Metric Thickenings of Euclidean Submanifolds

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Introduction
Data has topological structure:

Energy landscape of cyclo-octane:

- **Martin, Thompson, Coutsiias, Watson ’10**
- “A reducible algebraic variety, composed of the union of a sphere and a Klein bottle, intersecting in two rings.”

Given a data set, can we describe the underlying space?
Reconstructing a Manifold
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Underlying Questions:

- Persistent Homology:
  - What information is contained at different scale parameters?

- Manifold Reconstruction:
  - Does any scale parameter give a simplicial complex with the “correct” homology (or homotopy type)?
  - Does the simplicial complex have predictable structure at “bad” scale parameters?
Technical Aside #1

- Čech complex, $\check{C}(X; r)$, contains an $n$-simplex for every $(n + 1)$-fold intersection of balls of radius $r$.
- Vietoris–Rips complex, $VR(X; r)$, contains an $n$-simplex for every set of $n + 1$ points with diameter $< r$.
- Write $K(X; r)$ when the distinction is unimportant.
Model Theorems

**Theorem (Hausmann ’95)**

Let $M$ be a compact Riemannian manifold and $r > 0$ be sufficiently small (depending on curvature of $M$). Then $\text{VR}(M; r) \simeq M$.

**Theorem (Latschev ’01)**

Let $M$ be a compact Riemannian manifold and $r > 0$ be sufficiently small. Then there exists a $\delta > 0$ such that for any metric space with $d_{\text{GH}}(Y, M) < \delta$, $\text{VR}(Y; r) \simeq M$.

**Theorem (Niyogi, Smale, Weinberger ’05)**

Let $Y$ be a sufficiently dense sampling (possibly with noise) of a Euclidean submanifold $M$, and $r > 0$ sufficiently small. Then $\check{C}(Y; r) \simeq M$. 
Metric Thickenings
Let $X$ be a metric space, $r \geq 0$, and $K(X; r)$ either a Vietoris–Rips or Čech complex.

**Definition**

The **Metric thickening** $K^m(X; r)$ is the set

$$\left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N}, \lambda_i \geq 0, \sum_i \lambda_i = 1, [x_0, \ldots, x_k] \text{ a simplex in } K(X; r) \right\},$$

equipped with the 1-Wasserstein metric.

**Definition**

The **1-Wasserstein metric** on $\text{VR}^m(X; r)$ is the distance defined by

$$d_W(x, x') = \inf \{ \text{cost}(p) \mid p \text{ is a matching between } x \text{ and } x' \}.$$
Established Facts

Theorem (Adamaszek, Adams, Frick ’17)

- $\text{VR}^m(X; r) \cong \text{VR}(X; r)$ if and only if $\text{VR}(X; r)$ is locally finite.
- $\text{VR}^m(X; r)$ is an $r$-thickening of $X$: The metric of $X$ extends to that of $\text{VR}^m(X; r)$ and $d(x.\text{VR}^m(X; r)) < r$ for all $x \in X$.
- Hausmann’s theorem holds: if $X$ is a Riemannian manifold, then for $r$ sufficiently small $\text{VR}^m(X; r) \simeq X$. 
Results
Theorem (Adams, M. ’17)

Let \( X \subseteq \mathbb{R}^n \) and suppose the reach, \( \tau \), of \( X \) is positive. Then for all \( r < \tau \), the metric Vietoris–Rips thickening \( \text{VR}^m(X; r) \) is homotopy equivalent to \( X \).

For all \( r < 2\tau \), the metric Čech thickening \( \tilde{\text{C}}^m(X; r) \) is homotopy equivalent to \( X \).
The medial axis of $X \subseteq \mathbb{R}^n$ is the closure, $\overline{Y}$, of

$$Y = \{ y \in \mathbb{R}^n \mid \exists x_1 \neq x_2 \in M \text{ with } d(y, x_1) = d(y, x_2) = d(y, X) \}.$$ 

The reach, $\tau$, of $X$ is the minimal distance $\tau = d(X, \overline{Y})$ between $X$ and its medial axis.

Smooth manifolds (embedded in $\mathbb{R}^n$) have positive reach. Sets with corners have zero reach.
Proof.

\( \pi \circ f \) and \( i \) are homotopy inverses:
Future Work

• Use these methods to compute homotopy types of $\text{VR}^m(X; r)$ at larger scale parameters for particular classes of $X$.

• Understand the structure of $\text{VR}^m(X; r)$ at larger $r$. (In particular, the higher homotopy groups.)

• Similar results for (infinite) dense samplings.

• Show stability with regard to persistence.
References


Slides will be available at

http://www.math.colostate.edu/~mirth/talks.html