Comparison Tests for Convergence or Divergence of Improper Integrals

Consider the improper integral \( \int_{a}^{\infty} f(x) \, dx \)

If \( f(x) \) tends to a nonzero limit \( L \neq 0 \) as \( x \) tends to \( \infty \), then the integral is clearly divergent. On the other hand, if \( L = 0 \), then we must compare \( f(x) \) to a suitable comparison function in order to determine the convergence or divergence of the integral.

Examples

1. \( \int_{1}^{\infty} \frac{\ln x}{x^{3/2}} \, dx \)
   Here \( \lim_{x \to \infty} \frac{\ln x}{x^{3/2}} = 0 \) so we need a comparison function.

   Note first that
   \[
   \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{1/x}{2 \sqrt{x}} = \lim_{x \to \infty} \frac{2}{2 \sqrt{x}} = 0
   \]

   In fact, L’Hôpital’s rule shows: for any \( p > 0 \) \( \lim_{x \to \infty} \frac{\ln x}{x^p} = 0 \)

   This is just the "relative rate of growth" result that says
   
   - \( \ln x \) grows more slowly at infinity than any positive power of \( x \); i.e.,
     for any \( p > 0 \) \( \lim_{x \to \infty} \frac{\ln x}{x^p} = 0 \)

   We use this fact in connection with
   
   - \( \int_{a}^{\infty} \frac{1}{x^p} \, dx \) converges for any \( p > 1 \).

   Then we can combine these two results in finding a comparison function for
   \( f(x) = \frac{\ln x}{x^{3/2}} = \frac{\ln x}{x^{1/4}} \cdot \frac{1}{x^{5/4}} \)

   We have: (i) \( \frac{\ln x}{x^{1/4}} \to 0 \) as \( x \to \infty \) and (ii) \( \int_{a}^{\infty} \frac{1}{x^{5/4}} \, dx \) converges

   The fact that \( \lim_{x \to \infty} \frac{\ln x}{x^{1/4}} = 0 \) implies there exists a constant \( M > 0 \) such that \( \frac{\ln x}{x^{1/4}} < M \) for all \( x > a \).

   To see why this must be true, look at the graph of \( f(x) = \frac{\ln x}{x^{1/4}} \). We see that the graph increases from \( f(1) = 0 \), reaches a maximum value and then decreases steadily as \( f(x) \to 0 \) as \( x \to \infty \). Since the maximum value is less than 2, we can see that \( \frac{\ln x}{x^{1/4}} < 2 \) for all \( x > 1 \).
Now we have \[ \int_1^\infty \frac{\ln x}{x^{1/4}} \, dx = \int_1^\infty \frac{\ln x}{x^{1/4}} \frac{1}{x^{5/4}} \, dx < \int_1^\infty \frac{2}{x^{5/4}} \, dx \]

and since \[ \int_1^\infty \frac{2}{x^{5/4}} \, dx \]
is convergent, the integral in question must also be convergent.

We could have equally well split \[ \frac{\ln x}{x^{3/2}} = \frac{\ln x}{x^{1.5}} \]
into the pieces \[ \frac{\ln x}{x^{1.1}} \frac{1}{x^{1.4}} \], since \[ \frac{\ln x}{x^{1.1}} \to 0 \] as \( x \to \infty \) (so it is bounded by some constant \( M \)) and \[ \int_1^\infty \frac{1}{x^{1.4}} \, dx \]
is convergent by the p-test.

The same approach could be used on the integral \[ \int_1^\infty \frac{\ln x}{x^{1.001}} \, dx. \]

We would just write \[ \frac{\ln x}{x^{1.001}} = \frac{\ln x}{x^{1.0005}} \frac{1}{x^{1.0005}} \leq M \frac{1}{x^{1.0005}} \] for \( x > a \)

Then \[ \int_1^\infty \frac{\ln x}{x^{1.001}} \, dx \leq \int_1^\infty \frac{M}{x^{1.0005}} \, dx \] and since \( 1.0005 > 1 \), the second integral converges.

On the other hand, \[ \int_1^\infty \frac{\ln x}{\sqrt[3]{x}} \, dx \]
is divergent. It is true that \[ \frac{\ln x}{\sqrt[3]{x}} \to 0 \] as \( x \to \infty \) but this integrand does not tend to zero fast enough to make the integral converge. To see this, note that

\[ \frac{\ln x}{\sqrt[3]{x}} > \frac{1}{\sqrt[3]{x}} \] for \( x > 3 \)

Then \[ \int_3^\infty \frac{\ln x}{\sqrt[3]{x}} \, dx > \int_3^\infty \frac{1}{\sqrt[3]{x}} \, dx \]
and since the second integral is divergent, the first one is too.

2. \[ \int_1^\infty \frac{4}{x^4} \, dx \]
The integrand here goes to zero as \( x \to \infty \), so the question is, does it go to zero fast enough to make the integral convergent. To find a suitable comparison function for this integral, we use the following facts.
Another "relative rate of growth" result says

- If \( a > 1 \), then \( a^x \) grows more rapidly at infinity than any positive power of \( x \); i.e., for any \( p > 0 \)
  \[
  \lim_{x \to \infty} \frac{x^p}{a^x} = 0
  \]

We use this fact in connection with

- \[
  \int_1^\infty \frac{1}{ax} \, dx = \int_1^\infty a^{-x} \, dx = \int_1^\infty e^{-kx} \, dx \quad (k = \ln a)
  \]
  converges for any \( a > 1 \).

Then we can combine these two results in finding a comparison function for \( \int_1^\infty \frac{x^4}{4^x} \, dx \).

We write \( \frac{x^4}{4^x} = \frac{x^4}{2^x} \). Then \( \frac{x^4}{2^x} \to 0 \) as \( x \to \infty \), and \( \int_1^\infty \frac{1}{2^x} \, dx \) is convergent.

The fact that \( \lim_{x \to \infty} \frac{x^4}{2^x} = 0 \) implies there exists a constant \( M > 0 \) such that \( \frac{x^4}{2^x} < M \) for all \( x \) large enough.

Then \( \int_1^\infty \frac{x^4}{4^x} \, dx = \int_1^\infty \frac{x^4}{2^x} \frac{1}{2^x} \, dx \leq \int_1^\infty \frac{M}{2^x} \, dx \) and since the last integral is convergent, the integral in question is too.

3. \( \int_1^\infty \sin(1/x) \, dx \)

Here \( \lim \sin(1/x) = \sin\left(\lim 1/x\right) = 0 \), but in order to tell if the integrand goes to zero fast enough to make the integral convergent, we need a comparison function.

What about the comparison \( \sin(1/x) \) with \( \frac{1}{x} \)? (How we thought of this comparison will become clear after we have studied Taylor’s series)

In order to see how these two functions compare as \( x \) tends to \( \infty \), note that,

\[
\lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \to \infty} \frac{\cos \left( \frac{1}{x^2} \right)}{-\frac{1}{x^2}} = \lim_{x \to \infty} \cos \frac{1}{x} = 1
\]

Now we apply the following theorem:

- If \( \lim \frac{f(x)}{g(x)} = L \), where \( L \) is a finite number and not zero, then the integrals
  \[
  \int_1^\infty f(x) \, dx \quad \text{and} \quad \int_1^\infty g(x) \, dx
  \]
  either both converge or both diverge.

Since \( \int_1^\infty \frac{1}{x} \, dx \) is divergent, \( \int_1^\infty \sin(1/x) \, dx \) is also divergent.

Now consider \( \int_1^\infty \sin(1/x^p) \, dx \). In view of the previous example, we would expect to compare the integrand with \( \frac{1}{x^p} \). That is,
\[
\lim_{x \to \infty} \frac{\sin(1/x^p)}{1/x^p} = \lim_{x \to \infty} \cos \frac{1}{x^p} \frac{d}{dx} \left( \frac{1}{x^p} \right) = \lim_{x \to \infty} \cos \frac{1}{x} = 1
\]

Then the theorem implies that \(\int_1^\infty \sin(1/x^p)\,dx\) and \(\int_1^\infty 1/x^p\,dx\) are either both convergent or both divergent. Then the integrals converge for \(p > 1\) and diverge otherwise.

4. \(\int_a^\infty \frac{1}{(\ln x)^2}\,dx\) \(a > 1\). Here \(\lim_{x \to \infty} \frac{1}{(\ln x)^2} = 0\) so, to determine the convergence or divergence, we need a comparison function.

We use again, the fact that for any \(p > 0\) \(\lim_{x \to \infty} \frac{\ln x}{x^p} = 0\). This implies that there is a constant \(M > 0\) such that for any \(p > 0\) \(\frac{\ln x}{x^p} < M\) \(\text{for } x > a\)

Then for any \(p > 0\) \(\frac{1}{M^2} \frac{1}{x^{2p}} < \frac{1}{(\ln x)^2}\) \(\text{for } x > a\)

and \(\int_a^\infty \frac{1}{x^{2p}}\,dx\) is divergent if \(2p \leq 1\).

In particular, taking \(p = 1/2\) we have \(\frac{1}{M^2} \int_a^\infty \frac{1}{x^{2p}}\,dx = \frac{1}{M^2} \int_a^\infty \frac{1}{x}\,dx < \int_a^\infty \frac{1}{(\ln x)^2}\,dx\)

and since \(\int_a^\infty \frac{1}{x}\,dx\) is divergent, the integral in question is also divergent.

More generally, if we have \(\int_a^\infty \frac{1}{(\ln x)^q}\,dx\) \(\text{where } q > 0\), then for any \(p > 0\) \(\frac{1}{M^q} \frac{1}{x^{2pq}} < \frac{1}{(\ln x)^q}\) \(\text{for } x > a\)

and for any \(p > 0\) \(\frac{1}{M^q} \frac{1}{x^{2pq}} < \frac{1}{(\ln x)^q}\) \(\text{for } x > a\).

If we choose \(p = 1/q\), then \(\frac{1}{M^q} \int_a^\infty \frac{dx}{x^{2p}} = \frac{1}{M^q} \int_a^\infty \frac{dx}{x} < \int_a^\infty \frac{dx}{(\ln x)^q}\)

and since \(\int_a^\infty \frac{1}{x}\,dx\) is divergent, the integral in question is also divergent.