

## 5. Similarity Solutions for PDE's

For linear partial differential equations there are various techniques for reducing the pde to an ode (or at least a pde in a smaller number of independent variables). These include various integral transforms and eigenfunction expansions. Such techniques are much less prevalent in dealing with nonlinear pde's. However, there is an approach which identifies equations for which the solution depends on certain groupings of the independent variables rather than depending on each of the independent variables separately. We will illustrate this technique first for a linear pde. Consider the heat equation

$$\partial_t u(x, t) - D \partial_{xx} u(x, t) = 0 \quad (5.1)$$

and introduce the dilation transformation

$$z = \varepsilon^a x, \quad s = \varepsilon^b t, \quad v(z, s) = \varepsilon^c u(\varepsilon^{-a} z, \varepsilon^{-b} s) \quad (5.2)$$

Then  $\partial_t u(x, t) = \varepsilon^{-c} \partial_s v(z, s) s'(t) = \varepsilon^{b-c} \partial_s v(z, s)$ , etc and so (5.1) becomes

$$\varepsilon^{b-c} \partial_s v(z, s) - D \varepsilon^{2a-c} \partial_{zz} v(z, s) = 0.$$

Now if  $b - c = 2a - c$ , (i.e.,  $b = 2a$ ) then the equation (5.1) is invariant under this transformation; i.e., if  $u(x, t)$  solves the heat equation in the variables  $x, t$  then for  $z, s, v(z, s)$  given by (5.2),  $v(z, s)$  solves the heat equation in the variables  $z, s$ . Note that

$$v s^{-c/b} = (\varepsilon^c u)(\varepsilon^b t)^{-c/b} = u t^{-c/b}$$

and

$$\frac{z}{s^{a/b}} = \frac{\varepsilon^a x}{(\varepsilon^b t)^{a/b}} = \frac{x}{t^{a/b}}.$$

i.e., both groupings of variables are invariant under the transformation (5.2) for all choices of  $a, b, c$ . This suggests that we look for a solution for (5.1) that is of the form

$$u = t^{c/b} y(\xi) \quad \text{for} \quad \xi = \frac{x}{t^{a/b}} = \frac{x}{\sqrt{t}} \quad \text{since} \quad a/b = \frac{1}{2}. \quad (5.3)$$

i.e.,

$$\begin{aligned} \partial_t u &= \frac{c}{2a} t^{c/2a-1} y(\xi) + t^{c/2a} y'(\xi) \partial_t \xi \\ &= \frac{c}{2a} t^{c/2a-1} y(\xi) + t^{c/2a} y'(\xi) \left( -\frac{1}{2} \frac{x}{\sqrt{t^3}} \right) \\ &= t^{c/2a-1} \left( \frac{c}{2a} y(\xi) - \frac{\xi}{2} y'(\xi) \right) \end{aligned}$$

and

$$\partial_x u = t^{c/2a} y'(\xi) \partial_x \xi = t^{c/2a-1/2} y'(\xi)$$

$$\partial_{xx} u = t^{c/2a-1/2} y''(\xi) \partial_x \xi = t^{c/2a-1} y''(\xi).$$

Then (5.1) becomes

$$t^{c/2a-1} \left[ D y''(\xi) - \frac{c}{2a} y(\xi) + \frac{\xi}{2} y'(\xi) \right] = 0; \quad (5.4)$$

i.e., the pde has been reduced to an ode. If (5.1) is satisfied for  $x > 0$ ,  $t > 0$  and if, in addition,  $u(x,t)$  satisfies

$$u(x,0) = 0, \quad x > 0, \quad u(x,t) \rightarrow 0, \quad x \rightarrow \infty, \quad \partial_x u(0,t) = Q, \quad t > 0,$$

then since  $y(\xi) = t^{-c/2a} u(x,t)$  and  $\xi = \frac{x}{\sqrt{t}}$ , it follows that

$$\xi \rightarrow \infty \text{ if } x \rightarrow \infty, \text{ or } t \rightarrow 0 \quad \text{and} \quad \xi = 0 \text{ if } x = 0$$

hence the first two side conditions reduce to  $y(\infty) = 0$ . Now  $\partial_x u(0,t) = t^{c/2a-1/2} y'(0)$  and this can equal the constant  $Q$  if and only if  $c = a$ . Then the initial boundary value problem for  $u(x,t)$  reduces to the following problem for  $y(\xi)$

$$D y''(\xi) - \frac{1}{2} y(\xi) + \frac{\xi}{2} y'(\xi) = 0, \quad y'(0) = Q, \quad y(\xi) \rightarrow 0 \text{ as } \xi \rightarrow \infty \quad (5.5)$$

On the other hand, if the boundary condition at  $x = 0$  is  $u(0,t) = u_0$ , then  $u(0,t) = t^{c/2a} y(0)$  can equal the constant  $u_0$  if and only if  $c = 0$ . In this case the initial boundary value problem for  $u(x,t)$  reduces to

$$D y''(\xi) + \frac{\xi}{2} y'(\xi) = 0, \quad y(0) = u_0, \quad y(\xi) \rightarrow 0 \text{ as } \xi \rightarrow \infty \quad (5.6).$$

We can integrate this once to obtain

$$y'(\xi) = C_1 e^{-\xi^2/4D},$$

and  $y(\xi) = C_1 \int_0^\xi e^{-\lambda^2/4D} d\lambda + C_2$

$$= C_3 \operatorname{erf}\left(\frac{\xi}{\sqrt{4D}}\right) + C_2$$

where  $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} d\eta$ .

Then the boundary conditions lead to

$$y(\xi) = u_0 - u_0 \operatorname{erf}\left(\frac{\xi}{\sqrt{4D}}\right),$$

and

$$u(x, t) = u_0 - u_0 \operatorname{erf}\left(\frac{x}{\sqrt{4Dt}}\right) = u_0 \operatorname{erfc}\left(\frac{x}{\sqrt{4Dt}}\right).$$

where  $\operatorname{erfc}(s) = 1 - \operatorname{erf}(s)$ . The solution of the equation (5.5) is given by

$$y(\xi) = C_1 \xi + C_2 \left[ 2\sqrt{\pi D} e^{-\xi^2/4D} + \pi \xi \operatorname{erf}\left(\frac{\xi}{\sqrt{4D}}\right) \right]$$

and since  $y'(\xi) = C_1 + C_2 \left[ \pi \operatorname{erf}\left(\frac{\xi}{\sqrt{4D}}\right) \right]$

the auxiliary conditions are satisfied for

$$\begin{aligned} y(\xi) &= Q\xi - \frac{Q}{\pi} \left[ 2\sqrt{\pi D} e^{-\xi^2/4D} + \pi \xi \operatorname{erf}\left(\frac{\xi}{\sqrt{4D}}\right) \right] \\ &= Q\xi \operatorname{erfc}\left(\frac{\xi}{\sqrt{4D}}\right) - 2Q\sqrt{\frac{D}{\pi}} e^{-\xi^2/4D}. \end{aligned}$$

Of course these problems could have been solved by integral transform methods as well, and in fact the integral transform is able to deal with much more general boundary conditions than the ones treated here. On the other hand, consider the example

$$\begin{aligned} \partial_t u(x, t) - \partial_x [u \partial_x u(x, t)] &= 0, & x \in R, t > 0, \\ u(x, 0) &= \delta(x), \\ u(x, t) &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty \end{aligned} \tag{5.7}$$

and  $\int_R u(x, t) dx = 1, \text{ for } t > 0. \tag{5.8}$

The transformation (5.2) reduces the pde to

$$\varepsilon^{b-c} \partial_s v - \varepsilon^{2a} \partial_{zz} \left( \frac{1}{2} \varepsilon^{-2c} v^2 \right) = 0,$$

and the equation is invariant under the transformation if  $b - c = 2(a - c)$ , i.e.,  $c = 2a - b$ . Then if we let

$$u(x, t) = t^{(2a-b)/b} y(\xi), \quad \xi = \frac{x}{t^{a/b}},$$

the condition (5.8) becomes

$$\int_R u(x, t) dx = t^{2(a/b)-1} \int_R y\left(\frac{x}{t^{a/b}}\right) t^{a/b} d\xi = t^{3(a/b)-1} \int_R y(\xi) d\xi = 1$$

This condition can hold for all  $t > 0$  if  $3(a/b) = 1$ ; i.e.,  $3a = b$  and then

$$u(x, t) = t^{-1/3} y(\xi), \quad \xi = \frac{x}{\sqrt[3]{t}},$$

so that

$$\partial_t u = -\frac{1}{3} t^{-4/3} y(\xi) + t^{-1/3} y'(\xi) \partial_t \xi = -\frac{1}{3} t^{-4/3} [y(\xi) + \xi y'(\xi)]$$

$$\partial_{xx} \left( \frac{1}{2} u^2 \right) = \frac{d^2}{d\xi^2} \left( \frac{1}{2} t^{-2/3} y^2 \right) (\partial_x \xi)^2 = \frac{1}{2} t^{-2/3} (2y y'' + 2y') t^{-2/3}$$

and

$$-\frac{1}{3} t^{-4/3} [y(\xi) + \xi y'(\xi)] - \frac{1}{2} t^{-4/3} (2y y'' + 2y') = 0;$$

i.e.,

$$y y'' + y' + \frac{1}{3} [y(\xi) + \xi y'(\xi)] = 0$$

or

$$3(y(\xi) y'(\xi))' + (\xi y(\xi))' = 0.$$

Then  $3y(\xi) y'(\xi) + \xi y(\xi) = C_0$ ,

and since we want the solution to be an even function of  $\xi$ , we impose the condition  $y'(0) = 0$ , which leads to  $C_0 = 0$ . Then  $3y'(\xi) + \xi = 0$ , or  $y(\xi) = -\frac{1}{6} \xi^2 + C_1$ . If we write this solution in the form

$$y(\xi) = \frac{1}{6} (A^2 - \xi^2)$$

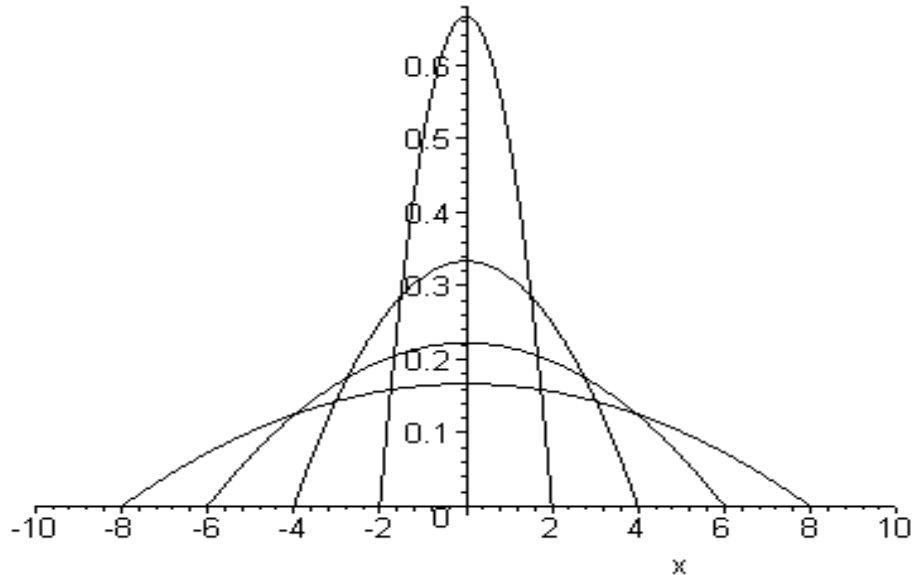
then in order to have  $y(\xi) \rightarrow 0$  as  $\xi^2 \rightarrow \infty$ , we must have

$$y(\xi) = \begin{cases} \frac{1}{6} (A^2 - \xi^2) & \text{if } \xi^2 < A^2 \\ 0 & \text{if } \xi^2 > A^2 \end{cases}$$

Then this is a weak solution since the derivative is discontinuous at  $\xi^2 = A^2$ . The condition (5.8) requires that  $A = \sqrt[3]{\frac{9}{2}}$  and then

$$u(x,t) = t^{-1/3} y(\xi) = \left\{ \begin{array}{ll} \frac{1}{6} t^{-1/3} \left( A^2 - \left( \frac{x}{\sqrt[3]{t}} \right)^2 \right) & \text{if } |x| < A t^{1/3} \\ 0 & \text{if } |x| > A t^{1/3} \end{array} \right\}$$

The solution is plotted versus x for several times



Note that unlike solutions to linear parabolic equations, solutions to nonlinear equations may have finite speed of propagation.

### **Burger's Equation**

We will consider the effect of the transformation (5.2) on the Burger's equation

$$\partial_t u(x,t) + u(x,t) \partial_x u(x,t) = 0.$$

Under the transformation (5.2), this becomes

$$\varepsilon^{b-c} \partial_s v(z,s) + \varepsilon^{a-2c} v(z,s) \partial_z v(z,s) = 0,$$

and the equation is invariant if  $c = a - b$ . Then we look for a solution of the form (5.3)

$$u = t^{a/b-1} y(\xi), \quad \xi = \frac{x}{t^{a/b}},$$

or, letting  $a/b - 1 = m$ , so that m is the free parameter,

$$u = t^m y(\xi), \quad \xi = \frac{x}{t^{m+1}}.$$

Then 
$$\partial_t u = m t^{m-1} y(\xi) + t^m y'(\xi) \left( -(m+1) \frac{x}{t^{m+2}} \right)$$

$$= t^{m-1} [m y(\xi) - (m+1) \xi y'(\xi)],$$

$$\partial_x u = t^m y'(\xi) (t^{-m-1}),$$

and 
$$\partial_t u + u \partial_x u = t^{m-1} [m y(\xi) - (m+1) \xi y'(\xi) + y(\xi) y'(\xi)] = 0.$$

For  $m = 0$  this reduces to

$$y(\xi) y'(\xi) - \xi y'(\xi) = 0, \quad \text{or} \quad y(\xi) = \xi$$

$$u(x, t) = \frac{x}{t}.$$

A more interesting example is provided by the KdV equation

$$\partial_t u + 6u \partial_x u + \partial_{xxx} u = 0$$

which is transformed by (5.2) into

$$\varepsilon^{b-c} \partial_s v + 6\varepsilon^{a-2c} v \partial_z v + \varepsilon^{3a-c} \partial_{zzz} v = 0.$$

For invariance, we must have

$$b - c = a - 2c = 3a - c,$$

or

$$c = a - b, \quad 3a = b;$$

i.e.,

$$a = \frac{b}{3} \quad \text{and} \quad c = -\frac{2}{3}b, \quad b = \text{free parameter.}$$

Looking for a solution of the form

$$u = (3t)^{-2/3} y(\xi) \quad \xi = \frac{x}{(3t)^{1/3}},$$

leads to

$$\partial_t u + 6u \partial_x u + \partial_{xxx} u = -(3t)^{-5/3} [2y(\xi) + \xi y'(\xi) - 6y(\xi) y'(\xi) - y'''(\xi)] = 0$$

i.e., 
$$y'''(\xi) + [6y(\xi) + \xi] y'(\xi) - 2y(\xi) = 0.$$

This ordinary differential equation can be shown to have the so called **Painlevé property**, meaning that it does not have a **movable singular point**. A movable singular point is a point where the solution becomes singular at a point whose location depends on the arbitrary constants of integration. The equation  $y'(\xi) = y(\xi)^2$ , has the solution

$$y(\xi) = \frac{1}{C - \xi}$$

which has a singular point whose location depends on the arbitrary constant of integration, C. Then this equation does not have the Painleve property. There is a conjecture, as yet not proved, that pde's that reduce under transformations of the form (5.2) to ode's having the Painleve property are equations that admit soliton solutions and are solvable by the inverse scattering transform.

## 6. Darcy's Law and the Richard's Equation

In deriving a mathematical model for flow in a porous medium, we begin with a conservation statement expressed in terms of the state variables

$\theta(x, t)$  = moisture content (percentage of available pore space at position x that is filled with fluid at time t)

$Q(x, t)$  = fluid flow rate at x, t

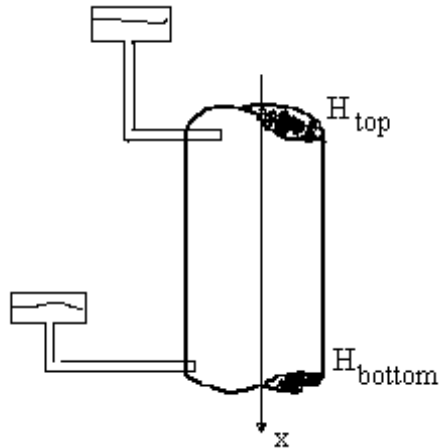
Conservation of fluid requires that in each ball, B, in the porous medium, we have

$$\int_B \partial_t \theta(x, t) dx + \int_{\partial B} Q \cdot n dS = 0.$$

Then, using the divergence theorem to convert the boundary integral to a volume integral, and using the fact that B is arbitrary, we arrive at the conservation equation

$$\partial_t \theta(x, t) + \text{div} Q = 0 \quad \text{at each } (x, t).$$

To complete the model, we need a constitutive relation between the fluid flux, Q, and the other state variables. An empirical law known as Darcy's law provides this relationship. Just as Maxwell initiated the mathematical treatment of electromagnetism, a French engineer named H. Darcy initiated in 1856 the mathematical theory of flow in porous media. He conducted a series of experiments on columns of various soils that led to the formulation of what is now called Darcy's law. Imagine a vertical soil column which is totally saturated with water. The flow of water through the column is assumed to be driven by gravity and external pressure and we can define the **hydraulic head**, H(x), at a point, x, in the column to be the sum of the gravity head (proportional to the elevation of the water above some datum level) and the pressure head (due to atmospheric or other applied pressure fields)



If we denote the hydraulic head by  $H = H(x)$ , and the flow rate through the saturated column by  $Q$ , then Darcy's law asserts that

$$Q = -K \frac{H_{Top} - H_{Bottom}}{\Delta x} \quad \text{or} \quad Q = -K \nabla H$$

where  $K$  denotes a soil dependent parameter called the **hydraulic conductivity**. Note the similarity with Fourier's law of heat conduction, Fick's law of diffusion and Ohm's law relating current flow to potential gradient.

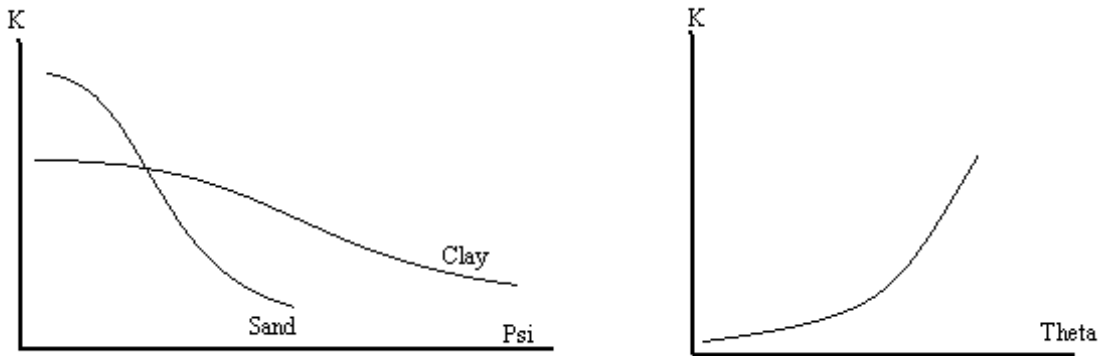
For saturated flow, the moisture content is constant and the flow equation obtained by combining Darcy's law with the conservation equation therefore becomes

$$\text{div}(-K \nabla H) = 0.$$

In order to describe partially saturated flow, we note first that in the saturated situation the fluid flow is driven by the hydraulic head. To extend Darcy's law to the case of partially saturated flow, it is necessary to postulate the existence of some other driving potential. For this purpose we can assert that fluid flow in a partially saturated medium is driven by a capillary suction force that propels fluid from regions of low suction toward regions where the suction is high. Empirically, we expect the highest suction at the interface between a wet and a dry region, at a so called "wetting front". We define the **total head** as the sum of the suction head,  $\psi$ , the hydraulic head,  $H$ . Having done this, we immediately decide to neglect the gravity and pressure heads in order to concentrate on the role of the capillary suction head. In a physical setting, it is legitimate to neglect the gravity head if the flow is in a horizontal direction (since in this case the entire flow is at the same elevation above the datum) and we assume further that the external pressure is the same at each point in the porous medium. Then Darcy's law for partially saturated flow is assumed the form

$$Q = -K \nabla \psi \quad \text{where } K \text{ is a function of suction, } \psi, \text{ or of water content } \theta..$$





Typical K vs Psi or K vs Theta curves

Then, combining this version of Darcy's law with the conservation equation leads to the so called Richard's equation. If we choose to view moisture content as a function of suction,  $\theta = \theta(\psi)$ , we obtain

$$\partial_t \theta(\psi(x, t)) - \nabla(K(\psi) \nabla \psi) = 0,$$

or

$$C(\psi) \partial_t \psi(x, t) - \nabla(K(\psi) \nabla \psi) = 0, \quad \text{where} \quad C(\psi) = \theta'(\psi). \quad (6.1)$$

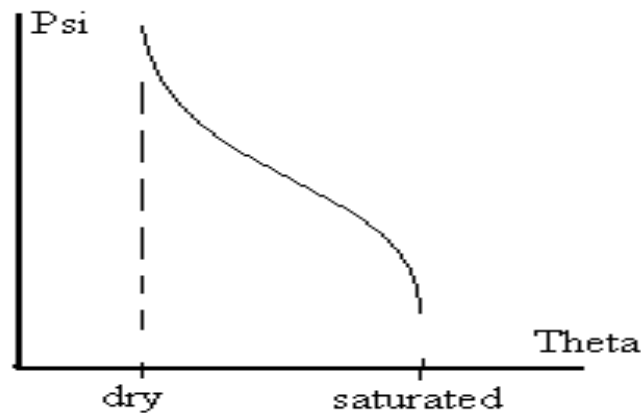
On the other hand, if we view the suction as a function of moisture content,  $\psi = \psi(\theta)$ , then

$$\partial_t \theta(x, t) - \nabla(K(\psi(\theta)) \nabla \psi) = \partial_t \theta(x, t) - \nabla(K(\psi(\theta)) \psi'(\theta) \nabla \theta) = 0$$

or

$$\partial_t \theta(x, t) - \nabla(D(\theta) \nabla \theta) = 0 \quad \text{where} \quad D(\theta) = K(\psi(\theta)) \psi'(\theta). \quad (6.2)$$

Empirical measurements for  $\theta$  and  $\psi$  suggest that the relationship between these two quantities has the following form



The equations (6.1) and (6.2) are referred to respectively as the pressure formulation and moisture content formulation of Richard's equation. Empirical evidence suggests that the

diffusivity coefficient,  $D = D(\theta)$ , in the moisture content formulation behaves approximately like  $\theta^m$  for some  $m > 1$ . We will use similarity transformations to show that if this is the case, then solutions to Richard's equation behave much differently than solutions to the heat equation which is superficially similar to Richard's equation.

Consider 
$$\partial_t \theta(x, t) - \partial_{xx}(\theta^2) = \partial_t \theta(x, t) - \partial_x(2\theta \partial_x \theta) = 0. \quad (6.3)$$

Guided by previous examples, we make the ansatz

$$\theta(x, t) = t^a y(z) \quad z = \frac{x}{t^b}. \quad (6.4)$$

Then 
$$t^{a-1}(ay(z) - bzy'(z)) - 2t^{2(a-b)}(y'(z)^2 + y(z)y''(z)) = 0$$

and equation (6.3) reduces to an ode in  $z$  if  $a - 1 = 2(a - b)$ , i.e.,  $a = 2b - 1$ . In this case

$$2(y'(z)^2 + y(z)y''(z)) = (2b - 1)y(z) - bzy'(z),$$

and, in particular, if  $b = \frac{1}{3}$ , we have  $2b - 1 = -b$  and  $(2b - 1)y(z) - bzy'(z) = \frac{1}{3}(zy(z))'$ . Then the ode becomes the exact equation,

$$2(y(z)y'(z))' + \frac{1}{3}(zy(z))' = 0.$$

If we add the conditions,  $y'(0) = 0$ , and  $y(z) \rightarrow 0$  as  $z^2 \rightarrow \infty$ , to the differential equation, then

$$y(z) = \left\{ \begin{array}{ll} \frac{A^2 - z^2}{12} & \text{if } z^2 < A^2 \\ 0 & \text{if } z^2 > A^2 \end{array} \right\}$$

satisfies the equation and these conditions. Then

$$\theta(x, t) = \frac{1}{\sqrt[3]{t}} y\left(\frac{x}{\sqrt[3]{t}}\right)$$

is a weak solution for the partial differential equation (6.3) (which is Richard's equation with  $D(\theta) = 2\theta$ ). Note particularly, this solution has compact support for each  $t > 0$  and the "front" where  $x = A\sqrt[3]{t}$ , travels with finite speed equal to  $x'(t) = \frac{1}{3}At^{-2/3}$ ,  $t > 0$ .

More generally, consider the equation

$$\partial_t \theta(x, t) - \partial_{xx}(\theta^m) = 0, \quad m > 1, \quad (6.5)$$

where  
and

$$\theta(x, t) = \theta(-x, t), \quad \theta(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty$$

$$\int_R \theta(x, t) dx = 1 \text{ for all } t > 0.$$

We make the ansatz (6.4) which leads to

$$t^{a-1}[ay(z) - by'(z)] - t^{ma-2b}[m(m-1)y^{m-2}(z)y'(z)^2 + my^{m-1}y''(z)] = 0.$$

The equation reduces to an ODE in  $z$  if  $a - 1 = ma - 2b$ ; i.e.,  $a = \frac{2b-1}{m-1}$ .

Additionally,

$$\int_R \theta(x, t) dx = t^a \int_R y(z) t^b dz = t^{a+b} \int_R y(z) dz = 1,$$

and this holds, independent of  $t$ , provided  $a + b = 0$ . Then

$$-b[y(z) + zy'(z)] - [m(m-1)y^{m-2} + my^{m-1}y''(z)] = 0,$$

or 
$$-b(zy(z))' - (my^{m-1}(z)y'(z))' = 0.$$

We integrate and use the fact that  $\theta(x, t) = \theta(-x, t)$ , to obtain

$$my^{m-1}(z)y'(z) + bzy(z) = 0.$$

We separate and integrate again, obtaining

$$\frac{m}{m-1} y^{m-1}(z) = -\frac{1}{2} bz^2 + C_0.$$

Now the previously discovered relations

$$a = \frac{2b-1}{m-1} = -b$$

imply  $b = \frac{1}{m+1}$  and so

$$\begin{aligned} y(z) &= \left( A^2 - \frac{m-1}{m} \frac{b}{2} z^2 \right)^{\frac{1}{m-1}} \\ &= \left( A^2 - \frac{m-1}{m+1} \frac{z^2}{2m} \right)^{\frac{1}{m-1}} \\ &= \left( \frac{m-1}{m+1} \frac{B^2 - z^2}{2m} \right)^{\frac{1}{m-1}}. \end{aligned}$$

The condition  $y(z) \rightarrow 0$  as  $|z| \rightarrow \infty$  is satisfied if

$$y(z) = \left\{ \begin{array}{ll} \left( \frac{m-1}{m+1} \frac{B^2 - z^2}{2m} \right)^{\frac{1}{m-1}} & \text{if } z^2 < B^2 \\ 0 & \text{if } z^2 > B^2 \end{array} \right\}$$

and then

$$\theta(x, t) = t^a y\left(\frac{x}{t^b}\right) = \frac{1}{m+1\sqrt{t}} \left\{ \begin{array}{ll} \left( \frac{m-1}{m+1} \frac{B^2 - \frac{x^2}{m+1\sqrt{t^2}}}{2m} \right)^{\frac{1}{m-1}} & \text{if } x^2 < B^2 m+1\sqrt{t^2} \\ 0 & \text{if } x^2 > B^2 m+1\sqrt{t^2} \end{array} \right\}$$

This is a weak solution for the equation (6.5) which is generally referred to as the **porous medium equation**. The notable features of the solution are the lack of smoothness and the compact support (also interpreted as finite speed of propagation). Both of these properties are in contrast to what we find in solving the heat equation. Note that in the first example, equation (6.3) we chose  $b = 1/3$  in order to make the ode exactly integrable while in the example (6.5) the choice  $b = -a$  was made in order to have the area under the graph of  $\theta(x, t)$  equal one for all positive  $t$ . In this case, the ode incidentally turned out to be integrable when  $b$  was chosen in this way. Since (6.3) is a special case of (6.5), it appears that the condition on the area under the graph automatically forces the ode obtained by the similarity transformation to become exactly integrable. Then it seems to be a fortuitous accident that the conditions that must be enforced to get the similarity transformation to go through are also conditions that make the resulting solution physically relevant.

It is mildly surprising that the similarity method we used to construct a solution for the 1-dimensional porous flow equation also succeeds in  $n$  dimensions. The  $n$  dimensional porous flow equation is

$$\partial_t u(x, t) - \text{div grad}(u^m) = 0 \quad x \in R^n, \quad t > 0.$$

We assume  $u(x, t) = t^a y(z) \quad z = \frac{x}{t^b}, \quad x \in R^n, \quad t > 0.$  Then

$$\partial_t(t^a y(z)) = t^{a-1}[a y(z) - b \nabla_z y(z) \cdot z]$$

$$\nabla_x^2(u^m) = \nabla_z^2(t^{am} y(z)^m) |\nabla_x z|^2 = t^{ma-2b} \nabla_z^2(y(z)^m)$$

In order to simplify the spatial term, let us now suppose that  $y(z) = w(r)$  where  $r = \sqrt{z \cdot z}$ , i.e., the solution is radial. Then we are left with just the radial part of the Laplacian,

$$\nabla_z^2(y(z)^m) = (w(r)^m)'' + \frac{n-1}{r}(w(r)^m)'$$

and the equation reduces to

$$t^{a-1}[aw(r) - brw'(r)] - t^{ma-2b} \left[ (w(r)^m)'' + \frac{n-1}{r}(w(r)^m)' \right] = 0$$

The equation becomes independent of  $t$  if  $a-1 = am-2b$ ; i.e.,  $a = \frac{2b-1}{m-1}$ . In addition, the area under the graph of  $u$  equals one for all  $t > 0$  if  $a+nb = 0$ . Then  $b = \frac{1}{(m-1)n+2}$  and if we multiply the equation by  $r^{n-1}$ , we get

$$b[nr^{n-1}w(r) + r^n w'(r)] + [r^{n-1}(w(r)^m)'' + (n-1)r^{n-2}(w(r)^m)'] = 0.$$

This can be rewritten as

$$(r^{n-1}(w(r)^m)')' + b(r^n w(r))' = 0.$$

Then  $r^{n-1}(w(r)^m)' + br^n w(r) = C_0$ ,

and the conditions  $w(r), w'(r) \rightarrow 0$  as  $r \rightarrow \infty$  imply that  $C_0 = 0$ . Then

$$(w(r)^m)' = m w(r)^{m-1} w'(r) = -br w(r)$$

and

$$m w(r)^{m-2} w'(r) = \frac{m}{m-1} (w(r)^{m-1})' = -br$$

so

$$w(r)^{m-1} = C_1 - \frac{m-1}{2m} br^2$$

Finally, the conditions at infinity lead to the solution

$$w(r) = \begin{cases} \left( C_1 - \frac{m-1}{2m} br^2 \right)^{\frac{1}{m-1}} & \text{if } r^2 < C_1 \frac{2m}{(m-1)b} \\ 0 & \text{if } r^2 > C_1 \frac{2m}{(m-1)b} \end{cases}$$

and

$$u(x, t) = t^{-nb} \begin{cases} \left( C_1 - \frac{m-1}{2m} b \frac{|x|^2}{t^b} \right)^{\frac{1}{m-1}} & \text{if } \frac{|x|^2}{t^b} < C_1 \frac{2m}{(m-1)b} \\ 0 & \text{if } \frac{|x|^2}{t^b} > C_1 \frac{2m}{(m-1)b} \end{cases}$$

where

$$b = \frac{1}{(m-1)n+2}.$$

This is called Barenblatt's solution for the PM equation. Like the 1-dimensional example, it has compact support for each  $t > 0$  and the set where  $u > 0$  grows at a finite speed for all  $t > 0$ .