Simplicial Sets

Motivation: Simplicial Complexes $\rightarrow$ Homology

Simplicial Sets $\rightarrow$ Homotopy

Basics: Defn. The simplex category, $\Delta$.

- (Nonempty) finite ordered sets, $\{0, \ldots, n\}$ (simplices)
- $\rightarrow$ (weakly) order-preserving functions.

Generating morphisms: there are $n+1$ of each of:

- $D_i : \{0, \ldots, n\} \rightarrow \{0, \ldots, n+1\}$
  - by $[0, \ldots, n] \mapsto [0, \ldots, i, \ldots, n+1]$

- $S_i : \{0, \ldots, n\} \rightarrow \{0, \ldots, i, \ldots, n\}$
  - by $[0, \ldots, n] \mapsto [0, \ldots, i, 1, \ldots, n]$

Eq.: $D_i \circ S_i = 1_{\{0, \ldots, n\}}$

Obvious fact: all morphisms are compositions of $S_i$ and $D_i$.

- Opposite category $\Delta^{op}$ has morphisms generated by $D_i^{op} = d_i$ and $S_i^{op} = s_i$

Eq.: $d_i \circ d_i = d_{i-1} d_i$

Not set maps. $d_i$ = "fusing" $s_i$ = "degeneracy"

Follow some rules:

- $d_i d_j = d_i d_{j-1}$ if $i < j$

- $d_i s_j = s_{j-1} d_i$ if $i < j$

- $s_i s_j = s_i s_{j-1}$ if $i < j$

- $d_i s_i = d_i$ if $i < j$

- $s_i d_i = s_i$ if $i < j$

- $d_i s_i = 1_{\{0, \ldots, i\}}$ if $i < j+1$

Calculations:

- $d_0 d_2 [0, 1, 2] = d_0 [0, 1] = [1, 2]$

- $s_2 s_2 [0, 1, 2] = s_2 [0, 1, 2] = [0, 1, 2]$

- $s_2 s_1 [0, 1, 2] = s_2 [0, 1, 2] = [0, 1, 2]$

- $s_1 s_0 [0, 1, 2] = s_0 [0, 1, 2] = [0, 1, 2]$
Defn: A simplicial set is a functor $\mathcal{X}: \Delta \to \text{Set}$. The category $s\text{Set}$ is the functor category $\text{Set}^{\Delta^{o}}$.

Eq. Write $\mathcal{X}_i$ for $\mathcal{X}[i]$. The standard 1-simplex as a $s\text{Set}$ is

$\mathcal{X}_0 = \{[0], [1]\}$ \hspace{1cm} $\mathcal{X}_1 = \{[0,0], [0,1], [1,1]\}$ \hspace{1cm} $\mathcal{X}_2 = \{[0,0,0], [0,0,1], [0,1,1], [1,1,1]\}$, etc.

"Degenerate simplices". The standard n-simplex is similar: $\Delta^n = [0, \ldots, n]$

Eq. (Singular Set) $\Lambda(\mathcal{X})$ w/ $\mathcal{X} \in \text{Top}$. The set $\Lambda(\mathcal{X})_i = \text{set of its functions}$

$|\Delta^i| \to \mathcal{X}$. Morphisms:

- $d_i: \sigma = \sigma \upharpoonright \text{ith face of } \Delta^n$
- $s_i: \sigma = \sigma \circ \text{collapse of ith vertex}$

Eq. (Nerve) $\mathcal{C}$ a small category. The nerve of $\mathcal{C}$, $N(\mathcal{C})$ is $s\text{Set}$ with

$N_0 = \text{Ob}(\mathcal{C})$ \hspace{1cm} $N_1 = \overset{\longrightarrow}{\text{in } \mathcal{C}}$ \hspace{1cm} $N_2 = \text{composable pairs}$ \hspace{1cm} $\overset{\Rightarrow}{\longrightarrow}$ \hspace{1cm} etc.

$N_3 = \overset{\Rightarrow}{\longrightarrow}$ \hspace{1cm} $d_i$: is deletion of $i$-th arrow in composition.

- $d_i(\mathcal{E}gh) = (\mathcal{E}gh)$_{i}$
- $s_i$: inserts the identity \hspace{1cm} $s_i(\mathcal{E}g) = \mathcal{E}id_ig$

Realization:

$\mathcal{X} \mapsto X : s\text{Set} \to \text{Top}$ \hspace{1cm} by $|\mathcal{X}| = \frac{\times_{n \geq 0} \mathcal{X}_n \times |\Delta^n|}{\sim}$ \hspace{1cm} w/ $\sim$ given by face and degeneracy maps.

- In fact, gives a CW complex with an $n$-cell for each non-degenerate simplex.

Eq. $S^n = |\Delta^n|$ but also easier! Take $[0,1,2] = \overset{\Rightarrow}{\longrightarrow}$ but with all $d_i = [0,0]$.

Non-degenerate are $[0]$ and $[0,1,2]$. Realization is $\overset{\Rightarrow}{\longrightarrow}$

Thm: $|-| \text{ adjoint to } \mathcal{L}(-)$, i.e. \hspace{1cm} $\text{Hom}_{\text{Top}}(|\mathcal{X}|, Y) \cong \text{Hom}_{s\text{Set}}(\mathcal{X}, \mathcal{L}(Y))$, and $|\mathcal{L}(Y)| \cong Y$.

(for $Y$ a CW)
Example: Let \( G \) be a group, as a category, \( \text{N}(G) \) is the classifying space of \( G \). 
\[
G = \mathbb{Z}/2\mathbb{Z} \times \langle a | a^2 = e \rangle, \quad N_0 = \{0, 1\}, \quad N_1 = \{e, a\}, \quad N_2 = \{(e, 0), (a, 0), (e, 1), (a, 1)\} \quad \&c.
\]

One non-degen, simplex in each \( (q_k, a) \). \( BG = \mathbb{RP}^\infty \).

**Kan Condition:** The \( k \)-th horn of \( \Delta^n \), \( \Lambda_k \Delta^n \) is \( \partial \Delta^n \setminus \partial_k \Delta^n \).

\[
\Lambda_2 = \begin{tikzpicture} \draw[black] (0,0) -- (0,1) (0,1) -- (1,1) (1,1) -- (1,0) (0,0) -- (1,0) (0,1) -- (0,2); \end{tikzpicture} \quad = \begin{bmatrix} [0, 1], [1, 2]; [0, 2], [1, 2]; \end{bmatrix} \quad \text{degeneracies as standard}
\]

*The Kan condition is satisfied by \( X \) if \( \Lambda_k \Delta^n \to X \).

"Every horn has a filler." "Fibrant." \( \to \text{not unique} \)

**Eq.:** \( \text{S}(Y) \) is a Kan complex. Choose any retract \( \Delta^n \to \Lambda_k \Delta^n \). This serves as filler.

**Eq.:** The \( n \)-simplex is not Kan! Take \( \Lambda_2 = \begin{tikzpicture} \draw[black] (0,0) -- (0,1) (0,1) -- (1,1) (1,1) -- (1,0) (0,0) -- (1,0) (0,1) -- (0,2); \end{tikzpicture} \to \begin{bmatrix} [0, 1], [1, 2]; [0, 2], [1, 2]; \end{bmatrix} \) by \( [0, 2] \mapsto [0, 0] \), \( [0, 1] \mapsto [0, 0] \).

This does not extend to \( \Delta^2 \). (Where does \( [1, 2], [0, 2] \)?) [Complexes are not Kan]

**Eq.:** \( \text{N}(C) \) is Kan \( \iff \) \( C \) is a groupoid. Here all fillers are unique.

**Homotopy:** \( x, x' \in X \) homotopic if \( d_i x = d_i x' \quad \forall i \), \( d_n y = x \) and \( d_n y' = x' \), and \( \frac{d_i y}{d_i y'} = s_{m-1} \frac{d_i x}{d_i x'} \) for some \( y \in X_{m+1} \).

**Eq.:** \( \Delta^1 \to \begin{bmatrix} [0, 1], [1, 2]; [0, 2], [1, 2]; \end{bmatrix} \to \begin{bmatrix} [0, 1], [1, 2]; [0, 2], [1, 2]; \end{bmatrix} \to X \).

\( x \simeq x' \)

**Defn.:** \( \text{Kan.} \quad \pi_n(X, *) \) is set of homotopy classes of \( n \)-simplices with \( d_i x = * \quad \forall i \).

\( (n > 0) \)

Any \( x, y \in X_n \) form a horn. Define \( x \cdot y \) as the other face of the filler.

**Thm.:** Homotopy theory of Kan complexes is equivalent to that of CW complexes.