The Transport Equation

Consider a fluid, flowing with velocity, V, in a thin straight tube whose cross section will be denoted by A. Suppose the fluid contains a contaminant whose concentration at position x at time t will be denoted by u(x,t). Then at time t, the amount of contaminant in a section of the tube between positions, x_1 and x_2 is given by the expression

$\int_{x_1}^{x_2} u(x,t)\,A\,dx =$ amount of contaminant in (x_1,x_2) at time t

Similarly, we can write an expression for the amount of contaminant that flows through a plane located at position, x, during the time interval from t_1 to t_2

$\int_{t_1}^{t_2} u(x,t)\,A\,V\,dt =$ amount of contaminant flowing through a plane at position, x, during the interval (t_1,t_2)

Then an equation expressing a material balance for the contaminant can be written as follows,

$$\int_{x_1}^{x_2} u(x,t_2)\,A\,dx = \int_{x_1}^{x_2} u(x,t_1)\,A\,dx + \int_{t_1}^{t_2} u(x_1,t)\,A\,V\,dt - \int_{t_1}^{t_2} u(x_2,t)\,A\,V\,dt$$

i.e., the amount of contaminant in the section (x_1,x_2) at time t_2 equals the amount of contaminant in the section (x_1,x_2) at time t_1, plus the amount of contaminant that flowed through the plane at position x_1 during the time interval (t_1,t_2) minus the amount of contaminant that flowed through the plane at position x_2 during the time interval (t_1,t_2). Of course this equation is based on the assumption that there are no other sources of contaminant in the tube and there is no loss of contaminant through the walls of the tube.

Now the fundamental theorem of calculus implies that

$$\int_{x_1}^{x_2} u(x,t_2)\,A\,dx - \int_{x_1}^{x_2} u(x,t_1)\,A\,dx = \int_{x_1}^{x_2}\int_{t_1}^{t_2} \partial_t u(x,t)\,A\,dt\,dx$$

$$\int_{t_1}^{t_2} u(x_1,t)\,A\,V\,dt - \int_{t_1}^{t_2} u(x_2,t)\,A\,V\,dt = -\int_{t_1}^{t_2}\int_{x_1}^{x_2} \partial_x u(x,t)\,A\,V\,dx\,dt$$

and, combining these results with the balance equation leads to

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} \partial_t[u(x,t)A]\,dt\,dx + \int_{t_1}^{t_2}\int_{x_1}^{x_2} \partial_x[u(x,t)\,A\,V]\,dx\,dt = 0.$$

If we assume that this equality holds for every segment (x_1,x_2) in the tube and for each time interval (t_1,t_2), and if the function $u(x,t)$ and its partial derivatives of order one are continuous functions of x and t, then it follows from an elementary property of continuous functions that

$$\partial_t[u(x,t)A] + \partial_x[u(x,t)\,A\,V] = 0 \qquad \text{for all } (x,t).$$

If the fluid velocity V and the cross section of the tube, A, are constants, then this equation reduces to

$$\partial_t u(x,t) + V\ \partial_x u(x,t) = 0 \qquad \text{for all } (x,t).$$

This is the so called "transport equation" in one dimension.

Since this equation contains partial derivatives of order at most equal to one, it is called a ***first order partial differential equation***. It is, moreover, a ***linear partial differential equation***. This terminology relates to the fact that if we define an operator, L, as follows

$$L[u(x,t)] = \partial_t u(x,t) + V\ \partial_x u(x,t)$$

then it is a simple matter to verify that

$$L[C_1u_1(x,t) + C_2u_2(x,t)] = C_1L[u_1(x,t)] + C_2L[u_2(x,t)].$$

Any operator with this property is called a ***linear operator*** (any function of one variable $f(x)$ with the property that $f(c_1x_1 + c_2x_2) = c_1f(x_1) + c_2f(x_2)$ is a function whose graph is a straight line), and any partial differential equation (PDE) expressing an equality for a linear partial differential operator is called a linear equation.

Problem 1 Show that any PDE that contains terms involving products of the unknown function with its derivatives is not a linear PDE. What is the form of the most general linear first order PDE for a function of 2 variables, $u(x,t)$? What is it for a function of 4 variables, $u(x,y,z,t)$?

Problem 2 Show that for any smooth function of one variable, $F(x)$, we have that

$$u(x,t) = F(x - Vt)$$

solves

$$\partial_t u(x,t) + V\ \partial_x u(x,t) = 0$$

Devise a similar such solution for the equation,

$$\partial_t u(x,y,z,t) + V_1\ \partial_x u(x,y,z,t) + +V_2\ \partial_y u(x,y,z,t) + +V_3\ \partial_z u(x,y,z,t) = 0.$$

Consider now the more general situation involving the flow of a fluid, flowing with velocity, $\vec{v}$, in a region U in R^n. Suppose the fluid contains a contaminant whose concentration at position $\vec{x}$ at time t will be denoted by $u(\vec{x},t)$. Then at time t, the amount of contaminant in an arbitrary ball B in U is given by the expression

$$\int_B u(\vec{x},t)\,dx = \text{amount of contaminant in B at time t}$$

Similarly, the outflow through the boundary of the ball during the time interval (t_1,t_2) is given by

$$\int_{t_1}^{t_2}\int_{\partial B} u(\vec{x},t)\vec{v}\bullet\vec{n}\ dS\,dt = \text{outflow through the boundary of the ball during the time interval } (t_1,t_2)$$

where $\vec{n}$ denotes the outward unit normal to ∂B, the boundary the surface of the ball. Now the material balance equation becomes

$$\int_B u(\vec{x},t_2)\,dx = \int_B u(\vec{x},t_1)\,dx - \int_{t_1}^{t_2}\int_{\partial B} u(\vec{x},t)\vec{v}\bullet\vec{n}\ dS\,dt$$

expressing the fact that the amount of contaminant in the ball at time t_2 equals the amount of contaminant in the ball at time t_1, less that amount that has flowed out of the ball through the boundary. As before, we can use the fundamental theorem to write

$$\int_B u(\vec{x},t_2)\,dx - \int_B u(\vec{x},t_1)\,dx = \int_{t_1}^{t_2}\int_B \partial_t\ u(\vec{x},t)\ dxdt$$

Recall now that the divergence theorem asserts that for an arbitrary smooth vector field, $\vec{G}(\vec{x})$,

$$\int_{\partial B}\vec{G}(\vec{x})\bullet\vec{n}\ dS = \int_B div\,\vec{G}\,dx.$$

Then for $\vec{G} = u(\vec{x},t)\vec{v}$ we get

$$\int_{t_1}^{t_2}\int_B \partial_t\ u(\vec{x},t)\ dxdt + \int_{t_1}^{t_2}\int_B div[u(\vec{x},t)\vec{v}]\ dx\,dt = 0.$$

Problem 3 Show that for any smooth scalar function, $u(\vec{x},t)$, and any constant vector $\vec{v}$, $div[u(\vec{x},t)\vec{v}] = \vec{v} \bullet grad\, u(\vec{x},t)$

It follows from the result of the problem that since B is an arbitrary ball in U, and (t_1,t_2) is similarly arbitrary, then if u and its derivatives of order one are all continuous in U,

$$\partial_t\, u(\vec{x},t) + \vec{v} \bullet grad\, u(\vec{x},t) = 0, \quad \text{in U for all t.}$$

This is the transport equation in n-dimensions.

Solutions for First Order Equations

Consider first the problem of finding the general solution for the equation

$$\partial_t u(x,t) + V\; \partial_x u(x,t) = 0 \quad \text{for all } (x,t).$$

By a solution to the equation, we mean a function, $u(x,t)$, that is continuous and has continuous first derivatives at all points (x,t), and in addition is such that the continuous function $\partial_t u(x,t) + V\; \partial_x u(x,t)$ is equal to zero at all points. If $u(x,t)$ has all these properties, we say that $u(x,t)$ is a ***classical solution*** for the PDE.

Suppose that

$$\left\{ \begin{array}{l} x = x(s) \\ t = t(s) \end{array} \quad -\infty < s < \infty \right\}$$

is the parametric description of some curve C in the x-t plane. Then for $u = u(x,t)$ a smooth function of x and t, it is clear that $du/ds = \partial_x u(x,t)\,x'(s) + \partial_t u(x,t)\,t'(s)$ expresses the derivative of u along the curve C. In particular, if the curve C is such that

$$x'(s) = V \quad and \quad t'(s) = 1, \quad i.e., \quad x(s) = Vs + x_0, \quad and \quad t(s) = s + t_0$$

Then $\quad du/ds = \partial_x u(x,t)\,V + \partial_t u(x,t) = 0,\quad$ asserts that u is constant along C from which it is apparent that $u = u(x,t)$ solves the PDE if and only if u is constant along C. It is also clear that if V is a constant, then C is a straight line. The parametric representation for this straight line is then

$$\left\{ \begin{array}{l} x(s) = Vs + x_0 \\ t(s) = s + t_0 \end{array} \quad -\infty < s < \infty \right\}$$

Eliminating the parameter, s, leads to the implicit equation $x - Vt = x_0 - Vt_0 =: C_0$, for the straight line passing through the point (x_0,t_0) in the x-t plane. Clearly, for $F(x)$ any smooth function of one variable, $u(x,t) = F(x - Vt)$ is a smooth function of x and t which is constant along C. Then by our previous observation, $u = u(x,t)$ solves the PDE if and only if $u(x,t) = F(x - Vt)$ for $F(x)$ any smooth function of one variable. Note that the general solution to a linear first order ***partial*** differential equation contains an arbitrary function, in contrast to the general solution for a linear first order ***ordinary*** differential equation which contains an arbitrary constant.

Now consider the more general problem of finding the most general solution for the equation

$$\partial_t u(\vec{x},t) + \vec{v} \bullet \nabla u(\vec{x},t) = 0 \quad \text{for all } (\vec{x},t).$$

If $$\left\{ \begin{array}{c} \vec{x} = \vec{x}(s) = (x_1(s),...,x_N(s)) \\ t = t(s) \end{array} \quad -\infty < s < \infty \right\}$$

is the parametric description of some curve C in R^{N+1} then for $u = u(x,t)$ a smooth function of x and t, it is clear that

$$du/ds = \partial_1 u(x,t)\, x_1'(s) + ... + \partial_N u(x,t)\, x_N'(s) + \partial_t u(x,t)\, t'(s)$$

expresses the derivative of u along the curve C. If the curve C is such that

$$\vec{x}'(s) = \vec{v} \quad and \quad t'(s) = 1, \quad i.e., \quad \vec{x}(s) = \vec{v}s + x_0, \quad and \quad t(s) = s + t_0$$

then $$du/ds = \vec{v} \bullet \nabla u + \partial_t u(x,t) = 0,$$

asserts that u is constant along C, (obviously C is a straight line in R^{N+1}). By the previous argument, the following statements are equivalent:

1) $u = u(\vec{x},t)$ is a solution of $\partial_t u(\vec{x},t) + \vec{v} \bullet \nabla u(\vec{x},t) = 0$
2) $u = u(\vec{x},t)$ is constant along the straight line $\vec{x}(s) = \vec{v}s + x_0, \quad and \quad t(s) = s + t_0$
3) $u(\vec{x},t) = f(\vec{x} - \vec{v}t)$ for $f(\vec{x})$ any smooth scalar valued function of N variables

Problem 4 Show that the three statements are, in fact, equivalent and that C is a straight line if $\vec{v}$ is a constant.

Statement 3 here asserts that the general solution for the equation given in statement 1 is any function of the form $u(\vec{x},t) = f(\vec{x} - \vec{v}t)$ where f is any smooth function from R^N into R^1. Evidently the solution to this equation is a very long way from being unique. On the other hand, consider the problem of finding a function $u(\vec{x},t)$ which satisfies the conditions

a) $\partial_t u(\vec{x},t) + \vec{v} \bullet \nabla u(\vec{x},t) = F(\vec{x},t) \quad for\ all\ \vec{x}, \quad and\ all\ t > 0,$
b) $u(\vec{x},0) = g(\vec{x}), \quad for\ all\ \vec{x}$

The condition a) is an ***inhomogeneous*** version of the equation in statement 1). The term ***inhomogeneous*** is used to express the fact that the right side of the equation is not zero and, in fact, contains the ***forcing function***, $F(\vec{x},t)$. The equation in statement 1) is said to be a ***homogeneous*** equation, since the right side of the equation is zero. The condition b) is referred to as an ***initial condition*** since it specifies the state variable, $u(\vec{x},t)$, at the initial time t=0. The functions $F(\vec{x},t)$ and $g(\vec{x})$ are called the ***data*** for the problem and are assumed to be given. We will show that for each pair of data functions, $F(\vec{x},t)$ and $g(\vec{x})$, we can find a function $u(\vec{x},t)$ which satisfies both of the conditions a) and b). Then we say, $u(\vec{x},t)$ is a solution to the ***initial value problem***. Note that this construction does not gaurantee that the solution is unique (although we may be able to show later that the solution is unique).

We will describe the construction in the simplest case that N=1. Then the initial value problem (IVP) assumes the form

$$\partial_t u(x,t) + V\ \partial_x u(x,t) = F(x,t), \qquad -\infty < x < \infty, \qquad t > 0,$$
$$u(x,0) = g(x), \qquad -\infty < x < \infty$$

First, let $u_1(x,t)$ denote the solution of this IVP in the case that $F(x,t) = 0$. The general solution of the homogeneous PDE is any function of the form, $u_1(x,t) = f(x - Vt)$, where f denotes a smooth function of one variable. In order to satisfy the initial condition we must have, $u_1(x,0) = f(x) = g(x)$, from which it follows that $u_1(x,t) = g(x - Vt)$. Here we see that in order for the homogeneous initial value problem to have a solution, the given initial function must be continuously differentiable.

Now let $u_2(x,t)$ denote the solution of the IVP in the case that $g(x) = 0$ but $F(x,t)$ is not zero. Then

$$\partial_t u_2(x,t) + V\ \partial_x u_2(x,t) = F(x,t), \qquad -\infty < x < \infty, \qquad t > 0,$$
$$u_2(x,0) = 0, \qquad -\infty < x < \infty$$

The inhomogeneous equation is equivalent to

$$d/ds\,\{u_2(x(s),t(s))\} = F(x(s),t(s)),$$

where

$$x'(s) = V \quad and \quad t'(s) = 1;\ i.e., \quad x(s) = Vs + x_0, \quad and \quad t(s) = s + t_0$$

Then for arbitrary parameter values $s_2 > s_1$,

$$u_2(x(s_2),t(s_2)) = u_2(x(s_1),t(s_1)) + \int_{s_1}^{s_2} d/ds\,\{u_2(x(s),t(s))\}ds$$
$$= u_2(x(s_1),t(s_1)) + \int_{s_1}^{s_2} F(x(s),t(s))\,ds$$

If we choose $s_2 = 0 > s_1 = -t$, then $x_0 = x(s_2), \quad t_0 = t(s_2)$. That is,

$$x(s_2) = V0 + x, \quad and \quad t(s_2) = 0 + t$$
$$x(s_1) = V(-t) + x, \quad and \quad t(s_1) = -t + t$$

and

$$u_2(x,t) = u_2(x - Vt, 0) + \int_{-t}^{0} F(x + Vs, t + s)\,ds = 0 + \int_0^t F(x + V(\theta - t),\theta)\,d\theta$$

where we made the change of variable, $\theta = t + s$ in the integral. Now it is easy to show that

$$u(x,t) = u_1(x,t) + u_2(x,t) = g(x - Vt) + \int_0^t F(x - V(t - \theta),\theta)\,d\theta$$

satisfies both of the conditions a) and b); i.e., this is a solution of the initial value problem.

Problem 5 Verify that the solution constructed here does, in fact, satisfy both conditions in the IVP. What are the conditions on F(x,t) in order for this solution to be valid? Verify that in the case of general N, the solution of the IVP is given by

$$u(\vec{x},t) = g(\vec{x} - \vec{V}t) + \int_0^t F(\vec{x} - \vec{V}(t - \theta),\theta)\,d\theta$$

Suppose N = 1, F = 0 and that the initial data function, $g(x)$, is such that

$$g(x) = \begin{Bmatrix} 1 & if\ \ x < 1 \\ 2 & if\ \ x > 1 \end{Bmatrix}$$

Then

$$u(x,t) = \left\{ \begin{array}{lll} 1 & if & x - Vt < 1 \\ 2 & if & x - Vt > 1 \end{array} \right\}$$

formally satisfies the IVP but since the first derivatives of $u(x,t)$ fail to exist at points on the line $x - Vt = 1,$ it cannot be said to be a solution in the classical sense. In the coming weeks we will try to determine whether there is some weaker sense in which this can be said to be a solution of the IVP.

Method of Characteristics

The method of finding a solution for the transport equation in the previous examples is a special case of the so called "method of characteristics". If we consider a more general first order equation in 2 variables,

$$A(x,t)\,\partial_t u(x,t) + B(x,t)\,\partial_x u(x,t) + C(x,t)\,u(x,t) = F(x,t)\,,$$

then a curve C in the x-t plane is said to be a characteristic curve for this PDE if C is a solution curve for the following system of ordinary differential equations,

$$t'(s) = A(x(s),t(s)), \qquad x'(s) = B(x(s),t(s)).$$

Note that along any characteristic curve C, the PDE reduces to an ordinary differential equation

$$d/ds\{u(x(s),t(s))\} + C(x(s),t(s))\,u(x(s),t(s)) = F(x(s),t(s)).$$

Note that although the linear PDE reduces to a linear ODE along characteristic curves, the ordinary differential equations which produce the characteristics may be nonlinear. In the examples we have considered, the coefficients A, B and C were constants and correspondingly, the characteristics turned out to be straight lines. In general, when the coefficients in the PDE are not constant, then the characteristics turn out to be a family of curves. We will return to the method of characteristics later when considering conservation laws.