The Dual of a Hilbert Space

Let V denote a real Hilbert space with inner product, $(u,v)_V$. Let V' denote the space of bounded linear functionals on V, equipped with the norm,

$$\|F\|_{V'} = \sup\{|F(v)| : \|v\|_V = 1\}.$$

Then for any $u \in V,\ \ F_u(v) := (u,v)_V$ defines a unique bounded linear functional

$$F_u \in V' \qquad \text{with} \qquad \|F_u\|_{V'} = \|u\|_V$$

Similarly, it follows from the Riesz theorem, that for any $F \in V'$ there is a unique $u_F \in V$ such that

$$F(v) = (u_F, v)_V \quad \forall v \in V.$$

Moreover,
$$\|F\|_{V'} = \|u_F\|_V$$

Evidently there exists an isometric isomorphism J from V onto V'

$$V \ni u \rightsquigarrow J(u) = F_u \in V'$$

In addition, J induces an inner product on V',

$$(F,G)_{V'} = (J^{-1}F, J^{-1}G)_V = (u_F, v_G)_V$$

One can define an abstract duality pairing on $V' \times V$ by

$$< F,v >_{V'\times V} = F(v) \text{ for } F \in V',\ \ v \in V$$

Then
$$(F,Jv)_{V'} = < F,v >_{V'\times V} = (J^{-1}F,v)_V \qquad \text{for } F \in V',\ \ v \in V$$

Now V' is an abstract dual space in the sense that we have no characterization for the elements of V' other than the assertion that they are continuous linear functionals on V. In order to have a concrete characterization for the elements in the dual space, we consider a concrete Hilbert space, W. Then W is called a *realization* for the abstract space, V' if there exists an isometry j of W onto V'; i.e.,

$$\begin{array}{ccccc} & & & J & \\ & & V & \longrightarrow & V' \\ K = j^{-1}\circ J & & & \searrow & \uparrow j \\ & & & & W \end{array}$$

Then for all f in W, $\langle f, v\rangle_{W\times V} = jf(v)$ for all v in V, and $\langle f, v\rangle_{W\times V}$ is the realization of the abstract duality pairing.

Examples

1. Take $W = V$. Then $j = J$ and $K = identity$. In this case the duality pairing is just the V inner product. This is what is meant when we say we "identify a Hilbert space with its dual". Although this is frequently done, it is not the only realization for the dual of a Hilbert space and in the treatment of problems in partial differential equations by variational methods, it is not the best choice of realization for the dual.

2. Consider the Hilbert space

$$V = \{u \in L^2 : (1+|z|^2)^{s/2}\hat{u}(z) \in L^2\} = H^s, \ s \geq 0,$$

with

$$(u,v)_V = \int_{R^n} (1+|z|^2)^s \hat{u}(z)\hat{v}(z)dz,$$

Then V is dense in a larger Hilbert space $H = H^0$ and we can use the space H, called the pivot space, to show that

$$W = H^{-s} = \{f \in S' : (1+|z|^2)^{-s/2}\hat{f}(z) \in L^2\}$$

is a realization for the dual of $V = H^s$.
For $f \in W$ and $v \in V$ define

$$jf(v) = \langle jf, v\rangle_{W\times V} = \int_{R^n} f(x)v(x)dx = (f,v)_0$$

Then it follows that

$$|jf(v)| = |\int_{R^n} f(x)v(x)dx| = |\int_{R^n} \hat{f}(z)\hat{v}(z)dz|$$

$$= \left|\int_{R^n} (1+|z|^2)^{-s/2}\hat{f}(z)(1+|z|^2)^{s/2}\hat{v}(z)dz\right|$$

$$\leq \left(\int_{R^n} (1+|z|^2)^{-s}|\hat{f}(z)|^2 dz\right)^{1/2} \|v\|_V$$

This proves that $jf \in V'$ for all $f \in W$; i.e., j maps W *into* V'.
To show that j maps W onto V', let G be an element of V' with $G = Ju_G$ for some $u_G \in V$. Then for all $v \in V$,

$$G(v) = \langle G, v\rangle_{V'\times V} = (u_G, v)_V$$

$$= \int_{R^n} (1+|z|^2)^s \hat{u}_G(z)\hat{v}(z)dz$$

Now let $$g = T_F^{-1}\left[(1+|z|^2)^s \hat{u}_G(z)\right] = T_F^{-1}[\hat{g}(z)],$$

so $$\langle G, v\rangle_{V'\times V} = \int_{R^n} g(x)v(x)\,dx = \langle jg, v\rangle_{V'\times V}$$

Here G denotes the element of V' and $\hat{g}$ denotes the function $(1+|z|^2)^s\hat{u}_G(z)$ which is the Fourier transform of g. Then the fact that $(1+|z|^2)^s\hat{u}_G(z) = \hat{g}(z)$ and $u_G \in V = H^s$ implies

$$(1+|z|^2)^{-s/2}\hat{g}(z) = (1+|z|^2)^{s/2}\hat{u}_G(z) \in L^2;$$

i.e., g belongs to $W = H^{-s}$. We have shown that $G(v) = (u_G, v)_V = jg(v)$, and since $jg = G$, every G in V' is the image under j of an element g in W which shows j is onto. If we define

$$\|g\|_W = \sup\{|jg(u)| : \|u\|_V = 1\},$$

then j is an isometry. Note that since $jg = Ju_G$, where

$$g = j^{-1}Ju_G = T_F^{-1}\left[(1+|z|^2)^s\hat{u}_G(z)\right]$$

then $$Ku_G = j^{-1}Ju_G = T_F^{-1}\left[(1+|z|^2)^s\hat{u}_G(z)\right]$$

$$= T_F^{-1}\left[(1+|z|^2)^s T_F(u_G)\right]$$

is the canonical isometry of V onto W. i.e.,

$$\begin{array}{ccc} & P_s & \\ S' & \longrightarrow & S' \\ T_F \uparrow & & \downarrow T_F^{-1} \\ V & \longrightarrow & W \\ & K & \end{array}$$

where P_s denotes the multiplication by $(1+|z|^2)^s$. Since

$$jf(v) = \int_{R^n} f(x)v(x)dx = \langle f, v\rangle_{W\times V}$$

the realization of the duality pairing is just the H^0 inner product, extended to $W\times V$. This may be interpreted to mean that the space $H = H^0(R^n)$ occupies a position precisely midway between the space H^s and its dual space, H^{-s}. We say that H^0 is the "*pivot space*" between H^s and H^{-s} . Functions in H^s may be viewed as being more regular than H^0 functions to precisely the same degree that H^0 functions are more regular than elements of H^{-s}.

3. Consider the Hilbert space $V = H_0^1(U)$ with

$$(u,v)_V = \int_U [u(x)v(x) + \nabla u \bullet \nabla v]dx.$$

Here we will use the pivot space $H = H^0(U)$ to show that a realization of the dual of V is given by

$$W = \{f \in D'(U) : f = f_0 + \sum_{j=1}^n \partial_j f_j,\ \ f_i \in H^0(U)\ \forall i\}$$

For $f \in W$, and $v \in V$ define

$$jf(v) = \int_U f(x)v(x)dx = (f,v)_0.$$

Then for any test function, ϕ

$$jf(\phi) = \int_U (f_0 + \sum_{j=1}^n \partial_j f)\phi(x)dx. = \int_U [f_0\phi - \sum_{j=1}^n f_j \partial_j \phi(x)]dx.$$

and

$$|jf(\phi)| \leq \|f_0\|_0 \|\phi\|_0 + \sum_{j=1}^n \|f_j\|_0 \|\partial_j \phi\|_0 \leq \left(\sum_{j=0}^n \|f_j\|_0^2\right)^{1/2} \|\phi\|_V \quad \forall \phi \in D(U)$$

But $D(U)$ is dense in V, hence this estimate extends to all of V, showing that $jf \in V'$ for all $f \in W$.
To show that j is onto V', let F be an element of V' with $F = Ju_F$ for some u_F in V. Then for any test function, ϕ

$$F(\phi) = (u_F, \phi)_V = \int_U [u_F(x)\phi(x) + \nabla u_F \bullet \nabla \phi]dx$$

$$= \int_U [(u_F(x) - \nabla^2 u_F)\phi(x)]dx$$

$$= \int_U f(x)\phi(x)dx \qquad \text{for} \qquad f(x) = (u_F(x) - \nabla^2 u_F).$$

Since $u_F \in V$, it follows that $f \in D'(U)$. In addition,

$$f(x) = u_F - \nabla^2 u_F = u_F - \sum_{j=1}^n \partial_j(\partial_j u_F)$$

and $u_F \in V$ implies $\partial_j u_F \in H^0(U)$ for each $j = 1,\ldots,n$. Thus we have shown,

$$f \in D'(U) \ \text{and} \ f(x) = u_F - \sum_{j=1}^n \partial_j(\partial_j u_F) \in W$$

Moreover, for all test functions ϕ,

$$F(\phi) = (u_F, \phi)_V = (f, \phi)_0 = jf(\phi)$$

Once again, since $D(U)$ is dense in V, the result extends to all of V; i.e., $F = jf$ for f in W given by $f = u_F - \nabla^2 u_F.,\ \ u_F \in V$.

Note that $jf = Ju_F$ implies

$$Ku_F = j^{-1}Ju_F = (1 - \nabla^2)u_F = f.$$

i.e., $K = 1 - \nabla^2$ is the canonical isometry of $V = H_0^1(U)$ onto its dual W which we denote by $H^{-1}(U)$. The realization of the duality pairing is

$$\langle f, v\rangle_{W\times V} = \int_U f(x)v(x)dx. = (f, v)_0;$$

showing that the duality pairing is just the extension of the $H^0(U)$ inner product to $W \times V$. That is to say that $H^0(U)$ acts as the *pivot space* between the spaces $V = H_0^1(U)$ and its dual, $H^{-1}(U)$,

$$H_0^1(U) \hookrightarrow H^0(U) \hookrightarrow H^{-1}(U).$$

As in the case of H^s and its dual, H^{-s}, the pivot space $H^0(U)$occupies a position that is precisely midway (in terms of regularity) between$H_0^1(U)$ and its dual, $H^{-1}(U)$. The functions in $H^0(U)$ are first derivatives of functions from $H_0^1(U)$, and elements of $H^{-1}(U)$ are first derivatives of functions from $H^0(U)$.

We have considered here some examples of the situation where $X \text{ and } Y$ are linear spaces where the notion of convergence is defined and

$X \hookrightarrow Y$ means that a) $X \subset Y$
and b) $x_n \to 0 \text{ in } X \text{ implies } i(x_n) \to 0 \text{ in } Y.$

In addition, $F \in X'$ means that $x_n \to 0 \text{ in } X \text{ implies } F(x_n) \to 0 \text{ in } R,$ and it is then evident that

$$X \hookrightarrow Y \text{ implies } Y' \hookrightarrow X';$$

$$\begin{array}{lll} i.e., \text{ if} & i : X \to Y & \text{is continuous,} \\ \text{then} & {}^t i : Y' \to X' & \text{is also continuous.} \end{array}$$

Note, however, that the transpose inclusion, ${}^t i$, need not be injective. In particular, if X is a closed subspace of Hilbert space, Y, consider

$$z_F \in Y \qquad with \qquad F = J(z_F) \in Y'.$$

The projection theorem implies $z_F = z_1 + z_2,\ with\ z_1 \in X\ and\ z_2 \perp X,$ so that ${}^t i(F) = J(z_1)$ which can be zero even if F is not zero.

However, if $i : X \to Y$ has a dense image, then ${}^t i : Y' \to X'$ is injective. To see this, suppose that $i(X)$ is dense in $Y,$ and ${}^t i(F) = 0.$ Then,

$$\left\langle x,\ {}^t i(F) \right\rangle = 0 \ \ for\ all\ x \in X.$$

But

$$\left\langle x,\ {}^t i(F) \right\rangle = \langle i(x), F \rangle$$

and since the elements $i(x)$ are dense in Y, and F is continuous on Y, F must be zero whenever ${}^t i(F)$ is zero.

It follows from this discussion that the dual space any function space in which the test functions are densely included (e.g. $H_0^m(U), H^m(R^n)$) can be identified with a subspace of the space of distributions. That is, if the inclusion, $C_0^\infty(U) \hookrightarrow X$ has a dense image, then $X' \hookrightarrow D'(U)$ is a continuous injection. For example, $H^{-1}(U)$ is a subspace of the distributions, which means that equality in $H^{-1}(U)$ can be interpreted to mean equality in the sense of distributions. On the other hand, the dual of $H^1(U)$ is not a subspace of the distributions since the test functions are not dense in $H^1(U)$. In fact, we have already seen that the orthogonal complement of $H_0^1(U)\ in\ H^1(U)$ is the subspace

$$N = \{z \in H^1(U) : z - \nabla^2 z \in H^0(U), \text{ and } z - \nabla^2 z = 0\ \}$$

and it is clear that N is the null space of the duality mapping

$$J : H^1(U) = H_0^1(U) \oplus N \to H^{-1}(U)\ .$$