Morse Theory for Wasserstein Spaces
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Motivation

Applied topology uses simplicial complexes to approximate a manifold based on data. This approximation is known not to always recover the homotopy type of the manifold. In this work-in-progress we investigate how to compute the homotopy type in such settings using techniques inspired by Morse theory.

Background

Points in simplices can be described with barycentric coordinates:

\[
\begin{align*}
(a, b, c),
(b, a, c),
(c, a, b),
(c, b, a),
\end{align*}
\]

These can be interpreted as probability measures:

\[
\sum_{i=0}^{n} \lambda_i x_i \iff \sum_{i=0}^{n} \lambda_i \delta_{x_i}
\]

The set of finitely-supported probability measures on a metric space X admits a natural metric.

Definition: Let μ and ν be probability measures on a metric space X. Denote by Γ(μ, ν) the set of all measures on X × X with marginals μ and ν. The p-Wasserstein distance is defined to be

\[
d_W(\mu, \nu) = \inf_{\pi \in \Gamma(\mu, \nu)} \left( \int d(x, y)^p d\pi \right)^{1/p}.
\]

Definition: A metric simplicial complex on X is a metric space (S, d_W) where S is a collection of finitely-supported probability measures on X which satisfies:

- For all x ∈ X, the point mass δ_x is in S, and
- If μ ∈ S and ν ≤ μ, then ν ∈ S.

Main Example: The Vietoris–Rips metric complex, VR(X; r), contains all finitely-supported measures, μ, such that the diameter of the support of μ is less than r.

Questions

Main Question: Given a known metric space (e.g. a compact Riemannian manifold), M, what is the homotopy type of VR(X; r) for all values of r?

Question: How is VR(X; r) related to the ordinary Vietoris–Rips simplicial complex, VR(X; r), with the simplicial complex topology?

(Partial answer: if X is finite then VR(X; r) ≃ VR(X; r).)

Question: Given X and Y and some operation on metric spaces ∗, how is VR(X ∗ Y; r) related to VR(X; r) ∗ VR(Y; r)?

Morse Theory

Classical Morse theory is based on two lemmas [6]. Given a smooth manifold, M, and a smooth function F: M → R with no degenerate critical points, then

- If [a, b] ⊆ R contains no critical values of F, then F^−1(−∞, a) ≃ F^−1(−∞, b), and
- If a is an index-k critical point of F, then F^−1(−∞, a + ε) ≃ F^−1(−∞, a − ε) ∪ D_k where D_k is a k-cell.

We propose to answer the questions above using a form of Morse theory for metric simplicial complexes. In particular, [4] and [5] develop a form of differential geometry for Wasserstein spaces, which should be amenable to Morse theory.

Preliminary Results

Theorem: For small r, VR(M; r) ≃ M.

Proof sketch: Appears in [2] and [3] for different types of M.

Theorem: For any metric spaces X and Y, and any r ∈ [0, +∞], we have VR(X × Y; r) ≃ VR(X; r) × VR(Y; r) and VR(X ∨ Y; r) ≃ VR(X; r) ∨ VR(Y; r).

Proof sketch: For products, the homotopy equivalence is given by forming the product measure:

\[
\left( \sum_{i} \lambda_i \delta_{x_i}, \sum_{j} \lambda_j \delta_{y_j} \right) \mapsto \left( \sum_{i,j} \lambda_i \lambda_j \delta_{x_i(y_j)} \right)
\]

given by taking the marginals of a distribution.

Additional Known Results:

- For any convex K ⊆ R^d, VR(K; r) is contractible for all r.
- For 0 ≤ r < 1/3, VR(S^1; r) ≃ S^1, and VR(S^1; 1/3) ≃ S^1.
- If X is a simply-connected space of non-positive curvature, then VR(X; r) ≃ X all r ∈ [0, +∞].

References