M 317 Problems Chapter 5

1. For $f(x) = 1+x$ on $I = \{0 \le x \le 2\}$, let P denote a uniform partition of I with $N = 4$. Then compute each of the quantities, $m|I|$, $\sigma[f,P]$, $RS[f,P]$, $\Sigma[f,P]$, and $M|I|$, where the tag points for the Riemann sum are the midpoints of the intervals I_k in the partition. Compare these numbers with $\int_0^2 f$.

2. Using the results of problem 1, sketch the graph of f on I and indicate the quantities $m|I|$, $\sigma[f,P]$, $RS[f,P]$, $\Sigma[f,P]$, and $M|I|$, on the sketch.

3. Refine the partition P of the previous problem by adding the midpoint of the first interval in P. Call this refined partition P^*. Compute $\sigma[f,P^*]$, and $\Sigma[f,P^*]$.

4. Using the results of problem 3, sketch the graph of f on I and indicate the quantities $\sigma[f,P]$, $\Sigma[f,P]$, and ,$\sigma[f,P^*]$, $\Sigma[f,P^*]$, on the sketch. Explain why $\sigma[f,P] \le \sigma[f,P^*]$, and $\Sigma[f,P^*] \le \Sigma[f,P]$. Explain why these results apply for any partitions P and P^* where P^* is a refinement of P.

5. Repeat problem 1 with $f(x) = 3-x$ on $I = \{0 \le x \le 2\}$. How do the numbers $\sigma[f,P]$, $RS[f,P]$, $\Sigma[f,P]$ in this case relate to the same quantities in problem 1?

6. For $f(x) = 1+x$ on $I = \{0 \le x \le 2\}$, let Q denote a uniform partition of I with $N = 8$. Is Q a refinement of P? Use a sketch as in problem 4 to explain why $\sigma[f,P] \le \sigma[f,Q] \le \Sigma[f,Q] \le \Sigma[f,P]$. Explain why, $\sigma[f,Q] \le \Sigma[f,P]$ holds for any partitions P and Q for I.

7. For $f(x) = 1+x$ on $I = \{0 \le x \le 2\}$, let P denote a uniform partition of I with $N = 4$. Compute $\sum_{k=1}^{4}(M_k - m_k)|I_k|$ and indicate on a sketch of the graph of the function f, what this sum represents. If we refine the partition P, will the corresponding sum increase, decrease, or stay the same.?

8. Let $f(x) = 1$ if $x \in [0,2] \cap Q$ and let $f(x) = 0$ for any irrational number in $[0,2]$. Compute $\sum_{k=1}^{4}(M_k - m_k)|I_k|$ for the uniform partition P of the previous problem. If we refine the partition P, will the corresponding sum in this case increase, decrease, or stay the same.?

9. Let $f(x) = 1+x$ on $I = [0,2]$. Choose an arbitrary $\varepsilon > 0$, and find a partition P of $I = [0,2]$ such that $S[f,P] - s[f,P] < \varepsilon$ Does this imply that f is integrable on I?

10. Let $f(x) = 3-x$ on $I = [0,2]$. For an arbitrary given $\varepsilon > 0$, find a partition P of $I = [0,2]$ such that $S[f,P] - s[f,P] < \varepsilon$ Does this imply that f is integrable on I?

11. Let $f(x) = \begin{cases} 2x & if\ 0 \le x < 1 \\ 3 & if\ 1 \le x \le 2 \\ x & if\ 2 < x < 3 \\ 2(4-x)^2 & if\ 3 \le x \le 4 \end{cases}$ $0 \le x \le 4$. Choose an $\varepsilon > 0$, and find a partition P of $I = [0,4]$ such that $S[f,P] - s[f,P] < \varepsilon$ Does this imply that f is integrable on I?

12. Suppose $f \in \mathbb{C}[a,b]$ is non-negative and not identically zero on $[a,b]$.

a. Prove that $\int_a^b f > 0$

b. Is this result still true if we assume only that *f* is integrable but not continuous? If your answer is yes, prove it, if it is no, give a counterexample.

13. Suppose *f* is defined on $[a,b]$ and $|f(x)|$ is integrable on $[a,b]$. Does it necessarily follow that *f* is integrable on $[a,b]$? If your answer is yes, prove it, if it is no, give a counterexample.

14. Let $f(x) = \begin{cases} \frac{1}{2^n} & \textit{if } x = \frac{j}{2^n} \text{ for } j = \text{odd integer}, \\ & 0 < j < 2^n \quad n = 1,2,\ldots \\ 0 & \textit{otherwise} \end{cases} \qquad 0 \le x \le 1$

a. sketch the graph of *f* and find where *f* is continuous and where it is discontinuous

b. determine whether *f* is integrable on $[0,1]$ and prove it.

15. Suppose *f* and g are continuous on $I = [a,b]$ and that $\int_I f = \int_I g$. Prove there exists c in I such that $f(c) = g(c)$

16. Compute the average value for $f(x)$ on $[0,1]$,:

a. $f(x) = x(1-x^2) \qquad x \in [0,1]$

b. $f(x) = \begin{cases} x & \textit{if } x = \frac{1}{n}, \quad n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} . \qquad x \in [0,1]$

17. Suppose: (i) $f \in C[a,b]$, (ii) $f(x) \ge 0 \quad \forall x \in I$, and (iii) $s[f] = 0$. Then prove that $f(x) = 0$ for all x in I.

18. . Suppose *f* is integrable on $I = [a,b]$ and $m \le f(x) \le M$ for all x in I

a. prove that $m(b-a) \le \int_a^b f \le M(b-a)$

b. if *f* is continuous on I prove there is a c in I such that $f(c) = \frac{1}{b-a}\int_a^b f$

19. Suppose *f* is continuous on $I = [a,b]$, $f(x) \ge 0$ for all x in I

a. if, in addition, $s[f] = 0$, then prove that $f(x) = 0$ for all x in I.

b. if, instead of a we have $f(c) > 0$ for some c in I, prove that $\int_I f > 0$

20. Let $\{x_1, \ldots x_M\}$ denote a finite set of points in $[a,b]$ and let

$$f(x) = \begin{cases} k & if \quad x = x_k \\ 0 & if \quad x \neq x_k \end{cases}$$

Show that f is integrable on I and compute the integral.

21. Suppose f is integrable on $I = [a,b]$

a. Use the inequality $||x| - |y|| \leq |x - y|$ to prove that $|f|$ is integrable on I.

b. give an example of a function g that is not integrable on I but $|g|$ is integrable on I

22. Suppose f is integrable on $I = [a,b]$ and $m \leq f(x) \leq M$ for all x in I

a. prove that

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

b. if f is continuous on I prove there is a c in I such that

$$f(c) = \frac{1}{b-a} \int_a^b f$$

23. For f integrable on $I = [a,b]$ let $F(x) = \int_x^b f$

a. show that $F(x)$ is uniformly continuous on I

b. show that at each x in I where f is continuous, $\lim_{h \to 0} D_h F(x)$ exists. What does the limit equal?

24. Let $f(t) = \begin{cases} 3-t & if \quad 0 \leq t \leq 2 \\ 1 & if \quad 2 \leq t \leq 3 \end{cases}$ and $F(t) = \int_0^t f$

a. obtain an explicit formula for F

b. draw the graph of F and tell where you think F is differentiable

c. compute $F'(t)$ at each point where the derivative exists.

25. Suppose $f \in C[0,\infty)$ and $f(x) \neq 0$ for all $x > 0$. Then show that if $[f(x)]^2 = 2\int_0^x f$ *for* $x > 0$ *then* $f(x) = x$ for $x \geq 0$.

26. Suppose: $f \in C[a,b]$, and $\int_a^b fg = 0$ for all $g \in C[a,b]$. Then prove that $f(x) = 0$ for all x in I.

27. Let $\{x_1, \ldots, x_M\}$ denote a finite set of points in $[a,b]$ and suppose $f(x) = k$ if $x = x_k$ and $f(x) = 0$ otherwise. Show that f is integrable and compute $\int_I f$.

28. Give an example of an f that is not integrable on I but $|f|$ is integrable.

29. Suppose f is integrable on I and $m \leq f(x) \leq M$ for all x *in* I.

a. Prove that $m(b-a) \leq I = \int_I f \leq M(b-a)$

b. If $f \in C[a,b]$ prove there is a $c \in I$ such that $f(c)(b-a) = \int_I f$. Is this result true if f is not continuous?

30. For f integrable on $I = [a,b]$ let $F(x) = \int_x^b f$

a. show that F is uniformly continuous on I

b. Show that at each x in I where f is continuous, we have $\lim_{h\to 0} D_h F(x) = f(x)$

31. Let $f(x) = \begin{cases} 3-t & if \quad 0 < t < 2 \\ 1 & if \quad 2 < t < 3 \end{cases}$ and $F(x) = \int_a^x f$

a. obtain an explicit formula for F

b. sketch the graph of F and tell where F is differentiable

c. Compute $F'(t)$ at each point where the derivative exists.

32. Calculate: $(a) \quad \lim_{x\to 0} \frac{1}{x}\int_0^x e^{-t^2}dt \qquad (b) \quad \lim_{h\to 0} \frac{1}{h}\int_3^{3+h} e^{-t^2}dt$

33. Let $G(x) = \int_0^{\sin x} f(t)dt$. Is G differentiable? If so, find $G'(x)$.

34. Let $g : [0,1] \to [0,1]$ be continuous and strictly increasing on $[0,1]$. Use a sketch to explain why $\int_0^1 g(x)dx + \int_0^1 g^{-1}(y)dy = 1$.

35. Compute the value of $\int_0^\infty x^n e^{-x}dx$

36. Suppose f is continuous and strictly decreasing on $[0,\infty)$. Explain why $\int_0^\infty f$ and $\sum_{n=1}^{\infty} f(n)$ are either both convergent or both divergent.