Existence and Uniqueness of Solutions

Let

\[ L[y(t)] = y''(t) + by'(t) + cy(t) \]

and define \[ S = \{ y \in C^2 : L[y] = 0 \} = \text{the solution space for } L \]

Functions \[ W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t). \]

Here \[ W[y_1, y_2](t) \] is called the "Wronskian determinant" for the functions \[ y_1 \] and \[ y_2. \] Then we can prove the following results:

**Lemma 1**

If \[ y_1, y_2 \in S \] then either \[ W[y_1, y_2](t) = 0 \] for all \( t \), or else \[ W[y_1, y_2](t) \] is never zero for any \( t \).

Proof- Compute the derivative of \[ W[y_1, y_2](t) \],

\[
\frac{d}{dt} W[y_1, y_2](t) = \frac{d}{dt} (y_1(t)y_2'(t) - y_2(t)y_1'(t)) = y_1(t)y_2''(t) + y_1'(t)y_2'(t) - y_2(t)y_1''(t) - y_2'(t)y_1'(t) = y_1(t)y_2''(t) - y_2(t)y_1''(t) = -b[y_1(t)y_2'(t) - y_2(t)y_1'(t)] - c[y_1(t)y_2(t) - y_2(t)y_1(t)] = -bW[y_1, y_2](t) \]

where we have used the fact that \( y_1, y_2 \in S \). Now

\[
\frac{d}{dt} W[y_1, y_2](t) = -b W[y_1, y_2](t). \]

implies 

\[ W[y_1, y_2](t) = Ce^{bt} \]

and

\[ W[y_1, y_2](t) = W[y_1, y_2](t_0) e^{-b(t-t_0)}. \]

From this last equation we can see that if \( W[y_1, y_2](t_0) = 0 \), then \( W[y_1, y_2](t) = 0 \) for every \( t \), and if \( W[y_1, y_2](t_0) \neq 0 \), then \( W[y_1, y_2](t) \) is never zero. ■

**Lemma 2**

Functions \( y_1, y_2 \in S \) are independent if and only if \( W[y_1, y_2](t) \neq 0 \).

Proof- Suppose \( y_1, y_2 \in S \) satisfy

\[ C_1y_1(t) + C_2y_2(t) = 0. \]
Then, by differentiating with respect to $t$, we see that we also have

$$C_1 y_1'(t) + C_2 y_2'(t) = 0.$$  

This set of two equations in two unknowns, $C_1, C_2$, has a nontrivial solution ($C_1, C_2$, not both zero) if and only if the determinant of the system is equal to zero. But the determinant of this system is just the Wronskian, $W[y_1, y_2](t)$, which is either zero for all $t$ or zero for no $t$. If the Wronskian is zero for all $t$, then a nontrivial pair of constants exists and in this case $y_1, y_2 \in S$ are dependent.

If the Wronskian is nonzero for all $t$, then the trivial pair of constants is the only solution and in this case $y_1, y_2 \in S$ are independent. \[ \square \]

**Lemma 3**

If $y \in S$ then

$$Ce^{-kt} \leq \sqrt{y(t)^2 + y'(t)^2} \leq Ce^{kt}$$

where

$$C = \sqrt{y(0)^2 + y'(0)^2} \quad \text{and} \quad k = 1 + |b| + |c|.$$  

**Proof** - For $y \in S$ let $U(t) = y(t)^2 + y'(t)^2$. Then

$$U'(t) = 2y(t)y'(t) + 2y'(t)y''(t) = 2y(t)y'(t) + 2y'(t)[-by'(t) - cy(t)]$$  

$$= (2 - 2c)y(t)y'(t) - 2by'(t)^2,$$  

and

$$|U'(t)| \leq 2(1 + |c|)|y||y'| + 2|b||y'|^2.$$  

Now we use the result that

$$2 |y||y'| \leq |y|^2 + |y'|^2$$

which is a consequence of

$$(|y| - |y'|)^2 = |y|^2 - 2|y||y'| + |y'|^2 \geq 0.$$  

Then

$$|U'(t)| \leq (1 + |c|) \left( |y|^2 + |y'|^2 \right) + 2|b||y'|^2$$  

$$\leq 2(1 + |b| + |c|) \left( |y|^2 + |y'|^2 \right) = 2k \ U(t),$$

which is the same as saying

$$-2k \ U(t) \leq U'(t) \leq 2k \ U(t).$$

This implies that (check this out),

$$U(0) e^{-2kt} \leq U(t) \leq U(0) e^{2kt},$$

and

$$\sqrt{U(0)} e^{-kt} \leq \sqrt{U(t)} \leq \sqrt{U(0)} e^{kt},$$

which is the estimate we were trying to prove. \[ \square \]
Lemma 4
For arbitrary constants, A, B there exists at most one $y \in S$ such that $y(0) = A$ and $y'(0) = B$.

Proof- Suppose there are two functions $y_1, y_2 \in S$ which both satisfy $y(0) = A$ and $y'(0) = B$. Then the function $w(t) = y_1(t) - y_2(t)$ must satisfy
\[ L[w(t)] = 0 \quad \text{(why?)} \]
and
\[ w(0) = 0 \quad \text{and} \quad w'(0) = 0, \quad \text{(why?).} \]

But then Lemma 3 implies
\[ Ce^{-kt} \leq \sqrt{w(t)^2 + w'(t)^2} \leq Ce^{kt} \quad \text{for } C = 0. \]

Then $w(t) = w'(t) = 0$, and $w(t) = y_1(t) - y_2(t) = 0$ and it follows that $y_1(t) = y_2(t)$.

Lemma 5
For arbitrary constants, A, B there exists at least one $y \in S$ such that $y(0) = A$ and $y'(0) = B$.

Proof- Recall that $L[e^{rt}] = P(r)e^{rt} = 0$ implies $P(r) = 0$, and this polynomial equation has roots $r_1, r_2$. There are 3 possible cases, in each of which we can find independent solutions $y_1, y_2$ for the homogeneous equation. That is,
\[
\begin{cases}
    r_1 \neq r_2 & \text{both real} \\
    r_1 = r_2 \\
    r_1, r_2 = a \pm ib
\end{cases}
\]
then
\[
\begin{cases}
    y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t} \\
    y_1 = e^{r_1 t}, \quad y_2 = te^{r_1 t} \\
    y_1 = e^{at} \cos bt, \quad y_2 = e^{at} \sin bt,
\end{cases}
\]

In all three cases, we have
\[
\begin{align*}
    y(0) &= C_1y_1(0) + C_2y_2(0) = A \\
    y'(0) &= C_1y_1'(0) + C_2y_2'(0) = B
\end{align*}
\]
and the determinant of this set of two equations in two unknowns is just the Wronskian $W[y_1, y_2]$. Since $y_1, y_2 \in S$ are independent, this determinant is not zero and a unique solution for $C_1, C_2$ exists for all choices of A,B.

Combining the last two lemmas allows us to assert that

For arbitrary constants, A, B there exists one and only one solution for
\[ L[y(t)] = 0 \quad \text{with} \quad y(0) = A \quad \text{and} \quad y'(0) = B. \]

Lemma 6
Let $y_1, y_2 \in S$ be linearly independent. Then every $y \in S$ can be written uniquely in the form
\[ y(t) = C_1y_1(t) + C_2y_2(t). \]

The assertion of this lemma is that there are at least two independent functions in S but any set of three or more functions in S must be dependent. That is, $S$ is a two dimensional subspace of $C^2$. 

3
Proof- Let \( y_1, y_2 \in S \) be linearly independent and, for an arbitrary \( y \in S \), let \( y(0) = A, \ y'(0) = B. \) Then by lemma 5, there exists at least one choice of \( C_1, C_2 \) such that

\[
\dot{Y}(t) = C_1y_1(t) + C_2y_2(t)
\]

satisfies

\[
L[Y(t)] = 0 \quad \text{with} \quad Y(0) = A \quad \text{and} \quad Y'(0) = B.
\]

But by lemma 4, \( Y(t) = y(t) \)