3. Finite domain of dependence

\[ ds^2 = \frac{1}{c^2(x)} \, dx^2, \quad n=3, \quad dx^2 = dx_1^2 + dx_2^2 + dx_3^2 \]

\[ g = \frac{1}{c^2(x)} \, dx^2 \]

\[ L = \text{length of } Y \text{ is a } C^1 \text{-curve joining } x \text{ and } y \]

\[ dg(x,y) = \inf_{Y \in C^1(x,y)} L(Y), \quad L^2(Y) = \int_0^1 \frac{1}{c^2(y(t))} \, |Y'(t)|^2 \, dt \]

Exercise - show \( dg \) defines a metric

\[ dg(x,A) = \inf_{y \in A} dg(x,y) \]

Thm: Let \( u \) be a sol of the wave eq. Then \( u(x,t) = 0 \) if \( d(x, \text{supp } u) > |t| \)

(i.e., if you wait long enough, you don’t see the wave.)

Example if \( n = 1 \), \( u(x,t) = \frac{1}{2} \left( f(x-ct) + f(x+ct) \right) \) (propagates along characteristic curves)

if \( n = 2 \)

\[ u(x,t) = \text{const} \int \frac{f(y)}{\sqrt{c t^2 - |y|^2}} \, dy \]

if \( n = 3 \)

\[ u(x,t) = B \int \frac{f(x + ct w)}{|w|^2} \, dw \]

(propagates along spheres in \( 3 \)-D)

4. Propagation of singularities

\( u \in D'(\Omega), \) distribution, \( \Omega \) open set in \( \mathbb{R}^3 \)

\( \text{sing supp } u = \text{complement of largest open set where } f \)

is smooth

This is rough b/c \( S_0(x) + H(x) \) have the same

\( \text{singsupp}, B^3 \).
Prop: Let \( u \in D'(\mathbb{R}^n) \) w/ compact \( \text{supp.} \) (Notation \( u \in E'(\mathbb{R}^n) \))
\[ u \in C_0^\infty(\mathbb{R}^n) \iff \hat{u}(\xi) = O(\xi_j^{-\infty}) \text{ } \forall j \in \mathbb{N} \]
(i.e., decays faster than any poly.)

This led Hormander to define wavefront set

**Def:** \((x_0, \xi_0) \notin WF(u)\) if \( \exists \text{ U, nhd. of } x_0 + \xi \)

\( V \text{ nhd. of } \xi_0 \) (excl. 0) \text{ s.t.}

\[ \hat{\phi u}(\pm \xi) = O(t^{-\infty}) \text{ } \xi \in V, \forall \phi \in C_0^\infty(U) \]

**Ex:** \( WF(S_{x_0}) = \{ \} \)

such that \( \hat{\phi S_{x_0}} = \langle \phi, S_{x_0} \rangle e^{-i \cdot \xi} = \phi(0) \)

If \( \phi(0) \neq 0\), this does not decay in any \( \text{dir.} \xi \)

So \( WF_{x_0} = \{ (0, \xi_0) : \xi_0 \neq 0 \} \)

(Not very useful)

**Ex:** \( \mathbb{R}^2 : x = (x_1, x_2) \) \( S_{x,0} \)

\[ \langle S_{x_0}, \phi \rangle = \int \phi(0, x_2) \text{dx}_2 \]

(\( \phi \in C_0^\infty(\mathbb{R}^2) \))

What is \( WF(S_{x_0}) ? \) \( x \text{ is line } \perp \text{ to } y \)-axis

\[ \{ (0, x_2), (x_0, 0), x_0 \neq 0 \} \]

More generally, it will be the set of vec's normal to the surface (this is even true on manifolds)

**Exercise:** Let \( F \) be closed, conic set \( \subset \mathbb{R}^n - \{0\} \)

(Conic means \( (x, \lambda) \in F \Rightarrow (x, \lambda^2) \in F \) where \( \lambda > 0, |x| > 1 \))

Find \( u \in D'(\mathbb{R}^n) \) s.t. \( WF(u) = F \).

Hint: need a countable dense set of pts + take Dirac measures
Prop: \( WF(u) \subset X \times \mathbb{R}^n \setminus \{0\} \)

\[
\downarrow \\
\text{Sing supp} \\
\downarrow \\
X
\]

Reg for WF sets Grijish & Sjöstrand - Microlocal Analysis of Diff. Op's

Prop of Sing's (Hörmander)
\[
t^2 u - c^2(x) \Delta u = 0, \quad u \in D' (\mathbb{R}^n \times (0,T)) \\
c \in C^\omega (\mathbb{R}^n), \quad c = 1 \quad \text{if } |x| > R
\]

Princ symbol of wave eq \( \Pi(x,t,s,T) = -t^2 + c^2(x) |s|^{2} \)

Elliptic pt is a pt \( (x_0, t_0, s_0, T_0) \) s.t. \( \Pi(x_0, t_0, s_0, T_0) \neq 0 \).
Such pts are not in \( WF(u) \).

Char. variety \( \{(x,t,s,T) : \text{Princ. symbol } \Pi = 0\} \)
\[ t^2 = c^2(x) |s|^2 \text{ is light cone in phase sp} \]

On char. variety
\[ V_\Pi = \text{Hamiltonian vec.field assoc. to } \Pi, \text{princ. sym.} \]
\[ = \left( \begin{array}{c}
\frac{\partial H_\Pi}{\partial t} \\
\frac{\partial H_\Pi}{\partial s} \\
\end{array} \right), \quad \left( -\frac{\partial H_\Pi}{\partial t}, -\frac{\partial H_\Pi}{\partial x} \right) \]

\[ H_\Pi = t^2 - |s|^2 c^2(x) \]

Curves in phase space, the integral curves of \( V_\Pi \)
\[
\frac{dt}{ds} = \frac{\partial H_\Pi}{\partial t} \\
\frac{dx}{ds} = \frac{\partial H_\Pi}{\partial x} \\
\frac{ds}{ds} = -\frac{\partial H_\Pi}{\partial s} \\
\frac{d\gamma}{ds} = -\frac{\partial H_\Pi}{\partial t} \\
\frac{d\gamma}{ds} = -\frac{\partial H_\Pi}{\partial x} \\
\left\{ \text{bicharacteristic curves} \right\} \text{ (Classical Mechanics)}
Remark: Projections of bichar’s on x-sp are geodesics of $g = \frac{1}{c^2(x)} \, dx^2$ (nontrivial exercise).

Geodesics - curves that minimize length locally.

Note: if $c = 1$, bichar’s are just lines.

Thm. - (Hörmander) If $(x_0, t_0, \xi_0, T_0) \in WF(u)$, then whole bichar curve through $(x_0, t_0, \xi_0, T_0)$ is in $WF(u)$. (Prop. of sing’s along geodesics)

If you can see the sing’s in the data, the proof is likely to be stable. If not it is likely to be unstable.

$T_1 = \text{supremum of length of geodesics}$

If $T > T_{1/2}$, every sing. of $f$ has reached $\partial \Omega$.

We can see for $T > T_{1/2}$ all sing’s of $f$ at $\partial \Omega$. (Visible sing’s)

Then the IP is stable.

Assume no trapping

$\|f\|_{H^0(\Omega)} \leq C \|\Lambda f\|_{H^1(\mathbb{R}^n \times (0, T))}$

If $T > T_1$, then all sing’s of $f$ have left the domain.
5. Tataru's Thm (CPDE, 1995)

\[ \partial_t^2 u - c^2(x) \Delta u = 0 \quad \text{in} \quad \Omega \times (0, T) \]
\[ u \in H^1(\Omega \times (0, T)) \]

\[ \frac{\partial u}{\partial n} \bigg|_{\Gamma \times (0, T)} = 0 \]
\[ n \times (0, T) \]

\[ u = \text{unit outward normal to } \partial \Omega \]

\[ \Rightarrow u = 0 \text{ on } K_T = \text{double domain of influence} \]

Cor: Suppose \( u = 0 \) on a neighborhood of \( \{ x_0 \} \times (0, T) \)

Then \( u = 0 \) for \( |t| \leq d(x_0, x) \leq T \)

IP is to study \( \Delta f = \frac{\partial u}{\partial n} \bigg|_{\partial \Omega \times (0, T)} \)

a) Uniqueness

Suppose \( \Delta f = 0 \), \( S_1 > T_0 = \sup_{x \in \Omega} \| x \|_2 \), \( x \in \Omega \)

(All waves have arrived at \( \partial \Omega \))

(Prop. 3 - Finite speed of prop)

Uniqueness for solv. of BVP for wave eq. \( u = 0 \) on \( \partial \Omega \times (0, T) \)

Then \( \frac{\partial u}{\partial n} \bigg|_{\partial \Omega \times (0, T)} = 0 \).

Tataru's thm \( \Rightarrow u = 0 \) on \( \partial \Omega \times (0, T) \) \( \Rightarrow f = 0 \). (Exercise)

b) Reconstruction

(Modified time reversal)

Standard time reversal - \( \frac{\partial u}{\partial \tau} \bigg|_{\tau = 0} = \frac{\partial u}{\partial t} \bigg|_{t = T} = 0 \) on \( \partial \Omega \)

Solve wave eqn backwards.

However, this creates artifacts b/c data may not be 0 at \( t = T \).

Modified:

Take \( \frac{\partial^2 u}{\partial t^2} \bigg|_{t = T} = 0 \)

Energy \( E_{\Omega} (u(T)) = \int_\Omega |\nabla u(x, T)|^2 \, dx + \int_{\partial \Omega} \frac{\partial u(x, T)}{\partial n_1} \, ds \)
Want to minimize the "energy" of sol at time $t = T$, and $\int |\nabla v|^2$ is taken at $\phi$ where $\Delta \phi = 0$

$$v|_{t=T} = \phi(T)$$

(exercising - not hard)

Solve

$$\left( \partial_t^2 - c^2(x) \Delta \right) v = 0 \quad \text{on} \; \mathbb{R} \times (0, T)$$

$$v|_{\partial \mathbb{R} \times (0, T)} = h_0(x, t)$$

$$\partial_x v|_{t=T} = 0 \quad \Delta \phi = 0 \quad \text{on} \; \partial \mathbb{R}$$

$$\phi|_{t=T} = h_0(T)$$

Define $Av = v|_{t=0}$ (solve it backwards)

We have $A^2 f = f$ for $f \in H_0(D)$

Than $\|K\|_{H_0(D) \to H_0(D)} < 1$ for $T > T_{1/2}$

(You can invert the series)

(b) $T > T_{1/2} \Rightarrow K$ is comp+$^+$

(reason is that all sing's have left, so $K$ has a smooth kernel. So we can get "explicit" formula for $T > T_{1/2}$)

$$f = \sum_{m=0}^{\infty} k^{m} A^{m} f \quad \text{Neumann series}$$

Sketch of proof:

Prop: $\langle u(T, x) - \phi(x), \phi \rangle_{H^2_0(\mathbb{R})} = 0$

pf: $\int_{\mathbb{R}} \nabla (u(T, x) - \phi(x)) \nabla \phi(x) \, dx$

$$\partial_t^2 \phi = \int_{\mathbb{R}} (u(T, x) - \phi(x)) \Delta \phi = 0$$

$$\|w(T)\|_{H^2_0(\mathbb{R})} \leq \|w(T)\|_{H^2_0(\mathbb{R})} + \|\phi\|_{H^2_0(\mathbb{R})}$$

$$\|w(T) - \phi\|_{H^2_0(\mathbb{R})} \leq \|w(T)\|_{H^2_0(\mathbb{R})} + \|\phi\|_{H^2_0(\mathbb{R})}$$
\[ Kf = w(0, \cdot) \] where \( w \) solves

\[
\left( \frac{\partial^2}{\partial t^2} - c^2(\mathbf{x})\Delta \right) w = 0
\]

\[
\left. \frac{\partial w}{\partial t} \right|_{t=0} = 0
\]

\[
w(T_1, \cdot) = u(T, \cdot) - \Phi, \quad \frac{\partial w}{\partial t} (T_1, \cdot) = 0
\]

\[
\|Kf\|_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} (\text{energy by conservation of energy})
\]

\[
\leq \int_{\mathbb{R}} E_u(T, \cdot) - E_{\Phi} u(T, \cdot) = E_{\Phi} u(T, \cdot)
\]

Indicates when the integral is taken

\[
\|f\|_{H^1_0(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}
\]

\[
\text{So,} \quad \|Kf\|_{H^1_0(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}
\]

Want to show that \( \|K\|_{L(H^1_0(\mathbb{R}), H^1_0(\mathbb{R}))} < 1 \)

\[ \text{S not: Assume } \exists K \neq 0 \text{ s.t. } \|Kf\|_{H^1_0(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}
\]

Then every norm is an equality,

In particular, \( E_{\Phi} u(T, \cdot) = E_{\Phi} u(T, \cdot) \)

But this implies \( u = 0 \) outside \( \mathbb{R} \times (0, T) \)

By finite speed of propagation, \( u = 0 \) if \( d(x, 2\pi) > 3T \)

Also \( u = 0 \) if \( d(x, 2\pi) > 1T \)

From this, we can conclude that \( u = 0 \) when

\[
d(x, 2\pi) > T_2, \quad -T_2 \leq t \leq 3T_2 \quad \text{(easy exercise)}
\]

\[
\Rightarrow u = 0 \quad \text{for} \quad d(x, 2\pi) > T_2, \quad -3T_2 \leq t \leq 3T_2
\]

(since soil can be reflected in time)

By Tataru's theorem, \( u = 0 \) on \( \mathbb{R} \times (0, T) \)

\[
\Rightarrow f = 0, \quad \text{a contradiction}.
\]
If $T > T_1$, $K$ is smoothing, therefore exist in $H_0(\mathbb{R})$ and $\mathcal{L}(H_0(\mathbb{R}), H_0(\mathbb{R}))$

$\Rightarrow \|K\| < 1$

$\Rightarrow$ Neumann series converges

A more delicate argument shows that $T > \frac{T_1}{2}$ is enough.

Appl. Brain imaging (proposed by Lihong Wang)

Total internal reflection – trapped singularities

When geodesics go straight then only need $\sim 1^{st}$ term in Neumann series

However in trapped case (maybe partially trapped?)

ie, many reflections before exiting) need many terms