Differential Calculus of Vector Valued Functions

Functions of Several Variables

We are going to consider scalar valued and vector valued functions of several real variables. For example,

$$z = f(x,y), \qquad w = F(x,y,z), \qquad y = G(x_1,x_2,\ldots,x_n)$$
$$\vec{V} = v_1(x,y,z)\vec{i} + v_2(x,y,z)\vec{j} + v_3(x,y,z)\vec{k}$$

Here (x,y), (x,y,z) or $(x_1,x_2,\ldots,x_n)$ denote the independent variables for the functions f, F and G, and z, w, y and $\vec{V}$ are referred to as dependent variables. A real valued function like $z = f(x,y)$, assigns a unique value to each point (x,y) of a set D of the plane called the domain of the function. The set of values for $f(x,y)$ as (x,y) ranges over all points in the domain is called the range of the function f. For each (x,y) in D the function f assumes a scalar value (i.e., the value is a real number) and f is therefore called a scalar valued function or scalar field. The function $\vec{V} = v_1(x,y,z)\vec{i} + v_2(x,y,z)\vec{j} + v_3(x,y,z)\vec{k}$ assigns to each point (x,y,z) in its domain a unique value (v_1,v_2,v_3) in $3-space$ and since this value may be interpreted as a vector, this function is referred to as a vector valued function or vector field defined over its domain D.

Continuity

Let F denote a real or vector valued function of n real variables defined over domain D. We say that F is continuous at the point P in D if, for each Q that is "close to P", the value of $F(Q)$ is "close to" the value $F(P)$. The precise definition of this vague statement is the following; F is continuous at the point P in D if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|F(P) - F(Q)\| < \varepsilon$ whenever, $\|P - Q\| < \delta$. If F is continuous at each point P in its domain D, we say F is continuous on D.

Variation of a scalar field

Variations in the values of a real valued function of one variable are described in terms of its derivative. For a function of more than a singe variable there are several analogues of the derivative of a function of one variable. For example, let $\vec{u} = p\vec{i} + q\vec{j} + r\vec{k}$ denote a unit vector in R^3 and let $f = f(x,y,z)$ denote a scalar function defined on domain D with $P = (a,b,c)$ a point of D. Then if the following limit exists, it is defined to be the directional derivative of f at P in the direction $\vec{u}$,

$$\nabla_{\vec{u}} f(a,b,c) = \lim_{h\to 0}\frac{f(P+h\vec{u}) - f(P)}{h} = \lim_{h\to 0}\frac{f(a+ph,b+qh,c+rh) - f(a,b,c)}{h}$$

Clearly the directional derivative can be defined for functions of n variables for n other than 3. In the special case that the unit vector is in the direction of one of the coordinate axes, we refer to the derivative as the "partial derivative of f" with respect to that variable along whose axis the unit vector is directed. That is,

if $\vec{u} = (1,0,0)$ $\qquad \nabla_{\vec{u}} f = \dfrac{\partial f}{\partial x} = \partial_x f =$ the partial derivative of f with respect to x

if $\vec{u} = (0,1,0)$ $\quad \nabla_{\vec{u}} f = \dfrac{\partial f}{\partial y} = \partial_y f =$ the partial derivative of f with respect to y

if $\vec{u} = (0,0,1)$ $\quad \nabla_{\vec{u}} f = \dfrac{\partial f}{\partial z} = \partial_z f =$ the partial derivative of f with respect to z

We will use the notations $\dfrac{\partial f}{\partial x} = \partial_x f = f_x$ interchangeably to denote the partial derivative of f with respect to $x,$ with similar notations for the partial derivatives with respect to other independent variables. A function that is continuous on $D,$ together with all its partial derivatives will be said to be a smooth function on $D.$

The Del operator

It is convenient for what follows to define the vector differential operator

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

and to refer to this as the "del" operator. Then the following operations are defined for smooth scalar fields $f(x,y,z)$ or smooth vector fields $\vec{V}(x,y,z)$:

a) $\vec{\nabla} f = \vec{i}\dfrac{\partial f}{\partial x} + \vec{j}\dfrac{\partial f}{\partial y} + \vec{k}\dfrac{\partial f}{\partial z} =$" gradient of f"

b) $\vec{\nabla} \cdot \vec{V} = (\partial_x, \partial_y, \partial_z) \cdot (v_1, v_2, v_3) = \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z} =$"divergence of $\vec{V}$"

c) $\vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}(\partial_y v_3 - \partial_z v_2) - \vec{j}(\partial_x v_3 - \partial_z v_1) + \vec{k}(\partial_x v_2 - \partial_y v_1)$

$=$"curl of $\vec{V}$"

Clearly this definition of the del operator as an operator on functions of three variables can be generalized to an operator on functions of n variables for every n.

Example 1

1. For $f(x,y,z) = x^2 + y^2 + z^2,$ we have $\vec{\nabla} f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$ and for $g(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ we find

$$\partial_x g = \tfrac{1}{2}(x^2 + y^2 + z^2)^{-1/2} 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \text{ etc}$$

Then $\quad \vec{\nabla} g = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$

Note that if we let $\vec{r}$ denote the radial vector, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k},$ then $\|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$ equals the length of $\vec{r}$ and both of the functions f and g are functions of the scalar $R = \|\vec{r}\|$; i.e.

$f(R) = R^2$ and $g(R) = R.$ Then

$$\vec{\nabla} f = 2\vec{r} = 2R\frac{\vec{r}}{R} \qquad \text{and} \qquad \vec{\nabla} g = \frac{\vec{r}}{R}.$$

More generally, for $F = F(R),$ a scalar function of $R,$we have $\vec{\nabla} F = F'(R)\,\vec{u}_r$ where $\vec{u}_r = \frac{\vec{r}}{R}$ is a unit vector in the direction of $\vec{r}.$

2. For $\vec{V}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k} = \vec{r},$ we compute $div\vec{V} = \vec{\nabla} \cdot \vec{V} = \partial_x(x) + \partial_y(y) + \partial_z(z) = 3.$
For $\vec{W} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{R}\vec{r},$ we compute

$$\partial_x\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\sqrt{x^2+y^2+z^2} - x^2(x^2+y^2+z^2)^{-1/2}}{x^2+y^2+z^2} = \frac{1}{R} - \frac{x^2}{R^3}$$

$$\partial_y\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = \frac{1}{R} - \frac{y^2}{R^3} \qquad \partial_z\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{1}{R} - \frac{z^2}{R^3}$$

Then $\quad div\vec{W} = \dfrac{3}{R} - \dfrac{x^2+y^2+z^2}{R^3} = \dfrac{2}{R}$

3. For $\vec{V}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k} = \vec{r},$

$$\vec{\nabla} \times \vec{V} = curl\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0$$

Note also that $div\,\vec{V} = 1 + 1 + 1 = 3.$ Here $\vec{V}$ is an example of a "radial field"; i.e., the vector flow emanates out of the origin.
For $\vec{W} = \vec{\Omega} \times \vec{r}$ where $\vec{\Omega} = (w_1, w_2, w_3)$ is a constant vector,

$$\vec{W} = \vec{\Omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = \vec{i}(w_2 z - w_3 y) + \vec{j}(w_3 x - w_1 z) + \vec{k}(w_1 y - w_2 x)$$

The vector field $\vec{W}$ describes the velocity field for a rigid body rotation about the axis $\vec{\Omega}$ with angular speed equal to $\left\|\vec{\Omega}\right\| = \sqrt{w_1^2 + w_2^2 + w_3^2}.$ In this case, we compute $curl\vec{W} = 2\vec{\Omega}$ and $div\,\vec{W} = 0.$

Each of the quantities $\vec{\nabla} f = grad\,f,$ $\vec{\nabla} \cdot \vec{V} = div\,\vec{V},$ and $\vec{\nabla} \times \vec{V} = curl\,\vec{V}$ has physical meaning. The meanings for the divergence and curl must wait until necessary vector integration results have been derived. However, the meaning of the gradient is contained in the following theorem.

Theorem 1- Let $f = f(x,y,z)$ denote a smooth scalar field defined over D in R^3. Then

a) For a smooth curve C : $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ $a \le t \le b,$ the rate of change of f along C is given by

$$\frac{df}{dt} = \vec{\nabla}f \cdot \vec{V} = \vec{\nabla}f \cdot \vec{T}\,\frac{ds}{dt} \qquad \text{where } \vec{V} = (x'(t), y'(t), z'(t))$$

$$\frac{ds}{dt} = \left\|\vec{V}\right\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

and $\vec{T} = \dfrac{1}{\left\|\vec{V}\right\|}\vec{V}$ =unit tangent vector to C

b) For every unit vector $\vec{u},$ the directional derivative of f in the direction $\vec{u}$ is given by

$$\nabla_u f = \vec{\nabla}f \cdot \vec{u}$$

c) At each point P in D, f increases most rapidly in the direction of $\vec{\nabla}f$ and decreases most rapidly in the opposite direction, $-\vec{\nabla}f$.

d) At each interior point of D where f has a relative max or min, we have $\vec{\nabla}f(P) = \vec{0}$

e) Let S denote a level surface for f (i.e. $S = \{(x,y,z) \mid f(x,y,z) = \text{constant}\}$) Then at each point P of $S,$ the vector $\vec{\nabla}f(P)$ is normal to the surface.

Proof- (a) Suppose $w(t) = f(x(t), y(t), z(t))$. Then the chain rule implies

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \vec{\nabla}f \cdot \vec{V} \qquad \text{where } \vec{V} = (x'(t), y'(t), z'(t))$$

Since $\vec{V}$ is known to be tangent to C at each point of C,

$$\vec{V} = \frac{1}{\left\|\vec{V}\right\|}\vec{V}\,\left\|\vec{V}\right\| = \vec{T}\,\frac{ds}{dt}$$

(b) By definition $\nabla_{\vec{u}}f(a,b,c) = \lim\limits_{h\to 0}\dfrac{f(P + h\vec{u}) - f(P)}{h} = \lim\limits_{h\to 0}\dfrac{f(a + u_1h, b + u_2h, c + u_3h) - f(a,b,c)}{h}$

The mean value theorem for derivatives asserts that for $\lambda_1, \lambda_2, \lambda_3$ such that $0 < \lambda_j < h$,

$$f(P+h\vec{u}) = f(P) + \partial_x f(a+\lambda_1 u_1, b, c)u_1 h + \partial_y f(a, b+\lambda_2 u_2, c)u_2 h + \partial_x f(a, b, c+\lambda_3 u_3)u_3 h$$

Then $\displaystyle\lim_{h\to 0}\frac{f(P+h\vec{u})-f(P)}{h} = \partial_x f(a+\lambda_1 u_1, b, c)u_1 + \partial_y f(a, b+\lambda_2 u_2, c)u_2 + \partial_x f(a, b, c+\lambda_3 u_3)u_3$

i.e., $$\nabla_{\vec{u}} f(a,b,c) = \vec{\nabla} f(a,b,c) \cdot \vec{u}$$

(c) Since $\nabla_{\vec{u}} f(a,b,c) = \vec{\nabla} f(a,b,c)\cdot\vec{u} = \left\|\vec{\nabla} f(a,b,c)\right\| \|\vec{u}\| \cos\theta$

it is clear that $-\left\|\vec{\nabla} f(a,b,c)\right\| \le |\nabla_{\vec{u}} f(a,b,c)| \le \left\|\vec{\nabla} f(a,b,c)\right\|;$
i.e.

$$\begin{aligned}\nabla_{\vec{u}} f(a,b,c) &= \left\|\vec{\nabla} f(a,b,c)\right\| \quad \text{when } \theta = 0 \quad i.e.,\ \vec{u}\|\vec{\nabla} f(a,b,c)\\ &= -\left\|\vec{\nabla} f(a,b,c)\right\| \quad \text{when } \theta = \pi \quad i.e.,\ \vec{u}\|-\vec{\nabla} f(a,b,c)\end{aligned}$$

Then $\vec{\nabla} f(a,b,c)$ is in the direction of most rapid increase for f at $P = (a,b,c)$, while $-\vec{\nabla} f(a,b,c)$ is in the direction that f is most rapidly decreasing.

(d) Suppose $f(x,y,z)$ has an interior extreme point at $P = (a,b,c)$. Then $g_1(x) = f(x,b,c)$ has an interior extreme point at $x = a$, which means $g_1'(x) = \partial_x f(x,b,c) = 0$ at $x = a$. Similarly, $\partial_y f(a,y,c) = 0$ at $y = b$ and $\partial_z f(a,b,z) = 0$ at $z = c$. But then
$\vec{\nabla} f(a,b,c) = \partial_x f(a,b,c)\vec{i} + \partial_y f(a,b,c)\vec{j} + \partial_z f(a,b,c)\vec{k} = \vec{0}$

(e) Let $S = \{(x,y,z)|\, f(x,y,z) = A\}$ for A a constant and for P a point of S, let C denote a curve in S passing through P. Then $\vec{V} = (x'(t), y'(t), z'(t))$ is tangent to C, which is to say, $\vec{V}$ lies in the plane that is tangent to S at P. Since f is constant on S, f is constant on C and it follows from (a) that

$$\frac{df}{dt} = \vec{\nabla} f\cdot\vec{V} = 0 \qquad i.e.\quad \vec{\nabla} f \perp \vec{V}\ \ at\ \ P$$

Since this holds for any C lying in S and passing through P, $\vec{\nabla} f$ must, in fact, be normal to the plane that is tangent to S at P, which is the same as being normal to S.

Identities

Just as there are rules for the derivative of sums and products of differentiable functions of one variable, there are similar rules for applying the del operator to sums and various products of scalar and vector fields.

Theorem 2- Let f and g denote smooth scalar fields on domain D in R^3 and let $\vec{V}$ and $\vec{W}$ denote smooth vector fields on D.

a) $\vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g$; i.e., $grad(f+g) = grad\, f + grad\, g$

b) $\vec{\nabla}\cdot\left(\vec{V}+\vec{W}\right) = \vec{\nabla}\cdot\vec{V} + \vec{\nabla}\cdot\vec{W}$; i.e. $div\left(\vec{V}+\vec{W}\right) = div\vec{V} + div\vec{W}$

c) $\vec{\nabla} \times \left(\vec{V} + \vec{W}\right) = \vec{\nabla} \times \vec{V} + \vec{\nabla} \times \vec{W}$; i.e. $curl\left(\vec{V} + \vec{W}\right) = curl\vec{V} + curl\vec{W}$

d) $\vec{\nabla}(fg) = g\,\vec{\nabla}f + f\,\vec{\nabla}g$;

e) $\vec{\nabla}F(f(x,y,z)) = F'(f)\,\vec{\nabla}f$ for F a smooth function of one variable

f) $\vec{\nabla} \cdot \left(f\,\vec{V}\right) = grad\,f \cdot \vec{V} + f\,div\,\vec{V}$

g) $\vec{\nabla} \times \left(f\,\vec{V}\right) = grad\,f \times \vec{V} + f\,curl\,\vec{V}$

These rules can be seen to hold by using the definitions of the operations together with the product or chain rules for differentiation.

In addition to the rules for the del operator acting on sums and products, there are rules for combining the various operations with the del operator.

Theorem 3-Let f denote a sufficiently smooth scalar field on domain D in R^3 and let $\vec{V}$ denote a similarly smooth vector field on D. Then $grad\,f$ and $curl\,\vec{V}$ are vector fields so the following operations are defined:

$$div\left(\vec{\nabla}f\right) = \vec{\nabla} \cdot \left(\vec{\nabla}f\right) \quad \text{and} \quad div\left(curl\,\vec{V}\right) = \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{V}\right)$$

$$curl\left(\vec{\nabla}f\right) = \vec{\nabla} \times \left(\vec{\nabla}f\right) \quad \text{and} \quad curl\left(curl\,\vec{V}\right) = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right)$$

Similarly, $div\,\vec{V}$ is a scalar field so $grad\left(div\,\vec{V}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right)$ is a defined operation. We have the following identities

a) $\vec{\nabla} \cdot \left(grad\,f\right) = div(grad\,f) = \nabla^2 f = \partial_{xx}f + \partial_{yy}f + \partial_{zz}f$

b) $curl\left(curl\,\vec{V}\right) = grad\left(div\,\vec{V}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{V}$

c) $div\left(curl\,\vec{V}\right) = 0$

d) $curl\left(grad\,f\right) = \vec{0}$

Example 2-

1. Consider $f(x,y,z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{1}{R}$, where $R = \|\vec{r}\|$ *for* $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Then from problem 1.1, we have $\vec{\nabla}f = \dfrac{-1}{R^2}\,\dfrac{\vec{r}}{R} = -\dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{R^3}$

and $\partial_x\left(\dfrac{x}{R^3}\right) = \dfrac{R^3 - 3x^2R}{R^3}$, $\quad \partial_y\left(\dfrac{y}{R^3}\right) = \dfrac{R^3 - 3y^2R}{R^3}$, $\quad \partial_z\left(\dfrac{z}{R^3}\right) = \dfrac{R^3 - 3z^2R}{R^3}$

Therefore,

$$div(grad\,f) = -\frac{R^3 - 3(x^2 + y^2 + z^2)R}{R^3} = 0$$

A smooth function f that satisfies $div(grad\, f) = \nabla^2 f = 0$ at each point of a domain D is said to be harmonic in D.

2. For smooth vector field $\vec{V} = v_1(x,y,z)\vec{i} + v_2(x,y,z)\vec{j} + v_3(x,y,z)\vec{k}$,

$$\vec{\nabla} \times \vec{V} = curl\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}(\partial_y v_3 - \partial_z v_2) + \vec{j}(\partial_z v_1 - \partial_x v_3) + \vec{k}(\partial_x v_2 - \partial_y v_1)$$

Then $$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_y v_3 - \partial_z v_2 & \partial_z v_1 - \partial_x v_3 & \partial_x v_2 - \partial_y v_1 \end{vmatrix} =$$

$$= \vec{i}[\partial_y(\partial_x v_2 - \partial_y v_1) - \partial_z(\partial_z v_1 - \partial_x v_3)] - \vec{j}[\partial_x(\partial_x v_2 - \partial_y v_1) - \partial_z(\partial_y v_3 - \partial_z v_2)] \\ + \vec{k}[\partial_x(\partial_z v_1 - \partial_x v_3) - \partial_y(\partial_y v_3 - \partial_z v_2)]$$

$$= \vec{i}[\partial_x(\partial_x v_1 + \partial_y v_2 + \partial_z v_3) - \nabla^2 v_1] + \vec{j}[\partial_y(\partial_x v_1 + \partial_y v_2 + \partial_z v_3) - \nabla^2 v_2] \\ + \vec{k}[\partial_z(\partial_x v_1 + \partial_y v_2 + \partial_z v_3) - \nabla^2 v_3]$$

$$= (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})div\vec{V} - \nabla^2\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right) = grad\left(div\vec{V}\right) - (div\; grad)\vec{V}$$

3. For smooth vector field $\vec{V} = v_1(x,y,z)\vec{i} + v_2(x,y,z)\vec{j} + v_3(x,y,z)\vec{k}$,

$$\vec{\nabla} \times \vec{V} = \vec{i}(\partial_y v_3 - \partial_z v_2) + \vec{j}(\partial_z v_1 - \partial_x v_3) + \vec{k}(\partial_x v_2 - \partial_y v_1)$$

Then $$div\left(curl\;\vec{V}\right) = \partial_x(\partial_y v_3 - \partial_z v_2) + \partial_y(\partial_z v_1 - \partial_x v_3) + \partial_z(\partial_x v_2 - \partial_y v_1)$$

$$= \partial_{xy} v_3 - \partial_{yx} v_3 + \partial_{zx} v_2 - \partial_{xz} v_2 + \partial_{yz} v_1 - \partial_{zy} v_1$$

Since v_1, v_2 and v_3 are all smooth functions, the mixed partial derivatives are equal and it follows that $div\left(curl\;\vec{V}\right) = 0$. Thus if $\vec{W} = curl\;\vec{V}$ for some smooth vector field $\vec{V}$, then the divergence of $\vec{W}$ must vanish. The converse result can also be shown to be true. That is, if $div\;\vec{W} = 0$, then $\vec{W} = curl\;\vec{V}$ for some smooth vector field $\vec{V}$.

4. Let $f = f(x,y,z)$ denote a smooth scalar field defined over D in R^3. Then

$$\vec{\nabla} \times \vec{\nabla} f = curl\;\vec{\nabla} f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} = \vec{i}(\partial_{yz} f - \partial_{zy} f) - \vec{j}(\partial_{xz} f - \partial_{zx} f) + \vec{k}(\partial_{xy} f - \partial_{yx} f)$$

For f a smooth scalar field, the mixed partial are all equal and $curl\;\vec{\nabla} f = \vec{0}$. Thus if $\vec{W}$ is a so called gradient field (i.e., if $\vec{W} = \vec{\nabla} f$ for some smooth scalar field f) then $curl\ \vec{W}$ vanishes. The converse of the result is also true. If $curl\ \vec{W} = \vec{0},$ then $\vec{W}$ must be the gradient of some smooth scalar field. A vector field whose curl vanishes is said to be a conservative field or irrotational field. Then every conservative field is a gradient field.