Additional Problems Sequences

1. Consider the sequence of prime numbers, $1, 2, 3, 5, 7, 11, \ldots$ Is this really a sequence? How do you define a_n?

2. What is the next term in the sequence $3, 1, 5, 1, 7, \ldots$ Give a definition for a_n.

3. Find an N such that $|a_n - L| \le 10^{-3}$ for $n > N$
   - **a.** $a_n = \dfrac{2}{\sqrt{n+1}}$
   - **b.** $a_n = 1 - \dfrac{1}{n^3}$
   - **c.** $a_n = 2 + 2^{-n}$

4. Prove convergence/divergence for $a_n = \dfrac{2n^2 + 5n - 6}{n^3}$

5. Prove convergence/divergence for $a_n = \dfrac{3n+5}{6n+11}$.

6. Prove convergence/divergence for $a_n = \dfrac{n\sqrt{n+2} + 1}{n^2 + 4}$

7. Prove convergence/divergence for $a_n = \sqrt{n+1} - \sqrt{n}$

8. Prove convergence/divergence for $a_n = \sqrt{n}\,(\sqrt{n+1} - \sqrt{n})$

9. Suppose a_n assumes only integer values. Under what conditions does this sequence converge?

10. Show that the sequences a_n and $b_n = a_{n+10^6}$ either both converge or both diverge.

11. Let $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$. List the first few terms of this sequence. Prove that the sequence converges to $\left(1 + \sqrt{5}\right)/2$.

12. A subsequence $\{a_{n_k}\}$ is obtained from a sequence $\{a_n\}$ by deleting some of the terms a_n, and retaining the others in their original order. Explain why this implies that $n_k \ge k$ for every k.

13. Which statements are true? Explain your answer.
    - **a.** If $\{a_n\}$ is unbounded then either $\lim_n a_n = \infty$ or else $\lim_n a_n = -\infty$
    - **b.** If $\{a_n\}$ is unbounded then $\lim_n |a_n| = \infty$
    - **c.** If $\{a_n\}$ and $\{b_n\}$ are both bounded then so is $\{a_n + b_n\}$
    - **d.** If $\{a_n\}$ and $\{b_n\}$ are both unbounded then so is $\{a_n + b_n\}$
    - **e.** If $\{a_n\}$ and $\{b_n\}$ are both bounded then so is $\{a_n b_n\}$
    - **f.** If $\{a_n\}$ and $\{b_n\}$ are both unbounded then so is $\{a_n b_n\}$

14. Which statements are true? Explain your answer.
    - **a.** If $\{a_n\}$ and $\{b_n\}$ are both divergent then so is $\{a_n + b_n\}$
    - **b.** If $\{a_n\}$ and $\{b_n\}$ are both divergent then so is $\{a_n b_n\}$
    - **c.** If $\{a_n\}$ and $\{a_n + b_n\}$ are both convergent then so is $\{b_n\}$
    - **d.** If $\{a_n\}$ and $\{a_n b_n\}$ are both convergent then so is $\{b_n\}$
    - **e.** If $\{a_n\}$ is convergent then so is $\{a_n^2\}$

f. If $\{a_n\}$ is convergent then so is $\{1/a_n\}$

g. If $\{a_n^2\}$ is convergent then so is $\{a_n\}$

15. Either give an example of a sequence with the following property or else state a theorem that shows why no such example is possible.

a. a sequence that is monotone increasing but is not bounded

b. a seqence that converges to 6 but contains infinitely many terms that are not equal to 6 as well as infinitely many terms that are equal to 6

c. an increasing sequence that is bounded but is not convergent

d. a sequence that converges to 6 but no term of the sequence actually equals 6.

e. a sequence that converges to 6 but contains a subsequence converging to 0.

f. a convergent sequence with all negative terms whose limit is not negative

g. an unbounded increasing sequence containing a convergent subsequence

h. a convergent sequence whose terms are all irrational but whose limit is rational.

16. How are the notions of accumulation point of a set and limit point of a sequence related? How does this relate to the two formulations of the Bolzano-Weierstrass theorem?

17. Prove: If the Cauchy sequence $\{a_n\}$ contains a subsequence $\{a_{n_k}\}$ which converges to limit $L,$ then the original sequence must also converge to $L.$

18. Show that $1 + a + a^2 + \cdots + a^n = \dfrac{1 - a^{n+1}}{1 - a}$ for $a \neq 1$ and any positive integer $n.$ Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1.$ What is the limit if $|a| \geq 1?$

19. Let $\{s_n\}$ be such that $|s_{n+1} - s_n| \leq 2^{-n}$ for all $n \in N.$ Prove that this is a Cauchy sequence. Is this result true under the condition $|s_{n+1} - s_n| \leq \frac{1}{n}$?

20. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1.$ Find the first few terms of this sequence. Use induction to show that $s_n > \frac{1}{2}$ for all $n.$ Show that this sequence is nonincreasing. Prove that the sequence converges and find its limit.

21. Let $s_1 = 1$ and $s_{n+1} = \left(1 - \dfrac{1}{4n^2}\right)s_n$ for $n \geq 1.$ Determine if the sequence converges and, if it does, find the limit.

22. For each of the following sequences state a theorem which establishes the convergence/divergence:

a. $a_n = n^{1/3}$

b. $a_n = \dfrac{n^2 + 3}{n + 2}$

c. $a_n = (2 + 10^{-n})(1 + (-1)^n)$

- **d**. $a_n = \dfrac{1}{n^2 + 3n + 2}$
- **e**. $a_n = 1 + 2^{-n}$
- **f**. $a_n = \sqrt{n+1}$
- **g**. $a_n = \sum_{k=1}^{n} \frac{1}{k}$ (hint: show that $a_{2n} - a_n$ does not tend to 0 as $n \to \infty$)
- **h**. $\{a_n\} = \left\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots\right\}$

23. Let
$a_1 = 0.1,\ a_2 = 0.101,\ a_3 = 0.101001,\ a_4 = 0.1010010001,\ a_5 = 0.101001000100001, \ldots$
Show that this is a sequence of rational numbers that converges to a limit L. Is the limit L rational?

24. Which statements are true?:

- **a**. a sequence is convergent if and only if all its subsequences are convergent.
- **b**. a sequence is bounded if and only if all its subsequences are bounded.
- **c**. a sequence is monotone if and only if all its subsequences are monotone.
- **d**. a sequence is divergent if and only if all its subsequences are divergent.

25. The sequence $\{a_n\}$ has the property, $\forall \varepsilon > 0,\ \exists N_\varepsilon$ such that $|a_{n+1} - a_n| < \varepsilon$ when $n > N_\varepsilon$. Is the sequence necessarily a Cauchy sequence?