Chapter 4 Differentiation

In the study of calculus of functions of one variable, the notions of continuity, differentiability and integrability play a central role. The previous chapter was devoted to continuity and its consequences and the next chapter will focus on integrability. In this chapter we will define the derivative of a function of one variable and discuss several important consequences of differentiability. For example, we will show that differentiability implies continuity.

We will use the definition of derivative to derive a few differentiation formulas but we assume the formulas for differentiating the most common elementary functions are known from a previous course. Similarly, we assume that the rules for differentiating are already known although the chain rule and some of its corollaries are proved in the solved problems.

We shall not emphasize the various geometrical and physical applications of the derivative but will concentrate instead on the mathematical aspects of differentiation. In particular, we present several forms of the mean value theorem for derivatives, including the Cauchy mean value theorem which leads to L'Hôpital's rule. This latter result is useful in evaluating so called indeterminate limits of various kinds. Finally we will discuss the representation of a function by Taylor polynomials.

The Derivative

Let $f(x)$ denote a real valued function with domain D containing an $\varepsilon-$ neighborhood of a point $x_0 \in D$; i.e. x_0 is an ***interior point*** of D since there is an $\varepsilon > 0$ such that $N_\varepsilon[x_0] \subset D$. Then for any h such that $0 < |h| < \varepsilon$, we can define the ***difference quotient*** for f near x_0,

$$D_h f(x_0) = \frac{f(x_0+h) - f(x_0)}{h} \qquad (4.1)$$

It is well known from elementary calculus (and easy to see from a sketch of the graph of f near x_0) that $D_h f(x_0)$ represents the slope of a ***secant line*** through the points $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$. Then we have

Definition *(Derivative) The function $f(x)$ is said to be* ***differentiable*** *at the interior point $c \in D$ if $\lim_{h\to 0} D_h f(c)$ exists. We denote the value of this limit by $f'(c)$ and refer to this as the* ***derivative*** *of $f(x)$ at $x = c$.*

We may also define the derivative of $f(x)$ at the endpoints of an interval by using one sided limits for $D_h f(x)$. The set of points in D where the limit exists is the domain of a new function $f'(x)$, called the derivative of $f(x)$. We shall also use the notation $\frac{dy}{dx}$ for the derivative of $f(x)$. This derivative may be variously interpreted as:

- the ***slope of the tangent line*** to the graph of $y = f(x)$ at the point, $(x, f(x))$
- the ***instantaneous rate of change*** of $y = f(x)$ with respect to x.

It is important to note that while a secant line is a line through two points whose slope is then well defined, a tangent line is a line characterized by just one point, the point of tangency to the graph. Then the slope of the tangent line is not well defined but can only be defined in terms of a limit procedure. Similarly, the average speed of an object over a given time interval is well defined (it is the distance travelled during the time interval divided by the length of the time interval). Instantaneous speed at a given instant is not well defined but, like the slope of the tangent line to a curve, can only be defined by a limit procedure.

Definition *(Derivatives of higher order) If the function $f'(x)$ is differentiable, then its derivative is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$. This is called the* **second derivative** *of $y = f(x)$. We will denote derivatives of order higher than 2 by $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$ for $n \in N$.*

Example *Derivatives*

(a) In an elementary calculus course, we derive formulas for the derivatives of many elementary functions. The following functions are differentiable at each point where they are defined:

$$
\begin{array}{ll}
f(x) = x^p & f'(x) = px^{p-1} \\
f(x) = Sin(x) & f'(x) = Cos(x) \\
f(x) = Cos(x) & f'(x) = -Sin(x) \\
f(x) = a^x,\ a > 0 & f'(x) = a^x \ln a \\
f(x) = \ln(x) & f'(x) = \frac{1}{x}
\end{array}
$$

(b) The following functions are continuous for all x, but the derivative fails to exist at the indicated points:

- $f(x) = |x|$ is not differentiable at $x = 0$ since $\lim_{h\to 0} D_h f(0)$ fails to exist (see problem 4.4)
- $f(x) = \sqrt{x}$ s not differentiable at $x = 0$ since $\lim_{h\to 0} D_h f(0)$ tends to $+\infty$ as $h \to 0$

The examples in (b) show that there are continuous functions that are not differentiable. The following theorem shows that there are no differentiable functions that fail to be continuous.

Theorem ***4.1*** *If $f(x)$ is differentiable at $x = c$ in D, then $f(x)$ is continuous at $x = c$.*

Rules for Differentiation

In elementary calculus we learn differentiation formulas for commonly occurring functions like the ones in example 4.1(a). In addition, we learn differentiation rules which allow us to compute derivatives of various combinations of functions when the derivatives of the separate functions are known.

Theorem ***4.2*** *(Derivatives of sums, products, quotients) Suppose $f(x)$ and $g(x)$ are differentiable at $x = c$ in their common domain, D. Then*

1) $\frac{d}{dx}[C_1 f + C_2 g](c) = C_1 f'(c) + C_2 g'(c)$ for all constants C_1, C_2

2) $\frac{d}{dx}[f \cdot g](c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$ (4.2)

3) $\frac{d}{dx}\left[\frac{f}{g}\right](c) = \frac{f'(c) \cdot g(c) - f(c) \cdot g'(c)}{g(c)^2}$ if $g(c) \neq 0$.

Theorem ***4.3*** *(Chain Rule) Suppose $f(x)$ is differentiable at $x = c$ and $g(x)$ is differentiable at*

$y = f(c)$. *Then the composed function,* $F(x) = g(f(x))$ *is differentiable at* $x = c$ *and*

$$F'(c) = g'[f(c)] \cdot f'(c).$$

Corollary ***4.4*** *(Derivative of the inverse) Suppose f is strictly monotone and continuous on interval I. Then the inverse function* $g = f^{-1}$ *is strictly monotone and continuous on the interval* $J = f[I]$. *Moreover, if f is differentiable at* $x = c$ *in I and* $f'(c) \neq 0$, *then g is differentiable at* $d = f(c)$ *and* $g'(d) = \dfrac{1}{f'(c)}$.

Corollary ***4.5*** *(Derivative of parametric equations) Suppose* $x = f(t)$ *and* $y = g(t)$ *are differentiable functions of t for* $a \leq t \leq b$, *and that* $f'(t) \neq 0$ *for* $a \leq t \leq b$. *Then*

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

Consequences of Differentiability

Just as there were a number of useful consequences of the property of continuity, there are similar consequences associated with differentiability.

Local Extreme Points

A point c in the domain D of a function $f(x)$ is said to be a ***local maximum*** for $f(x)$ if for some $\delta > 0$, we have $f(c) \geq f(x)$ for all $x \in N_\delta[c] \cap D$. If $f(c) \leq f(x)$ for all $x \in N_\delta[c] \cap D$, then c is said to be a ***local minimum*** for $f(x)$. We say that $c \in D$ is a ***local extreme point*** for $f(x)$ if it is either a local maximum or a local minimum.

Theorem ***4.6*** *(Extreme Points) Suppose* $f(x)$ *is defined and continuous on the interval I and suppose* $c \in I$ *is a local extreme point for* $f(x)$. *Then exactly one of the following assertions must hold:*

i) c is an endpoint of the interval I

ii) c is an interior point of I and $f'(c) = 0$

iii) c is an interior point of I but $f'(c)$ fails to exist

In an elementary calculus course students are taught that in order to find the extreme points of a function it is just a matter of computing the function's derivative and setting it equal to zero. This theorem asserts that the extreme points of a function need not always be located at a point where its derivative is zero.(see problem 4.27).

Mean Value Theorem for Derivatives

There are several versions of one result, all of which are usually referred to as the mean value theorem for derivatives. The simplest version is known as Rolle's theorem. Corollaries 4.8 and 4.9 and theorem 4.10 are generalizations of Rolle's theorem.

Theorem ***4.7*** *(Rolle's theorem) Suppose* $f(x)$ *is continuous on the closed interval,* $[a, b]$, *and f is differentiable on the open interval* (a, b). *Suppose further that* $f(a) = f(b) = 0$. *Then there exists a point,* $c \in (a, b)$, *where* $f'(c) = 0$.

Corollary ***4.8*** *(Mean value theorem for derivatives) Suppose* $f(x)$ *is continuous on the closed* *interval, $[a,b]$, and f is differentiable on the open interval (a,b). Then there exists a point, $c \in (a,b)$, where*

$$f(b) - f(a) = f'(c)(b-a)$$

Corollary *4.9 Suppose $f(x)$ is continuous on the closed interval, $[a,b]$, and f is differentiable on the open interval (a,b). Suppose further that $f'(x) = 0$ for all $x \in (a,b)$. Then f is constant on $[a,b]$.*

Theorem ***4.10*** *(Cauchy's Mean Value theorem) Suppose $f(x)$ and $g(x)$ continuous on the closed interval, $[a,b]$, and are differentiable on the open interval (a,b). Suppose further that $f'(x), g'(x)$ never vanish simultaneously and that $g(a) \neq g(b)$. Then there exists a point, $c \in (a,b)$, where*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Indeterminate Forms

A limit of a quotient in which the numerator and denominator tend to zero simultaneously is said to be an indeterminate limit of the form $\frac{0}{0}$. Often, such limits can be evaluated by means of a corollary of the Cauchy mean value theorem known as ***L'Hôpital's rule***.

Corollary ***4.11*** *(L'Hôpital's rule) Suppose $f(x)$ and $g(x)$ continuous on the closed interval, $[a,b]$, and are differentiable on the open interval (a,b). Suppose further that $g(x) \neq 0$ and $g'(x) \neq 0$ for $x \in (a,b)$ and that $f(a) = g(a) = 0$. Then, either the one sided limits,*

$$\lim_{x\to a} \frac{f(x)}{g(x)} \qquad \text{and} \qquad \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

both exist and are equal, or else both limits fail to exist.

The limits as x tends to a in the corollary may be replaced by limits as x tends to b in the case that $f(b) = g(b) = 0$, and the conclusion holds. Moreover the conclusions still hold in the following situations which are also called indeterminate forms:

a) x tends to ∞ or $-\infty$ and f and g both tend to 0 as x tends to infinity
b) x tends to a, or b and f and g both tend to infinity as x tends to a or b.

Taylor Series Expansions

Corollary 4.8 can be extended to apply to derivatives of order higher than one. The result, known as Taylor's theorem, leads to polynomial expressions which can be used to approximate a given function in a neighborhood of a fixed point.

Theorem ***4.12*** *(Taylor's theorem) Suppose that $f(x)$, together with all its derivatives up to order n, $f'(x), f''(x), \ldots, f^{(n)}(x)$, are all continuous on $[a,b]$. Suppose further that $f^{(n+1)}(x)$ exists at each point of (a,b). Then for each fixed $x_0 \in (a,b)$ and all x in (a,b) there exists a point c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \qquad (4.3)$$

$$\cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

The sum of the first $n+1$ terms in the expression (4.3) is called the $n-th$ degree ***Taylor polynomial*** for $f(x)$, expanded about the point $x = x_0$. The final term in the expression is called the ***Lagrange form of the remainder term*** in the expansion. An integral form for the remainder term will be given in the next chapter. In the special case that $x_0 = 0$, we refer to (4.3) as the ***Maclaurin's expansion*** for $f(x)$. In any case, the Taylor polynomial for *f* can be used to approximate the values for *f* in a neighborhood of x_0.

Example *Maclaurin Expansions and Taylor Polynomials*

(a) The following expansions are some frequently used Maclaurin expansions:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$Sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$Cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

(b) The Taylor series expansion for $f(x) = ln(x)$ about the point $x_0 = 1$ takes the form

$$ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

Note that this is just the Maclaurin series for $ln(x) = ln(1 + x - 1)$.

Solved Problems

The Derivative

Problem 4.1 Use the definition of derivative to compute the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

Solution: For $x > 0$ we have from (4.1)

$$\begin{aligned} D_h f(x) &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \frac{1}{\sqrt{x+h} + \sqrt{x}} \end{aligned}$$

Let h tend to zero in the last expression, we can see that

$$\lim_{h\to 0} D_h f(x) \text{ exists and equals } \frac{1}{2\sqrt{x}}$$

Note, however, that for $x = 0$, we have

$$D_h f(0) = \frac{\sqrt{|0+h|} - 0}{h} = \frac{1}{\sqrt{h}}$$

and the limit as h tends to zero of $D_h f(0)$ does not exist. Then $f(x)$ is not differentiable at $x = 0$.

Problem 4.2 Use the definition of derivative to compute the derivative of $f(x) = Sin(x)$.
Solution: In this case, we have from (4.1),

$$\begin{aligned} D_h f(x) &= \frac{Sin(x+h) - Sin(x)}{h} \\ &= \frac{Sin(x)\,Cos(h) + Sin(h)Cos(x) - Sin(x)}{h} \\ &= Sin(x)\frac{Cos(h)-1}{h} + Cos(x)\frac{Sin(h)}{h} \end{aligned}$$

In problem 2.23 we showed

$$\lim_{h\to 0}\frac{Cos(h)-1}{h} = 0 \qquad \textit{and} \qquad \lim_{h\to 0}\frac{Sin(h)}{h} = 1$$

Then it follows that $\lim_{h\to 0} D_h f(x)$ exists and equals $Cos(x)$.

Problem 4.3 (The exponential function) Use the definition of derivative to compute the derivative of $f(x) = e^x$.

Solution: Let us first indicate how the exponential function is defined. For $A > 0$ and positive integers $m, n,$ we define

$$A^{m/n} = \sqrt[n]{A^m} = \left(\sqrt[n]{A}\right)^m.$$

This defines A^x for x a rational number. For an irrational exponent x, we suppose first that $A > 1$ and let $S(x)$ denote the following set,

$$S(x) = \left\{A^r : r \le x \textit{ and } r \in Q\right\}.$$

Then $S(x)$ has an upper bound since for any integer $m > x,$ it follows from $A > 1$ that $A^r \le A^m$ for all $r \in S(x).$ Then $S(x)$ has a least upper bound and we then define

Definition *For real numbers* $A,\ x$ *with* $A > 0,$

i) if $A > 1$ then $A^x = LUB\ S(x)$
ii) if $A = 1$ then $A^x = 1$
iii) if $A < 1$ then $\left(\frac{1}{A}\right) > 1$ and $A^x = 1/(1/A)^x$

Then A^x is defined for $A > 0$ and all real $x.$ Now let e denote the number between 2 and 3 found in problem 2.14 as the following limit

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \qquad (1)$$

For $f(x) = e^x$ we have from (4.1)

$$D_h f(x) = \frac{e^{x+h} - e^x}{h} = e^x\,\frac{e^h - 1}{h}.$$

If we can show

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1, \qquad (2)$$

then it will show that $f'(x) = e^x$.

In order to show (2), it will suffice to consider h such that $|h| \le \frac{1}{2}$. For $0 < h < \frac{1}{2}$, there exists an integer m such that

$$m \le \frac{1}{h} < m + 1,$$

$$i.e., \qquad \frac{1}{m+1} < h \le \frac{1}{m} \qquad (3)$$

Then

$$e^{\frac{1}{m+1}} < e^h \le e^{\frac{1}{m}} \qquad (4)$$

and, for e given by (1), $\left(1 + \frac{1}{n}\right)^n \le e$, and $e < \left(1 + \frac{1}{n-1}\right)^n$ for n sufficiently large.

This implies

$$\left(1 + \frac{1}{n}\right) < e^{1/n} < \left(1 + \frac{1}{n-1}\right). \qquad (5)$$

Using (4) and (5) together leads to

$$\left(1 + \frac{1}{m+1}\right) \le e^h \le \left(1 + \frac{1}{m-1}\right)$$

and (3) implies

$$\frac{h}{1+h} \le \frac{1}{m+1} \qquad \text{and} \qquad \frac{1}{m-1} < \frac{h}{1-2h}.$$

Then we obtain

$$\frac{1}{1+h} \le \frac{e^h - 1}{h} \le \frac{1}{1-2h} \qquad \text{for} \quad 0 < h < 1/2 \qquad (7)$$

$$\frac{1}{1+h} \ge \frac{e^h - 1}{h} \ge \frac{1}{1-2h} \qquad \text{for} \quad -1/2 < h < 0 \qquad (8)$$

Then (7) and (8) used with theorem 3.3 leads to the result (2).

Problem 4.4 Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Solution: For this function we have

$$D_h f(0) = \frac{|0+h| - 0}{h} = \frac{|h|}{h} = \begin{cases} 1 & if \quad h > 0 \\ -1 & if \quad h < 0 \end{cases}$$

Then $\lim_{h\to 0} D_h f(0)$ fails to exist and then the definition implies that f is not differentiable at zero.

Problem 4.5 Prove Theorem 4.1, that if $f(x)$ is differentiable at $x = c$ in D, then $f(x)$ is continuous at $x = c$.

Solution: The function $f(x)$ is continuous at $x = c$ if

$$\lim_{h\to 0}[f(c+h) - f(c)] = 0$$

But $f(c+h) - f(c) = D_h f(c) h$, and it follows that

$$\lim_{h\to 0}[f(c+h)-f(c)] = \lim_{h\to 0}[D_h f(c)h]$$
$$= \lim_{h\to 0} D_h f(c) \lim_{h\to 0} h$$
$$= f'(c)\,0 = 0.$$

Problem 4.6 Show that the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

is continuous at $x = 0$ but is not differentiable there.
Solution: Since $|\sin(\frac{1}{x})| \leq 1$ for all x, it follows that $|f(x)| \leq |x|$ for all x, and this implies $\lim_{x\to 0} f(x) = 0 = f(0)$. Then f is continuous at $x = 0$. On the other hand,

$$D_h f(0) = \frac{h\sin\left(\frac{1}{h}\right) - 0}{h} = \sin\left(\frac{1}{h}\right)$$

and we have shown previously that $\sin\left(\frac{1}{h}\right)$ tends to no limit as h tends to 0. Then f is not differentiable at $x = 0$.

Note that f is differentiable at all x not equal to zero. Using the rules of differentiation from theorem 4.2, we find

$$f'(x) = \sin\left(\frac{1}{x}\right) + x\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)$$

and, as long as x is not equal to zero, this is a valid expression for $f'(x)$.

Differentiation rules
Problem 4.7 Prove theorem 4.3, the chain rule for differentiation.
Solution: Suppose that $g(x)$ is differentiable at $x = c$ and $f(y)$ is differentiable at $y = g(c)$. If we denote the composed function by $F(x) = f(g(x))$, then the difference quotient has the form

$$D_h F(c) = \frac{f(g(c+h)) - f(g(c))}{h}$$
$$= \frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)}\,\frac{g(c+h) - g(c)}{h}$$

Since $g(x)$ is differentiable at $x = c$, it is continuous there by theorem 4.1. Therefore as h tends to zero, $g(c+h)$ tends to $g(c)$. Moreover, since $f(y)$ is differentiable at $y = g(c)$ it follows that

$$\frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)} \text{ tends to } f'(g(c)) \text{ as } g(c+h) \text{ tends to } g(c)$$

provided $g(c+h) - g(c)$ does not vanish in a neighborhood of $g(c)$. Then

$$\lim_{h\to 0} D_h F(c) = \lim_{h\to 0} \frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)} \lim_{h\to 0} \frac{g(c+h) - g(c)}{h}$$
$$= f'(g(c))\, g'(c).$$

There are more sophisticated arguments that avoid the necessity of assuming $g(c+h) - g(c)$ does not vanish in a neighborhood of $g(c)$. We define a function $G(y)$ by

$$G(y) = \begin{cases} \frac{f(y)-f(y_0)}{y-y_0} & if \quad y \neq y_0 \\ f'(y_0) & if \quad y = y_0 \end{cases}.$$

Then

$$\lim_{x \to c} G(y) = G(y_0)$$

so $G(y)$ is continuous at $y = y_0$. Then it follows from theorem 3.5 that

$$\lim_{x \to c} G(g(x)) = \lim_{y \to y_0} G(y) = G(y_0) = f'(g(c))$$

Now the definition of $G(y)$ implies

$$f(g(c+h)) - f(g(c)) = G(y)[g(c+h) - g(c)]$$

and this, together with the definition of $D_hF(c)$, leads to

$$\lim_{h \to 0} D_hF(c) = \lim_{h \to 0} G(g(c+h)) \lim_{h \to 0} D_hg(c)$$
$$= f'(g(c))\, g'(c). ■$$

Problem 4.8 Prove Corollary 4.4 on the derivative of the inverse of a function.

Solution: Suppose $f(x)$ is strictly monotone and continuous on the interval I. Then by a previous theorem, $g(y) = f^{-1}(y)$ is also strictly monotone and continuous on the interval $J = f[I]$. We want to show that if $f(x)$ is differentiable at c in I and if $f'(c) \neq 0$, then $g(y)$ is differentiable at $b = f(c)$ and moreover, $g'(b) = 1/f'(c)$. To show this, let $F(x)$ denote the composed function

$$F(x) = g[f(x)] = x \quad \text{for all } x \text{ in } I.$$

Then the chain rule implies that

$$F'(c) = g'[f(c)]f'(c) = g'(b)f'(c).$$

But $F'(c) = 1$ for all c in I, hence

$$g'(b) = \frac{1}{f'(c)}. ■$$

Consequences of Differentiability

Problem 4.9 Suppose $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(x)$ has a local extreme point at $x = c$ for some $c \in (a,b)$. Then prove that $f'(c) = 0$.

Solution: We shall suppose that $x = c$ is a local maximum for $f(x)$. The proof in the case of a local minimum is similar. If $x = c$ is a local maximum for $f(x)$, then for all h sufficiently small (h can assume both positive and negative values as long as they are small) we have $f(c+h) \leq f(c)$. But this implies

$$D_hf(c) = \frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{for} \quad h > 0$$
$$D_hf(c) = \frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{for} \quad h < 0$$

Since $\lim_{h \to 0} D_hf(c) = f'(c)$, the previous two lines imply $f'(c) \leq 0$ and $f'(c) \geq 0$. Together, these imply $f'(c)$ must vanish.■

Problem 4.10 Suppose $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b). Suppose further that $f(a) = f(b)$. Then show that there is a point $c,\ a < c < b,$ where $f'(c) = 0$. This is **Rolle's theorem**.

Solution: We have assumed that $f(a) = f(b)$ and if $f(x) = f(a)$ for all x in (a,b) then it is trivially true that $f'(c) = 0$ for $a < c < b$. Suppose then that f is not constant. Then under the hypothesis that $f(x)$ is continuous on $[a,b]$, the extreme value theorem asserts that f assumes is maximum and minimum values at points of $[a,b]$. Since f is not constant, at least one of these extreme values is different for $f(a) = f(b)$. Then it must be that the extreme value occurs at a point that lies strictly between a and b. But then the result of the previous problem implies that $f'(c) = 0$.■

Problem 4.11 Suppose $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b). Then prove that there exists a point $c,\ a < c < b,$ where $f(b) - f(a) = f'(c)(b - a)$. This is the **mean value theorem for derivatives**.

Solution: We begin by defining the function

$$g(x) = f(b) - f(x) - \frac{f(b) - f(a)}{b - a}(b - x) \quad \text{for} \quad a \leq x \leq b.$$

Then the assumptions on f imply that g is continuous on $[a,b]$ and is differentiable on (a,b). In addition, it is evident that $g(a) = g(b)$. Then Rolle's theorem implies the existence of a point c in (a,b) where $g'(c) = 0$. But

$$g'(c) = -f'(c) + \frac{f(b) - f(a)}{b - a} = 0$$

from which it immediately follows that $f(b) - f(a) = f'(c)(b - a)$.■

Problem 4.12 Suppose $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b). Suppose further that $f'(x) = 0$ for all x in (a,b). Then prove that f is constant on $[a,b]$.

Solution: We will suppose that f is not a constant and show that this leads to a contradiction of the hypothesis that $f'(x) = 0$. If f is not constant, then there exist points p and q in $[a,b]$ where $f(p) \neq f(q)$. Then by the MVT for derivatives, there is a point c between p and q such that

$$f'(c) = \frac{f(p) - f(q)}{p - q} \neq 0.$$

This contradicts the hypothesis that $f'(x) = 0$ for all x in (a,b), and we conclude that f is constant on $[a,b]$.

Problem 4.13 Suppose $f(x)$ and $g(x)$ are continuous on $[a,b]$ and are differentiable on (a,b). Suppose also that $f'(x) = g'(x)$ for all x in (a,b). Then prove that $f(x) = g(x) + C$ for some constant C.

Solution: The hypotheses imply that the result of the previous problem can be applied to $F(x) = f(x) - g(x)$, to conclude that $F(x) = C$.

Problem 4.14 Suppose $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b). Suppose further that $f'(x) \geq 0$ for all x in (a,b). Then prove that f is monotone increasing on $[a,b]$.

Solution: For arbitrary points x_1, x_2 such that $a \leq x_1 < x_2 \leq b$, the MVT for derivatives implies the existence of a point c in (x_1, x_2) such that $f(x_1) - f(x_2) = f'(c)(x_1 - x_2) \geq 0$. Then we have shown that $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$, which is to say, f is monotone increasing on (a,b). In the same way we can show that $f'(x) \leq 0$ for all x in (a,b). implies that f is monotone decreasing on $[a,b]$. Taken together, these results say that if $f'(x)$ is of one sign on (a,b) then f is monotone on $[a,b]$.

Problem 4.15 Show that for all $x,y,\ 0 \le y < x \le \pi,\ |\sin x - \sin y| < |x - y|$.
Solution: The MVT for derivatives asserts the existence of a point c, between x and y such that

$$\sin x - \sin y = \cos c\,(x - y).$$

Since c is between x and y, c is not equal to 0 or π, so $|\cos c| < 1$ and the result follows.

Problem 4.16 Show that for all $x \in R,\ e^x \ge 1 + x$, with equality occurring only for $x = 0$.
Solution: The expression is an equality when $x = 0$. For $x > 0$, use the MVT for derivatives applied to $f(x) = e^x$ on the interval $[0,x]$ to write

$$e^x - e^0 = e^c(x - 0) \quad \text{for } 0 \le c \le x,$$

$$\text{i.e.,} \quad e^x - 1 = e^c x > x \quad \text{for } x > 0$$

Problem 4.17 Use the mean value theorem to show that for $h > 0$, and $p > 1$, $(1 + h)^p > 1 + ph$.
Solution: We first rewrite the mean value theorem in an alternative form. Let $h = b - a > 0$ so that $b = a + h$. Then

$$f(b) - f(a) = f'(c)(b - a) \text{ for some } c \text{ between } a \text{ and } b$$

becomes

$$f(a + h) = f(a) + hf'(a + \lambda h) \quad \text{for some } \lambda, 0 \le \lambda \le 1.$$

In particular, for $f(x) = x^p$ and $a = 1$, this becomes

$$(1 + h)^p = 1 + hp(1 + \lambda h)^{p-1} > 1 + ph$$

since $p > 1$ and $1 + \lambda h > 1$.

Problem 4.18 Prove the Cauchy mean value theorem; i.e., Suppose $f(x)$ and $g(x)$ are

continuous on $[a,b]$ and differentiable on (a,b). Suppose further that $f'(x)$ and $g'(x)$ never vanish simultaneously and that $g(a) \ne g(b)$. Then prove there exists a point c in (a,b) where

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Solution: Let

$$\lambda = \frac{f(b) - f(a)}{g(b) - g(a)}$$

and define

$$F(x) = f(x) - f(a) - \lambda[g(x) - g(a)].$$

Then $F(a) = F(b) = 0$ and since f and g are continuous on $[a,b]$ and differentiable on (a,b), the same is true of F. Then Rolle's theorem applies and asserts that there exists a point c in (a,b) such that $F'(c) = 0$. But

$$F'(c) = f'(c) - \lambda g'(c) = 0$$

and since $f'(x)$ and $g'(x)$ never vanish simultaneously, this implies

$$\lambda = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

This is what we wanted to prove. Note that if $g(x) = x$, this reduces to the usual mean value theorem for derivatives.

Problem 4.19 Use the Cauchy mean value theorem to prove L'Hopital's rule; i.e., suppose $f(x)$ and $g(x)$ are continuous on $[a,b]$ and differentiable on (a,b). Suppose further that $f(a) = g(a) = 0$ and that $g(x) \neq 0$ and $g'(x) \neq 0$ for x in (a,b). Then show that the one sided limits

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{and} \quad \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

either both exist and are equal or else both limits fail to exist.

Solution: Suppose

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$$

Then by the definition of (one sided) function limit, it follows that for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|\frac{f'(x)}{g'(x)} - L| < \varepsilon \quad \text{if} \quad a < x < x + \delta.$$

Apply the Cauchy mean value theorem on the interval (a,x) to conclude that there exists a point c, $a < c < x$, such that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

We used the fact that $f(a) = g(a) = 0$ here. Since c is between a and x, it follows that,

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \varepsilon \quad \text{if} \quad a < x < x + \delta.$$

This implies $\lim_{x \to a} \frac{f(x)}{g(x)} = L$, and proves L'Hopital's rule. We have assumed here that both a and L are (finite) real numbers. The proof can be modified to show that the result also holds when a if infinite and when L is infinite.■

Problem 4.20 Use L'Hopital's rule to evaluate the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{2\sqrt{x}}.$$

Solution: The functions $f(x) = 1 - \cos x$, and $g(x) = 2\sqrt{x}$ are both continuous on $[0,1]$ and differentiable on $(0,1)$. Also, $f(0) = g(0) = 0$ so the hypotheses needed in order to apply L'Hopital's rule are satisfied. Note that g is not differentiable at $x = 0$ but this is not needed to apply the rule. We have $f'(x) = \sin x$ and $g'(x) = 1/\sqrt{x}$, so we consider the limit

$$\lim_{x \to 0} \sqrt{x} \sin x = 0.$$

Since this limit exists, the original limit also exists and both equal zero.■

Problem 4.21 Use L'Hopital's rule to evaluate the limit

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}$$

Solution: The functions $f(x) = \sin x - x$, and $g(x) = x^3$ are both continuous on $[0,1]$ and differentiable on $(0,1)$. Also, $f(0) = g(0) = 0$ so the hypotheses needed in order to apply L'Hopital's rule are satisfied. According to L'Hopital's rule, we consider the limit

$$\lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \frac{0}{0},$$

and apparently the new limit must also be treated using L'Hopital's rule; i.e., $p(x) = \cos x - 1$ and q$(x) = 3x^2$ satisfy all the conditions satisfied by f and g so we are justified in applying the rule again. We consider the limit

$$\lim_{x\to 0}\frac{-\sin x}{6x} = -\frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x} = -\frac{1}{6}.$$

Note that we could have applied the rule a third time to the limit $\frac{\sin x}{x}$, but since this limit is already known to exist and equal 1 from previous results, there was no need. It follows that all three limits exist and equal -1/6.

Problem 4.22 Use L'Hopital's rule to evaluate the limits

$$a)\ \lim_{x\to 0}\frac{e^{-1/x^2}}{x^p}\ \text{ for } p > 0 \quad \text{and} \quad b)\ \lim_{x\to 0} x^p \ln x\ \text{ for } p > 0.$$

Solution: Let $f(x)$ be defined as

$$f(x) = \begin{cases} e^{-1/x^2} & if \quad x > 0 \\ 0 & if \quad x = 0 \end{cases}$$

and let $g(x) = x^p$. Then both f and g continuous on $[0,1]$, both are differentiable on $(0,1)$, and each tends to zero as x tends to zero. This ensures that L'Hopital's rule applies. Since

$$f'(x) = \frac{2e^{-1/x^2}}{x^3} \quad \text{and} \quad g'(x) = px^{p-1}$$

we are obliged to consider the limit

$$\lim_{x\to 0}\frac{2e^{-1/x^2}}{px^{p+2}}.$$

This limit is again indeterminate and it is even worse than the original problem. Therefore, consider the following limit, which is equivalent to the original problem,

$$\lim_{x\to 0}\frac{x^{-p}}{e^{-1/x^2}}.$$

This is an indeterminate form of the type $\frac{\infty}{\infty}$. L'Hopital's rule applies to this type of indeterminate form as well so we consider the derived limit

$$\lim_{x\to 0}\frac{-px^{-p-1}}{e^{-1/x^2}(-2/x^3)} = \frac{p}{2}\lim_{x\to 0}\frac{x^{-p+2}}{e^{-1/x^2}}.$$

This limit is also an indeterminate form of the type $\frac{\infty}{\infty}$ but notice that the power of x in the numerator is now closer to zero than in the previous limit. By repeatedly applying the rule a sufficient number of times, the power of x in the numerator will eventually become positive, leading to a limit of the form $\frac{0}{\infty}$, which is equal to zero. Therefore, we have for the limit in $a)$,

$$\lim_{x\to 0}\frac{e^{-1/x^2}}{x^p} = 0 \quad \text{for } p > 0.$$

Note that this limit implies that although both f and g approach zero as x tends to zero, e^{-1/x^2} approaches zero more rapidly than any positive power of x.

The limit in $b)$ is not a quotient and does not seem to lend itself to treatment by L'Hopital's rule. However, if we rewrite the limit in the form

$$\lim_{x\to 0} \frac{\ln x}{x^{-p}}$$

then the limit is an indeterminate form of the type $\frac{\infty}{\infty}$, and L'Hopital's rule applies. We consider the derived limit

$$\lim_{x\to 0} \frac{1/x}{-px^{-p-1}} = \lim_{x\to 0} \frac{x^p}{-p} = 0.$$

Then it follows from the rule, that $\lim_{x\to 0} x^p \ln x = 0$ for $p > 0$. This limit tells us that although $\ln x$ tends to minus infinity as x tends to zero, it tends to minus infinity more slowly than any negative power of x, or equivalently, $\ln x$ times any positive power of x tends to zero as x tends to zero.■

Problem 4.23 Use L'Hopital's rule to evaluate the limits

$$a) \lim_{x\to\infty} \frac{\ln x}{x^p} \text{ for } p > 0 \quad \text{and} \quad b) \lim_{x\to\infty} \frac{x^p}{e^x} \text{ for } p > 0.$$

Solution: Note that in both of these limits, x tends to infinity rather than to a finite limit point. However, L'Hopital's rule can be shown to apply in this case as well, provided the quotients tend to $0/0$ or ∞/∞. Both $a)$ and $b)$ are of this latter type.

Applying the rule to the limit a), we consider the derived limit

$$\lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0 \quad \text{for } p > 0.$$

In limit b) we are led to consider the sequence of derived limits

$$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{x\to\infty} \frac{px^{p-1}}{e^x} = \lim_{x\to\infty} \frac{p(p-1)x^{p-2}}{e^x} = \cdots$$

Each time we apply the rule, the power of x in the numerator decreases by one while the denominator remains unchanged. At some point, (note that p need not be an integer) we arrive at a zero or negative exponent for x in the numerator. At that point, we can conclude that all of the limits in the sequence exist and are equal to zero. That is,

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0 \text{ for } p > 0.$$

We can interpret these two limits to mean that although $\ln x$, e^x and x^p all grow without bound as x tends to infinity, e^x grows more rapidly, and $\ln x$ grows more slowly than any positive power of x.

Problem 4.24 Use L'Hopital's rule to evaluate the limits

$$a) \lim_{x\to 0} x^x \quad \text{and} \quad b) \lim_{x\to\pi/2} (\sec x - \tan x).$$

Solution: Note that neither of these limits involves a quotient so it would seem that L'Hopital's rule does not apply. However, in case $a)$ we can write

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0} x\ln x} = e^0 = 1.$$

Here we have used the fact that e^x is continuous for all x, together with the result from problem 4.22b), which is based on L'Hopital's rule.

For limit b), we write

$$\sec x - \tan x = \frac{1-\sin x}{\cos x}$$

hence

$$\lim_{x\to\pi/2}(\sec x - \tan x) = \lim_{x\to\pi/2}\frac{1-\sin x}{\cos x}$$
$$= \lim_{x\to\pi/2}\frac{-\cos x}{-\sin x} = 0.$$

Taylor's Theorem

Problem 4.25 Prove theorem 4.12, Taylor's Theorem.

Solution: Suppose $f(x)$ satisfies all of the conditions of theorem 4.12, and let x and x_0 denote fixed points in the interval $[a,b]$. Then let I denote the interval whose endpoints are x and x_0. Now, for $t \in I,$ let

$$F(t) = f(x) - f(t) - (x-t)f'(t) - \cdots - \frac{(x-t)^n}{n!}f^{(n)}(t)$$

and $$G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1}F(x_0)$$

Then $G(x) = G(x_0) = 0$ and Rolle's theorem implies the existence of a point $c \in I$ such that $G'(c) = 0.$ That is,

$$G'(c) = F'(c) + (n+1)\left(\frac{x-c}{x-x_0}\right)^n\frac{F(x_0)}{x-x_0}.$$

But it follows from differentiating the expression for F that

$$F'(c) = -\frac{(x-c)^n}{n!}f^{(n+1)}(c),$$

and by combining these last two expressions, we find

$$F(x_0) = -\frac{1}{n+1}\left(\frac{x-x_0}{x-c}\right)^n(x-x_0)F'(c)$$
$$= \frac{1}{n+1}\left(\frac{x-x_0}{x-c}\right)^n(x-x_0)\frac{(x-c)^n}{n!}f^{(n+1)}(c)$$
$$= f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{(n+1)!}$$

Setting this expression for $F(x_0)$ equal to the expression obtained by substituting $t = x_0$ in the original definition for $F(x_0)$, leads to

$$f(x) - f(x_0) - (x-x_0)f'(x_0) - \cdots - \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) = f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{(n+1)!}$$

which is to say,

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \cdots - \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) + f^{(n+1)}(c)\frac{(x-x_0)^{n+1}}{(n+1)!}.$$

But this is what we needed to show in order to prove Taylor's theorem. Note that when $n = 0$ this reduces to

$$f(x) = f(x_0) + (x-x_0)f'(c)$$

which is the mean value theorem for derivatives. In other words, Taylor's theorem is the generalization of the MVT for derivatives to derivatives of order greater than one. Note also that the proof of each of these theorems involves a clever definition of a function F that reduces the proof to an application of Rolle's theorem.

Problem 4.26 Suppose f, f' and f'' are all continuous on (a,b) and that x_0 in (a,b) is an interior local extreme point for f. Suppose also that $f''(x_0) \neq 0$. Then show that necessarily, $f'(x_0) = 0$ and x_0 is a local maximum/minimum depending on whether $f''(x_0)$ is negative/positive.

Solution: We will show first that the hypotheses imply that $f'(x_0) = 0$. Let x denote an arbitrary point in a small neighborhood of x_0 and use Taylor's theorem to write

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(c)(x - x_0)^2 \text{ for some } c \text{ between } x \text{ and } x_0$$

Note that

$$f'(x_0)(x - x_0) + \frac{1}{2}f''(c)(x - x_0)^2 = (x - x_0)\left[f'(x_0) + \frac{1}{2}f''(c)(x - x_0)\right].$$

Thus if

$$|x - x_0| < \varepsilon \leq \left|\frac{f'(x_0)}{f''(c)}\right|$$

then $M(x,x_0) =: f'(x_0) + \frac{1}{2}f''(c)(x - x_0)$ has the same sign as $f'(x_0)$ for all x in a small neighborhood of x_0. Suppose, for example, $f'(x_0) > 0$. In this case

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0)M(x,x_0) \\ &> f(x_0) \text{ when } x > x_0 \\ &< f(x_0) \text{ when } x < x_0. \end{aligned}$$

But this contradicts the assumption that x_0 is a local extreme point for f. Assuming that $f'(x_0) < 0$ leads to the same contradiction and it follows that in order for x_0 to be a local extreme point for f, it is necessary to have $f'(x_0) = 0$. Note that we are assuming here that x_0 is an interior point of (a,b) and that f is differentiable on (a,b) so these results are consistent with theorem 4.6.

Now, given that $f'(x_0) = 0$ and $f''(x_0) \neq 0$, we have, for all x in a neighborhood of x_0,

$$f(x) = f(x_0) + \frac{1}{2}f''(c)(x - x_0)^2 \text{ for some } c \text{ between } x \text{ and } x_0.$$

If $f''(x_0) > 0$, then by the persistence of sign theorem (problem 3.20) it follows that $f''(c) > 0$ and

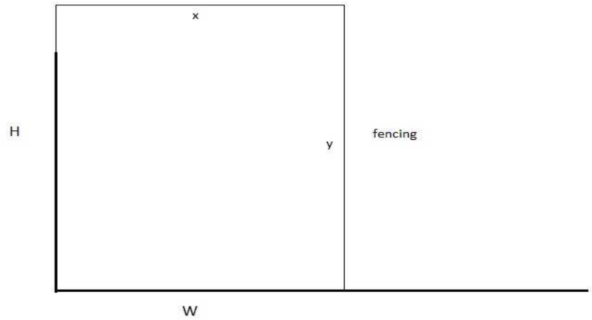
$$f(x) = f(x_0) + \frac{1}{2}f''(c)(x - x_0)^2 > f(x_0)$$

for all x in a neighborhood of x_0. Then x_0 is a local minimum. Similarly, if $f''(x_0) < 0$, we see that

$$f(x) = f(x_0) + \frac{1}{2}f''(c)(x - x_0)^2 < f(x_0)$$

for all x in a neighborhood of x_0 so that x_0 is a local maximum for f.■

Problem 4.27 Using L feet of fencing, build a rectangular pen of maximal area, making use of two fixed walls of height H and width W as shown in the following diagram.



Here $W+H > L.$ Note that there are three different configurations that make use of the fixed walls, one where $y \geq H,$ one where $y < H$ and $x < W,$ and one with $x \geq W.$ In each configuration, find the range of values for x and y and express y in terms of $x.$ The area in each case is $xy,$ so by expressing y in terms of $x,$ this reduces area to a function of one variable. Find the maximum area.

Solution: In the first configuration, $y \geq H$ and $x+y+(y-H) = L,$ so $y = \frac{1}{2}(L+H-x).$ In this case , we have $A(x) = \frac{1}{2}x(L+H-x),$ for $0 \leq x \leq L-H,$ and $H \leq y \leq \frac{1}{2}(L+H).$

In the second configuration, we have $x+y = L$ so $y = L-x$ and $A(x) = x(L-x)$ for $L-H \leq x \leq W.$

In the third configuration, we have $x+y+x-W = L$ so $y = L+W-2x$ and x varies between the value W (where $y = L-W$) and $\frac{1}{2}(L+W),$ (where $y = 0$). Then $A(x) = x(L+W-2x)$ for $W \leq x \leq \frac{1}{2}(L+W).$ Summarizing,

$$A(x) = \begin{cases} \frac{1}{2}x(L+H-x) & for\ \ 0 \leq x \leq L-H \\ x(L-x) & for\ \ L-H \leq x \leq W \\ x(L+W-2x) & for\ \ W \leq x \leq \frac{1}{2}(L+W) \end{cases}$$

For each of the three area expressions, the derivative equals zero for some positive value for $x,$ but this value for x may not lie in the interval where that area expression is valid. In this case, it is necessary to check the endpoints of the intervals where each area function is relevant and evaluate $A(x)$ there. The derivative of $A(x)$ is

$$A'(x) = \begin{cases} \frac{1}{2}(L+H)-x & for\ \ 0 \leq x \leq L-H \\ L-2x & for\ \ L-H \leq x \leq W \\ L+W-4x & for\ \ W \leq x \leq \frac{1}{2}(L+W) \end{cases}$$

and the zeroes of $A'(x)$ in each of the the configurations are $x_1 = \frac{1}{2}(L+H),$ $x_2 = \frac{1}{2}L,$ and $x_3 = \frac{1}{4}(L+W).$

In order to have x_1 lie in $(0, L-H)$ it is necessary to have $3H < L$ and $L < W+H;$ i.e., the max occurs in the first interval only if L, W and H satisfy $3HL < W+H.$ In order to have x_2 lie in $(L-H, W)$ it is necessary to have $L-H < \frac{1}{2}L < W,$ that is $W < L < 2W$ and $H < L < 2H.$ Finally, in order for x_3 to lie in the interval $\left(W, \frac{1}{2}(L+W)\right)$ the variables L, W and H must satisfy $3W < L < W+H.$ If none of these conditions is satisfied, then the max area must occur at one of the two endpoints $x = L-H$ or $x = W$ since $A = 0$ at $x = 0$ and $x = \frac{1}{2}(L+W).$ Note that these two interior endpoints are points where $A'(x)$ does not exist.■

Exercises

1. Use the definition of the derivative in terms of the difference quotient to find the derivative of: (a) $f(x) = \cos x$, (b) $g(x) = x^3$.
2. Let $f(x) = |x-3|$ and compute $D_h f(2)$ and $D_h f(4)$. Discuss $\lim_{h\to 0} D_h f(2)$.
3. Suppose $f(x)$ is differentiable for all x and let $g(x) = f(x+1)f(x-1)$. Find $D_h g(x)$ and $\lim_{h\to 0} D_h g(x)$.
4. Use the definition of the derivative in terms of the difference quotient to show that $f(x) = \sqrt{x-2}$ is not differentiable at $x = 2$.
5. Suppose $f(x)$ is such that for some $C > 0$, $\quad |f(x) - f(y)| < C|x-y|^2$. Then show that $f(x) =$ constant.
6. Use the definition of the derivative as the limit of the difference quotient to find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$.
7. Use the rules of differentiation to find the derivative of $f(x) = \sin\left(\sqrt{x^2 + \cos x + 1}\right)$
8. Explain the difference between differentiation rules and differentiation formulas.
9. Suppose $f \in \mathbb{C}^1(\mathbb{R})$ and $|f'(x)| < 1\ \forall x \in \mathbb{R}$. For $x_0 = 1$ let $x_n = f(x_{n-1})$ *for* $n = 1,2,\ldots$ Show that $\{x_n\}$ is a Cauchy sequence.
10. Suppose $f(x)$ is differentiable for all x, $f(0) = 0$, and $f'(x) \le p < 0$ for all $x \ge 0$. Show that $f(x)$ tends to minus infinity as x tends to plus infinity.
11. Consider the function $f(x) = x^3 \ln x$, for $x > 0$. Can $f(0)$ be given a value so that f is continuous for $x \ge 0$? For what values of x is f differentiable? For what x is f increasing and for which values is f decreasing? Does f have any local extreme points? Are these absolute extreme points?
12. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$$

For what values of p is $g(x) = x^p f(x)$ continuous and differentiable at $x = 0$?

13. For $f(x) = \sqrt{x}$ and x,y in $[0,10]$, does there exist a point c between x and y such that $|f(x) - f(y)| = |f'(c)||x-y|$? Does there exist a constant $M > 0$ such that $|f(x) - f(y)| = M\,|x-y|$ for all x,y in $[0,10]$? What about for all x,y in $[1,10]$?
14. Find all the extreme points for $f(x) = x^3 - 3x$, $-4 \le x \le 4$.
15. Find all the extreme points for $f(x) = |x^3 - 3x|$, $-4 \le x \le 4$.
16. Find all the extreme points for $f(x) = |x^2 - 1|$, $-3 \le x \le 3$.
17. Does Rolle's theorem predict the existence of a point c between 0 and 2π where $f'(c) = 0$ if $f(x) = |\sin x|$?
18. For what values of x is $f(x) = e^x/x$ is decreasing?
19. For what values of x is $f(x) = xe^{-x}$ is decreasing?
20. For what positive values of x is $f(x) = \sin(1/x)$ decreasing?
21. Evaluate the following limits:

$$a)\ \lim_{x\to 1}\left[\frac{4}{1-x^4}-\frac{2}{1-x^2}\right]$$

$$b)\ \lim_{x\to 0}\frac{\tan x - x}{x-\sin x}$$

$$c)\ \lim_{x\to 0}\frac{x}{x+\sin x}$$

22. Find numbers a and p such that

$$a)\ \lim_{x\to 0}\frac{1-\cos x}{ax^p} = 1$$

$$b)\ \lim_{x\to 0}\frac{\sin x - x}{ax^p} = 1$$

23. Suppose f and g are differentiable for all x and $F(x) = f(x)g(x)$. Show that

$$D_hF(x) = g(x+h)D_hf(x) + f(x)D_hg(x)$$

and find $\lim_{h\to 0} D_hF(x)$.

24. Suppose f and g are differentiable for all x and $G(x) = f(x)/g(x)$. Show that

$$D_hG(x) = \frac{1}{g(x)}D_hf(x) + \frac{f(x+h)}{g(x)g(x+h)}D_hg(x)$$

and find $\lim_{h\to 0} D_hG(x)$, assuming that $g(x)$ is never zero.