v \in H_0^1(U)\right\} = span[z_1, \ldots, z_p] \quad p < \infty,$$

and no solution for (1.2) exists unless $F(w_j) = 0 \quad 1 \le j \le p$. In that case

$$B\left[u_0 + \sum_{j=1}^p c_j z_j, v\right] = F(v) \quad \forall v \in H_0^1(U)$$

for all choices of $c_j's$ and any choice of $u_0$ satisfying (1.2).

Here $\qquad B^*[u,v] = B[v,u] \qquad u,v \in H_0^1(U)$

denotes the bilinear form associated with the adjoint BVP

$$\begin{aligned} L^*[v(x)] &= g(x) \qquad x \in U \\ v(x) &= 0 \qquad x \in \partial U = \Gamma \end{aligned}$$

where $\quad (L\phi,\psi)_0 = (\phi, L^*\psi)_0 \qquad \forall \phi,\psi \in C_c^\infty(U),$

i.e., $\quad L^*\psi(x) = L_0\psi(x) - \sum_{j=1}^n \partial_j(b_j(x)\psi(x)) + c(x)\psi(x).$

Note that since the matrix A is symmetric, $L_0^* = L_0$.

Now consider the Dirichlet problem with nonzero data on the boundary,

$$\begin{aligned} L[u(x)] &= f(x) \qquad x \in U \qquad\qquad (1.3) \\ u(x) &= g(x) \qquad x \in \Gamma \end{aligned}$$

where

$$f \in H^{-1}(U), \qquad g \in H^{1/2}(\Gamma)$$

and we are seeking a weak solution in $H^1(U)$. Recall that if $\Gamma$ is $C^1$ then $g \in H^{1/2}(\Gamma)$ can be extended to $U$ as a function in $H^1(U)$ by making use of the continuous right inverse of the trace map, $T_0$,

$$H^{1/2}(\Gamma) \ni g \; - - -A - - \rightarrow H^{1/2}(R^{n-1})^M$$

$$\downarrow \; K = right\ inv\ for\ T_0$$
$$H^1(U) \ni G \leftarrow --B--- H^{1/2}(R^n_+)$$

i.e., $g \in H^{1/2}(\Gamma)$ is extended to $U$ as an $H^1(U)$ function, and $T_0G = g.$

Let $w = u - G,$ and note that since both $u\ and\ G$ belong to $H^1(U),\ w \in H^1(U).$ In addition,

$$T_0w = T_0u - T_0G = 0,\ so\ w \in \ker T_0 = H^1_0(U)$$

and

$$L[w(x)] = L[u(x)] - L[G(x)] = f - L[G(x)].$$

But $\quad |(L[G],v)_0| = |B[G,v]| \leq \beta \|G\|_1 \|v\|_1$

which implies $L[G(x)] \in H^{-1}(U)\ and\ f - L[G(x)] = F \in H^{-1}(U),$

and $\ w \in H^1_0(U)\ solves\ \ B[w,v] = F(v) \quad \forall v \in H^1_0(U) \qquad (1.4).$

Then we say that $u \in H^1(U)$ is a weak solution of (1.3) if

i) $u = w + G\ \ for\ \ G \in H^1(U)\ such\ that\ T_0G = g$

ii) $w \in H^1_0(U)\ solves\ (1.4)\ where\ F(v) = (f,v)_0 - B[G,v]$

The Fredholm alternative applies to (1.4) when $B$ is not coercive.

## 2. The Neumann Problem

Consider the following weak boundary value problem of the previous section but with the solution space changed from $V = H^1_0(U)$ to $H^1(U)$; i.e., ,

find $u \in H^1(U) \quad satisfying \quad B_0[u,v] = F(v) \ \ \forall v \in H^1(U), \qquad (2.1)$

where $B_0$ denotes the symmetric bilinear form of the previous section, and $F$ is a continuous linear functional on $H^1(U);$ note that this is not the same as saying $F \in H^{-1}(U).$ We have that

$$B_0[\phi,\psi] = (L_0[\phi],\psi)_0 \quad \text{for} \quad \phi \in C^\infty(\bar{U}),\ \psi \in C^\infty_c(U)$$

and since $C^\infty(\bar{U})$ is dense in $H^1(U)$ and $B_0$ is continuous, we can extend by continuity to get

$$B_0[u,\psi] = (L_0[u],\psi)_0 \quad \text{for} \quad u \in H^1(U),\ \psi \in C^\infty_c(U)$$

Now if u solves (2.1), then

$$(L_0[u],\psi)_0 = F(\psi) = (f,\psi)_0 \quad \text{for all} \quad \psi \in C^\infty_c(U)$$

which is to say $\quad L_0[u] = f\ $ in the sense of $D'(U).$ Notice that we have continuous, dense injections

$$C^\infty_c(U) \subset C^\infty(\bar{U}) \quad and \quad C^\infty(\bar{U}) \subset H^1(U)$$

implying that the following inclusions are one to one,

$$(C^\infty(\bar{U}))' \subset D'(U) \quad and \quad (H^1(U))' \subset (C^\infty(\bar{U}))'.$$

However, since the inclusion $C^\infty_c(U) \subset H^1(U),$ does not have a dense image, we cannot conclude that the inclusion of the bounded linear functionals on $H^1(U)$ into the space of distributions is one to one. In fact, there are bounded linear functionals on $H^1(U)$ that are not zero but vanish when acting on any $C^\infty$ function with compact support. Of course, these functionals would then have to be "concentrated on the boundary of U".

Now for $\phi, \psi \in C^\infty(\bar{U})$,

$$(L_0[\phi], \psi)_0 = B_0[\phi, \psi] - \langle T_0\psi, T_1\phi\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$$

and since $T_1$ is continuous from $H^1(U)$ *to* $H^{-1/2}(\Gamma)$, we can extend by continuity to get

$$B_0[u,\psi] = (L_0[u], \psi)_0 + \langle T_0\psi, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}. \qquad \forall\psi \in C^\infty(\bar{U}) \quad \forall u \in H^1(U).$$

If u solves (2.1)

$$(f,\psi)_0 = (L_0[u], \psi)_0 + \langle T_0\psi, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \qquad \forall\psi \in C^\infty(\bar{U})$$

But we already know that if u solves (2.1) then $L_0[u] = f$ so this last equation reduces to

$$\langle T_0\psi, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = 0 \qquad \forall\psi \in C^\infty(\bar{U}).$$

Extending again by continuity, this becomes

$$\langle T_0 v, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = 0 \qquad \forall v \in H^1(U).$$

But $\quad T_0 : H^1(U) \to H^{1/2}(\Gamma)$ is onto so

$$\langle g, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = 0 \qquad \forall g \in H^{1/2}(\Gamma)$$

which is to say, $T_1 u = 0$ *in* $H^{-1/2}(\Gamma)$. It follows that any solution of (2.1) is a weak solution of the problem

$$\begin{aligned} L_0[u(x)] &= f(x) \qquad x \in U \\ \partial_N u(x) &= 0 \qquad x \in \Gamma \end{aligned} \qquad (2.2)$$

It is clear that $B_0$ is bounded on $H^1(U)$, *but*

$$B_0[u,u] \geq a_0 \|\nabla u\|_0^2 \qquad \forall u \in H^1(U)$$

does not imply that $B_0$ is coercive on $H^1(U)$ because the Poincare inequality does not hold in $H^1(U)$. In fact,

$$B_0[u,u] \geq a_0\|\nabla u\|_0^2 = a_0(\|u\|_1^2 - \|u\|_0^2) \qquad \forall u \in H^1(U),$$

so even the symmetric bilinear form $B_0$ is just V-H coercive on $V = H^1(U),\ H = H^0(U)$. Note that

$$B_0[u,u] = 0 \text{ if and only if } \nabla u = 0 \text{ in } H^0(U)$$

and $\nabla u = 0$ *in* $H^0(U)$ *if and only if* $u =$constant, assuming U is connected and $u \in H^1(U)$. Then

$$N^* = N = span[1]$$

and then (2.1) has no solution unless

$$F(1) = (f,1)_0 = \int_U f(x)\,dx = 0.$$

If this condition is satisfied then

$$B_0[u_0 + C, v] = F(v) \qquad \forall v \in H^1(U),$$

for all constants, C and any $u_0$ which solves (2.1).

In the more general case when $L_0$ and $B_0$ are replaced by $L$ and $B$ having lower order terms, the null spaces $N^*$ and $N$ may become more complicated to describe but the overall results are not significantly different from the more special case described here.

For $f \in H^0(U) \subset (H^1(U))'$ and $g \in H^{-1/2}(\Gamma)$ define

$$F(v) = (f, v)_0 + \langle T_0 v, g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \qquad \forall v \in H^1(U).$$

Then
$$|F(v)| \le \|f\|_0 \|v\|_0 + \|g\|_{H^{-1/2}(\Gamma)} \|T_0 v\|_{H^{1/2}(\Gamma)}$$
$$\le \|f\|_0 \|v\|_0 + \|g\|_{H^{-1/2}(\Gamma)} C\|v\|_1 \le C\|v\|_1$$

and it follows that F is a bounded linear functional on $H^1(U)$.

Now suppose $u \in H^1(U)$ satisfies

$$B[u, v] = F(v) = (f, v)_0 + \langle T_0 v, g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \qquad \forall v \in H^1(U) \qquad (2.3)$$

where $B$ is the usual bilinear form associated to an elliptic operator, $L$, involving lower order terms. Just as before, we can show that (2.3) implies $L[u] = f$ in the sense of distributions. Now for $u \in H^1(U)$ satisfying (2.3),

$$B_0[u, v] = (L_0[u], v)_0 + \langle T_0 v, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}. \qquad \forall v \in H^1(U)$$
$$= (f, v)_0 + \langle T_0 v, g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$$

and since $L[u] = f,$ it follows that

$$\langle T_0 v, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \langle T_0 v, g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \qquad \forall v \in H^1(U).$$

Since $T_0 : H^1(U) \to H^{1/2}(\Gamma)$ is a surjection, it follows that

$$\langle h, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \langle h, g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \qquad \forall h \in H^{1/2}(\Gamma)$$

i.e., $\qquad T_1 u = g.$

Then we have shown that a solution of (2.3) is a weak solution of the Neumann boundary value problem,

$$L[u(x)] = f(x) \quad x \in U$$
$$T_1 u = g(x) \qquad x \in \Gamma,$$

provided that U has a $C^1$ boundary.

## 3. A Mixed Boundary Value Problem

The setting for the Dirichlet problem was the Hilbert space $V = H_0^1(U)$ which was contained in the pivot space $H = H^0(U)$. The pivot space was identified with its dual which then became a subspace of $V' = H^{-1}(U),$ the dual of $V.$ In the Neumann problem we took the solution Hilbert spaceto be $V = H^1(U)$ with the pivot space again equal to $H^0(U)$. Now we are going to consider a solution space $V$ which is contained in $H^1(U)$ but contains $H_0^1(U),$ so that $H_0^1(U) \subset V \subset H^1(U)$ and this solution space lies "between" the solution spaces of the previous two examples.

Suppose that $\Gamma$, the boundary of the open set $U$, is composed of complementary parts, $\Gamma_1$ and $\Gamma_2$, and let $C_*^\infty(U) = \{\phi \in C^\infty(U) : \phi = 0$ in a neighborhood of $\Gamma_1\}$. Clearly $C_0^\infty(U) \subset C_*^\infty(U) \subset C^\infty(U)$ and if we complete each of these spaces in the $H^1 = norm$, then the completions satisfy $H_0^1(U) \subset V \subset H^1(U)$, where $V$ denotes the completion of $C_*^\infty(U)$. The dual spaces are then related as follows, $H \subset (H^1(U))' \subset V' \subset H^{-1}(U)$. The dual space $V'$ contains linear functionals which are continuous on $V$ and are not zero, but which vanish for every test function in $C_0^\infty(U)$. These would be functionals that are concentrated on $\Gamma_2$.

Now we consider the weak boundary value problem

$$\text{for } F \in V', \quad find \ \ u \in V \ \ such \ that \ \ B_0[u,v] = F(v) \quad \forall v \in V \qquad (3.1)$$

As in the previous two examples, we find that

$$L_0[u] = f \ \ in \ D'(U) \quad where \quad F(v) = (f,v)_0 \quad \forall v \in V.$$

Now for $\phi, \psi \in C^\infty(\bar{U})$,

$$(L_0[\phi],\psi)_0 = B_0[\phi,\psi] - \langle T_0\psi, T_1\phi\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$$

and since $T_1$ is continuous from $V \subset H^1(U)$ $to$ $H^{-1/2}(\Gamma)$, we can extend by continuity to get

$$B_0[u,\psi] = (L_0[u],\psi)_0 + \langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}. \qquad \forall \psi \in C^\infty(\bar{U}) \quad \forall u \in V.$$

Now we can writ

$$\langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma_1)\times H^{-1/2}(\Gamma_1)} + \langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)}$$

and if we choose $\psi \in C_*^\infty(U)$, $then$

$$\langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma_1)\times H^{-1/2}(\Gamma_1)} = 0 \quad and \qquad \langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)}$$

If u solves (3.1) then

$$(f,\psi)_0 = (L_0[u],\psi)_0 + \langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} \qquad \forall \psi \in C_*^\infty(U)$$

But we already know that if u solves (2.1) then $L_0[u] = f$ so this last equation reduces to

$$\langle T_0\psi, T_1u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} = 0 \qquad \forall \psi \in C_*^\infty(U)$$

Extending again by continuity, this becomes

$$\langle T_0v, T_1u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} = 0 \qquad \forall v \in V$$

But $\quad T_0 : V \subset H^1(U) \to H^{1/2}(\Gamma) = H^{1/2}(\Gamma_1) \oplus H^{1/2}(\Gamma_2)$ is onto so

$$\langle g, T_1u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} = 0 \qquad \forall g \in H^{1/2}(\Gamma_2)$$

which is to say, $T_1u = 0$ $in$ $H^{-1/2}(\Gamma_2)$. It follows that any solution of (3.1) is a weak solution of the problem

$$\begin{aligned} L_0[u(x)] &= f(x) \qquad x \in U \\ u(x) &= 0 \qquad\quad x \in \Gamma_1 \qquad\qquad (3.2) \end{aligned}$$

$$\partial_N u(x) = 0 \qquad x \in \Gamma_2$$

It is clear that $B_0$ is bounded on $V,$ and if U is connected and $u \in V$, then

$$B_0[u,u] = 0 \textit{ if and only if } \nabla u = 0 \textit{ in } H^0(U).$$

Now $\nabla u = 0$ *in* $H^0(U)$ *if and only if* $u =$constant, and for $u \in V$, this means $u = 0$ since $u = 0$ *in* neighborhood of $\Gamma_1$. This allows us to use the Poincare norm on $V$, in which case

$$B_0[u,u] \geq a_0 \|\nabla u\|_0^2 \qquad \forall u \in H^1(U)$$

does imply that $B_0$ is coercive on V. Then (3.1) has a unique solution in V for every F in V'.

Now for $g_1 \in H^{1/2}(\Gamma_1),$ *and* $g_2 \in H^{-1/2}(\Gamma_2)$ suppose $u \in H^1(U)$ satisfies

$$B[u,v] = F(v) = (f,v)_0 - B[G_1,v] + \langle T_0 v, g_2\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} \qquad \forall v \in V \qquad (3.3)$$

where $G_1 \in H^1(U)$ is such that $T_0 G_1 = g_1$ and $B$ is the usual bilinear form associated to an elliptic operator, $L,$ involving lower order terms. Just as in the first example, we can show that (3.3) implies $L[u] = f - L[G_1]$ in the sense of distributions. Now for $u \in V$ satisfying (3.3),

$$\begin{aligned} B_0[u,v] &= (L_0[u],v)_0 + \langle T_0 v, T_1 u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}. \qquad \forall v \in V \\ &= (L_0[u],v)_0 + \langle T_0 v, T_1 u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)}. \qquad \forall v \in V \\ &= (f,v)_0 - B[G_1,v] + \langle T_0 v, g_2\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} \qquad \forall v \in V \end{aligned}$$

and since $L[u] = f - L[G_1],$ it follows that

$$\langle T_0 v, T_1 u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} = \langle T_0 v, g_2\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} \qquad \forall v \in V..$$

Since $T_0 : V \subset H^1(U) \to H^{1/2}(\Gamma) = H^{1/2}(\Gamma_1) \oplus H^{1/2}(\Gamma_2)$ is a surjection, it follows that

$$\langle h, T_1 u\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} = \langle h, g_2\rangle_{H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)} \qquad \forall h \in H^{1/2}(\Gamma_2)$$

i.e., $\qquad T_1 u = g_2.$

Then we have shown that a solution of (2.3) is a weak solution of the Neumann boundary value problem,

$$\begin{aligned} L[u(x)] &= f(x) \quad & x \in U \\ u &= g_1(x) \quad & x \in \Gamma_1 \\ T_1 u &= g_2(x) \quad & x \in \Gamma_2, \end{aligned}$$

provided that U has a $C^1$ boundary.

## 4. A Robin Type Boundary Value Problem

We have seen in the previous examples that by varying the solution space from $V = H_0^1(U)$ at one extreme, to $V = H^1(U)$ at the other extreme, we could change the associated weak boundary value problem,

$$\text{find } u \in V \textit{ such that } B[u,v] = F(v) \quad \forall v \in V,$$

from a Dirichlet problem when $V = H_0^1(U)$, to a mixed problem when $H_0^1(U) \subset V \subset H^1(U)$, and finally when $V = H^1(U),$ to a Neumann problem. We have also seen that by modifying the functional F, we could achieve inhomogeneous Neumann conditions. Now we are going to modify the bilinear form $B[u,v]$ and find the corresponding modification of the associated weak boundary value problem.

For $u,v \in V = H^1(U)$, define

$$R[u,v] = B[u,v] + \int_\Gamma p(x)\,T_0u(x)\,T_0v(x)\,dS(x),$$

where B denotes the previously defined bilinear form on $V = H^1(U)$ and p denotes a function in $L^\infty(\Gamma)$. Clearly, since $T_0$ is continuous from $V$ into $H^{1/2}(\Gamma)$,

$$\begin{aligned}\left|\int_\Gamma p(x)\,T_0u(x)\,T_0v(x)\,dS(x)\right| &\le \|p\|_{L^\infty(\Gamma)}\,\|T_0u\|_{H^{1/2}(\Gamma)}\,\|T_0v\|_{H^{1/2}(\Gamma)}\\ &\le C\|u\|_V\,\|v\|_V\end{aligned}$$

which implies that $R[u,v]$ is a bounded bilinear form on $V\times V$. In addition,

$$\begin{aligned}R[u,u] = B[u,u] + \int_\Gamma p(x)\,T_0u(x)^2\,dS(x) &\ge C_0\|u\|_V^2 - C_1\|u\|_H^2 - \|p\|_{L^\infty(\Gamma)}\,\|T_0u\|_{H^{1/2}(\Gamma)}^2\\ &\ge C_0\|u\|_V^2 - C_2\|u\|_H^2\end{aligned}$$

so R is $V,H-coercive$. Then we can consider the weak boundary value problem,

find $u \in V$ *such that* $R[u,v] = F(v) \quad \forall v \in V, \qquad (4.1)$

where $\quad F(v) = (f,v)_0 + \langle T_0v, g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \qquad \forall v \in H^1(U).$

with $\quad f \in H^0(U) \subset (H^1(U))'$ *and* $g \in H^{-1/2}(\Gamma)$ given.

In order to interpret the meaning of (4.1), we note first that since

$$b[u,v] = \int_\Gamma p(x)\,T_0u(x)\,T_0v(x)\,dS(x)$$

is a bounded bilinear form on $V\times V$, it follows that for $u \in V$, *fixed*

$$H^{1/2}(\Gamma) \ni \mu \to; \int_\Gamma p(x)\,T_0[u]\,\mu(x)\,dS(x) \in R$$

defines a bounded linear functional on $H^{1/2}(\Gamma)$. Then the Riesz theorem implies the existence of an element $Pu \in H^{-1/2}(\Gamma)$ such that

$$\int_\Gamma p(x)\,T_0[u]\,\mu(x)\,dS(x) = \langle \mu, Pu\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \qquad \forall \mu \in H^{1/2}(\Gamma).$$

Then for $u,v \in V$,

$$b[u,v] = \langle T_0v, Pu\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}.$$

and we can now write,

$$\begin{aligned}R[u,v] &= B[u,v] + \langle T_0v, Pu\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}\\ &= (L[u],v)_0 + \langle T_0v, T_1u\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} + \langle T_0v, Pu\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}\\ &= (L[u],v)_0 + \langle T_0v, T_1u + Pu\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}.\end{aligned}$$

In the usual way, we show that if $u \in V$ satisfies (4.1), then

$$L[u] = f \qquad in\ D'(U)$$

and $T_1 u + Pu = g \quad in\ H^{-1/2}(\Gamma)$.

The interpretation of the abstract boundary condition must be $A\partial_N u + pu = g\ on\ \Gamma$.