NON EFFECTIVE PLANAR LINEAR SYSTEMS AT THE BOUNDARY OF THE MORI CONE

CIRO CILIBERTO AND RICK MIRANDA

We dedicate this paper to the memory of Alberto Collino

ABSTRACT. In this paper we prove that certain linear systems (and all their multiples) of plane curves with general base points and zero–self intersection are empty, thus exhibiting further examples of rays at the boundary of the Mori cone of a general blow–up of the plane.

INTRODUCTION

Let X_n be the blow–up of the projective plane at n general points. Let $\mathcal{L}_d(m_1, \ldots, m_n)$, with $d > 0$, be the linear system on X_n corresponding to plane curves of degree d with general points of multiplicities at least $m_1 \geqslant \cdots \geqslant m_n$ (we will use exponential notation for repeated multiplicities).

We assume $n \geqslant 3$. We define

$$N = \#\{j \mid m_j \geqslant 2\}, \quad h = \#\{j \mid m_j = 1\}$$

so that $N + h \leqslant n$.

Let us make the hypothesis $\mathcal{H}$:
(i) $d \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 0$,
(ii) $e = d - (m_1 + m_2 + m_3) \geqslant 0$,
(iii) $d^2 = \sum_{i=1}^n m_i^2$.

Note that condition (ii) implies that $\mathcal{L}_d(m_1, \ldots, m_n)$ is *Cremona reduced*, i.e., its degree d cannot be reduced by applying quadratic transformations based at its assigned base points.

Our goal in this article is to prove the following:

Theorem 1. *Suppose that $\mathcal{L}_d(m_1, \ldots, m_n)$ is a linear system satisfying the hypothesis $\mathcal{H}$, for which $N \leqslant 8$. Then for any $k \geqslant 1$, the system $\mathcal{L}_{kd}(km_1, \ldots, km_n)$ is empty, unless:*
(a) $\mathcal{L}_d(m_1, \ldots, m_n)$ is a multiple of $\mathcal{L}_1(1)$;
(b) $\mathcal{L}_d(m_1, \ldots, m_n)$ is a multiple of $\mathcal{L}_3(1^9)$.

If we set $N_1(X_n) = \mathrm{Pic}(X_n) \otimes_{\mathbb{Z}} \mathbb{R}$, then any system $\mathcal{L}_d(m_1, \ldots, m_n)$ such that $\mathcal{L}_{kd}(km_1, \ldots, km_n)$ is empty for any $k \geqslant 1$ determines a rational ray in $N_1(X_n)$ that is not effective (see [4, §3.1]). Therefore such a ray sits in the boundary of the Mori cone of X_n. Any such ray, if rational, is called a *good ray* in [4, §3.2] whereas, if irrational, it is called a *wonderful ray*. No wonderful ray has been discovered up to now[1]. Proving that a given ray is good seems in general to be difficult, and in [4] the authors were able to

Key words and phrases. Linear systems, Mori cone, nef rays.

[1]After this paper was finished wonderful rays have been exhibited in the preprint "Irrational nef rays at the boundary of the Mori cone for very general blowups of the plane" (arXiv:2201.08634), by J. Roé and the two authors of the present paper.

exhibit some examples. Other examples have been provided in [3] and they correspond to the case $N = 1$ in Theorem 1.

Our proof is inductive on N, and uses the degeneration introduced in [1]; we will recall the main lemma that provides the basic reduction step in §2 (see Lemma 6). In §1 we will separately treat the cases for which our general inductive strategy fails. The proof of Theorem 1 is in §3.

Acknowledgements: Ciro Ciliberto is a member of GNSAGA of INdAM.

1. The cases $\mathcal{L}_6(2^8, 1^4)$ and $\mathcal{L}_6(2^7, 1^8)$

In this section we prove the following:

Proposition 2. *For any positive integer k the linear system $\mathcal{L} = \mathcal{L}_{6k}((2k)^8, k^4)$ is empty.*

Proof. We use the collision technique introduced in [2], specifically the four points collision stated there in Proposition 3.1(c), which says that a general collision of four points of multiplicity k results in a point of multiplicity $2k$ with a matching condition, i.e., the $2k$ tangent directions on the exceptional $\mathbb{P}^1$ are invariant under an involution which can be taken to be general.

We apply this to the four k–tuple points of $\mathcal{L}$ and conclude that the emptiness of $\mathcal{L}$ follows from the emptiness of the subsystem of $\mathcal{L}_{6k}((2k)^9)$ which satisfies the aforementioned matching condition on the ninth point. However $\mathcal{L}_{6k}((2k)^9)$ has dimension zero, consisting of the unique cubic through the nine points with multiplicity $2k$, and this curve does not satisfy the matching condition. □

Corollary 3. *For any positive integer k the linear system $\mathcal{L} = \mathcal{L}_{6k}((2k)^7, k^8)$ is empty.*

Proof. We use again the collision technique as above, colliding four of the points of multiplicity k to a point of multiplicity $2k$. Then the emptiness of $\mathcal{L}$ follows from the emptiness of the system of $\mathcal{L}_{6k}((2k)^8, k^4)$, proved in Proposition 2. □

These two results are contained in [4, Remark 5.1.6, Remark 5.5.11 and Proposition 5.5.10]; we have included the brief proofs here for completeness.

2. The inductive strategy

In this section we will present the strategy of the proof of Theorem 1 and we will prove a couple of useful lemmas.

The proof will be by induction on N and we first prove the statement for $N = 0, 1$. Then, for $N \geqslant 2$, we start with a given system $\mathcal{L}_d(m_1, \ldots, m_n)$ satisfying $\mathcal{H}$ and different from multiples of the forbidden systems (a) and (b) of Theorem 1. Since $\mathcal{L}_d(m_1, \ldots, m_n)$ satisfies $\mathcal{H}$, we have $e \geqslant 0$. The first step is to reduce to $e = 0$. For this we use Lemma 6 below, which raises m_1 by 1, decreases e by 1 (and preserves $\mathcal{H}$). Repeated applications of this allow us to assume $e = 0$.

Next we define a *basic move* on $e = 0$ systems that satisfy $\mathcal{H}$.

Definition 4 (Basic move)**.** Suppose we have a system $\mathcal{L}_d(m_1, \ldots, m_n)$ satisfying $\mathcal{H}$ with $e = 0$ (for which we want to prove emptiness of all multiples). Consider then the system obtained using Lemma 6 below, by increasing m_1 by 1, and reducing appropriately the number of simple points. This system now has $e < 0$, and therefore we may reduce it via quadratic Cremona transformations. If the result has $e > 0$, we again apply Lemma 6 repeatedly to obtain a system with $e = 0$.

The result of a basic move still satisfies $\mathcal{H}$. The ability to apply Lemma 6 as needed is provided by Lemma 5 below. During the application of the basic moves, the hypotheses allow us to control the application of the required Cremona transformations, see Lemma 9.

The strategy is to apply basic moves iteratively and show that either N must decrease, so that we can apply induction, or $N = 8$ and we arrive at a system that we directly know is empty, such as a multiple of the degree 6 linear systems considered in §1.

Next we present the aforementioned useful lemmas.

Lemma 5. *Suppose $1 \leqslant N \leqslant 8$, $\mathcal{H}$ holds and the system is not $\mathcal{L}_6(2^8, 1^4)$. Then $h \geqslant 2m_1 + 1$, unless $N = 1$ and $d = m_1$ in which case $h = 0$.*

Proof. If $N = 1$ and $d > m_1$, then $h = d^2 - m_1^2 \geqslant (m_1 + 1)^2 - m_1^2 = 2m_1 + 1$ and we are done.

Suppose $N = 2$. Then $d \geqslant m_1 + m_2$, so $h = d^2 - m_1^2 - m_2^2 \geqslant 2m_1m_2 \geqslant 4m_1 \geqslant 2m_1 + 1$ as wanted.

Next suppose $N \geqslant 3$. Since $d \geqslant m_1 + m_2 + m_3$, we have

$$d^2 \geqslant (m_1 + m_2 + m_3)^2 = m_1^2 + m_2^2 + m_3^2 + 2m_1m_2 + 2m_1m_3 + 2m_2m_3.$$

As the m_is are in descending order, we have $2m_2m_3 \geqslant m_4^2 + m_5^2$, $2m_1m_3 \geqslant m_6^2 + m_7^2$, and $m_1m_2 \geqslant m_8^2$, so that $h \geqslant m_1m_2 \geqslant 2m_1$.

Suppose that $h = 2m_1$. Then all the above inequalities are equalities, which implies $N = 8$ and all m_is equal to 2. In that case $h = d^2 - 32$, forcing $d = 6$, which is forbidden by hypothesis. Hence $h \geqslant 2m_1 + 1$. □

Suppose next $2 \leqslant N \leqslant 8$, $\mathcal{L}_d(m_1, \ldots, m_n)$ satisfies $\mathcal{H}$ and it is different from $\mathcal{L}_6(2^8, 1^4)$.

Fix $k \geqslant 1$ and consider $\mathcal{L}_{kd}(km_1, \ldots, km_n)$. Then we can make the P–F degeneration described in [1, Section 2] with the limit line bundle of $\mathcal{L}_{kd}(km_1, \ldots, km_n)$ having aspects

$$\mathcal{L}_P = \mathcal{L}_{k(m_1+1)}(km_1, k^a)$$

and

$$\mathcal{L}_F = \mathcal{L}_{kd}(k(m_1 + 1), km_2, \ldots, km_N, k^b, 0^{h-b})$$

with $a = 2m_1 + 1$ and $b = h - 2m_1 - 1$ on P and F respectively. Note that Lemma 5 implies that a and b are positive which is the prerequisite to apply the P–F degeneration described above.

Lemma 6. *Suppose that, in the above setting, $\mathcal{L}_F$ is empty. Then so is $\mathcal{L}_{kd}(km_1, \ldots, km_n)$.*

Proof. First we notice that $\mathcal{L}_{m_1+1}(m_1, 1^a)$ is a pencil and no curve in this pencil contains a general line. Then $\mathcal{L}_{k(m_1+1)}(km_1, k^a)$ is composed with this pencil and no curve in this system contains a general line. Hence the system $\hat{\mathcal{L}}_P = \mathcal{L}_{km_1+k-1}(km_1, k^a)$ is empty. This implies the result because any section of the limit line bundle is identically zero. Indeed, such a section is identically zero on F by hypothesis, and it is also identically zero on P since its zero locus has to lie in $\hat{\mathcal{L}}_P$ and this is empty. □

An additional application of the P–F degeneration method enables us to prove the following statements. The first one is also a consequence of [4, Remark 5.5.11], where more general statements are made; we include the brief proof for the convenience of the reader.

Proposition 7. *For any positive integer k the linear system $\mathcal{L} = \mathcal{L}_{9k}((3k)^8, k^9)$ is empty.*

Proof. We make the P–F degeneration as above. The relevant systems are

$$\mathcal{L}_F = \mathcal{L}_{9k}(4k, (3k)^7, k^2) \quad \text{and} \quad \mathcal{L}_P = \mathcal{L}_{4k}(3k, k^7).$$

By Cremona reducing $\mathcal{L}_F$ we see that it consists of a unique curve with multiplicity k which is the Cremona image of a cubic through 9 general simple points. It meets the double curve $R = P \cap F$ at 4 points with multiplicity k which can be assumed to be general. The kernel system $\hat{\mathcal{L}}_F$ is empty. The system $\mathcal{L}_P$ is composed with the pencil $\mathcal{L}_4(3, 1^7)$, and it cuts out on R the linear system (a g^k_{4k}) composed with the g^1_4 cut out on R by $\mathcal{L}_4(3, 1^7)$. The kernel linear system $\hat{\mathcal{L}}_P$ is also empty. By the generality of the restriction of $\mathcal{L}_F$ to R, there can be no matching divisor in $\mathcal{L}_P$ with the unique curve in $\mathcal{L}_F$ and therefore no section of the limit line bundle, since both kernel linear systems are empty (see [1]). This ends the proof of the proposition. □

Before the next lemma we need a simple remark.

Remark 8. We notice that a linear system $\mathcal{L}_d(m_1, \ldots, m_n)$ cannot be a multiple of the system (a) of Theorem 1 if $m_2 \geqslant 1$ and it cannot be a multiple of the system (b) of Theorem 1 if $m_1 > m_8$.

Lemma 9. *Let $\mathcal{L}_d(m_1, \ldots, m_n)$ satisfy $\mathcal{H}$, with $8 \geqslant N \geqslant 2$ and $e = 0$ and assume it is different from a multiple of $\mathcal{L}_6(2^8, 1^4)$ or of $\mathcal{L}_9(3^8, 1^9)$ or of $\mathcal{L}_6(2^7, 1^8)$. Then the result of a basic move is different from a multiple of either of the two forbidden systems (a) and (b) of Theorem 1.*

Proof. Since the system is not a multiple of $\mathcal{L}_6(2^8, 1^4)$ we can do a basic move (see Lemma 5).

The first step in the basic move leads to the system $\mathcal{L}_d(m_1 + 1, m_2, \ldots, m_n)$. Then, applying the quadratic transformation based at the first three points, gives the linear system $\mathcal{L}_1 := \mathcal{L}_{d-1}(m_1, m_2 - 1, m_3 - 1, m_4 \ldots, m_n)$.

If $m_3 > m_4$, the three highest multiplicities are $m_1, m_2 - 1, m_3 - 1$ and therefore the system is Cremona reduced, with $e = 1$. At this point, to finish the basic move, we make one more application of Lemma 6 to reduce to the case $e = 0$. This leads to the system $\mathcal{L}_{d-1}(m_1 + 1, m_2 - 1, m_3 - 1, m_4, \ldots, m_n)$, and we conclude by applying Remark 8.

If $m_3 = m_4$ but $m_2 > m_5$ then the three highest multiplicities of $\mathcal{L}_1$ are $m_1, m_2 - 1, m_4 = m_3$ in some order. We see that $e = 0$ for $\mathcal{L}_1$ so the basic move is finished and we conclude again by using Remark 8.

We may now assume $m_3 = m_4$ and $m_2 = m_5$, hence $m_2 = \cdots = m_5 = m$. In this case reordering the multiplicities of $\mathcal{L}_1$ the three highest ones are m_1, m, m. This is not Cremona reduced. After making the quadratic transformation based at the three points of highest multiplicity, we get to the system $\mathcal{L}_2 = \mathcal{L}_{d-2}(m_1 - 1, (m-1)^4, m_6, \ldots, m_n)$. Now we have three cases.

The first case is $m_6 < m_2 = m$. In that case the three highest multiplicities of $\mathcal{L}_2$ are $m_1 - 1, m - 1, m - 1$, the system is Cremona reduced and $e = 1$. Moreover $\mathcal{L}_2$ does not coincide with a multiple of $\mathcal{L}_6(2^8, 1^4)$ that all have $e = 0$. Then we can apply Lemma 6 one more time to finish the basic move and we get the system $\mathcal{L}_{d-2}(m_1, (m-1)^4, m_6, \ldots, m_n)$. We conclude again by using Remark 8 because $m_1 > m - 1 \geqslant 1$.

In the second case we have $m_6 = m$ and $m_7 < m$. In that case the three highest multiplicities of $\mathcal{L}_2$ are $m_1 - 1, m, m - 1$ and the system is Cremona reduced with $e = 0$. This ends the basic move and we apply Remark 8 to finish.

In the final case we have $m_6 = m_7 = m$. Now the three highest multiplicities of $\mathcal{L}_2$ are $m_1 - 1, m, m$ and the system is not Cremona reduced. One more quadratic transformation gives $\mathcal{L}_3 = \mathcal{L}_{d-3}(m_1 - 2, (m-1)^6, m_8, \ldots, m_n)$.

If $m_1 > m$ and $m_8 < m$ the three highest multiplicities are $m_1 - 2, m - 1, m - 1$, the system is Cremona reduced, with $e = 1$. In this case we have to make a further step to accomplish the basic move and we get $\mathcal{L}_{d-3}(m_1 - 1, (m-1)^6, m_8, \ldots, m_n)$. We apply again Remark 8 to finish.

If $m_1 > m$ and $m_8 = m$ the three highest multiplicities of $\mathcal{L}_3$ are $m_1 - 2, m, m - 1$. Then $\mathcal{L}_3$ is Cremona reduced with $e = 0$, the basic move is finished and we conclude by applying Remark 8 since $m > m - 1$.

If $m_1 = m$ and $m_8 < m$, the three highest multiplicities of $\mathcal{L}_3$ are $m-1, m-1, m-1$. Then $\mathcal{L}_3$ is Cremona reduced with $e = 0$. The basic move is finished and by applying Remark 8 we see that this is not a multiple of the forbidden system (a) of Theorem 1 by Remark 8. It could be a multiple of the (b) system, but this only happens if $m = 2$, $d = 6$, and the original system is $\mathcal{L}_6(2^7, 1^8)$, which is forbidden by hypothesis.

If $m_1 = m$ and $m_8 = m$ the system we start with is $\mathcal{L}_{3m}(m^8, 1^{m^2})$ and we notice that $m \geqslant 3$ because $m = 2$ is forbidden by hypothesis. Then in $\mathcal{L}_3$ the three highest multiplicities are $m, m - 1, m - 1$. At this point the system is not Cremona reduced. Reducing it one gets $\mathcal{L}_4 := \mathcal{L}_{d-6}((m-2)^7, m - 3, m_9, \ldots, m_n)$. This again is not a multiple of the forbidden system (a) of Theorem 1. It could be a multiple of the (b) system, but this only happens if $m = 3$, $d = 9$, and the original system is $\mathcal{L}_9(3^8, 1^9)$, which is forbidden by hypothesis.

This ends the proof of the lemma. □

3. The proof of Theorem 1

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. If $N = 0$, then $\mathcal{L} = \mathcal{L}_d(1^{d^2})$. We have excluded the $d = 1$ case. The $d = 2$ case does not verify $\mathcal{H}$. We have excluded the $d = 3$ case. For $d \geqslant 4$, this is Nagata's Theorem (see [5]).

If $N = 1$, then $\mathcal{L} = \mathcal{L}_d(m_1, 1^h)$ where $h = d^2 - m_1^2$. If $h = 0$ then this is a multiple of $\mathcal{L}_1(1)$, which we have excluded. If $h \geqslant 1$ then by Lemma 5, we have $h \geqslant 2m_1 + 1$, so that there are at least five simple points. Since $\mathcal{H}$ holds, we must have $d \geqslant m_1 + 1 + 1 \geqslant 4$. Then the result is contained in [3].

Next we assume $N \geqslant 2$. By Propositions 2, 7, and Corollary 3, we can assume that the system is not a multiple of $\mathcal{L}_6(2^8, 1^4)$ or of $\mathcal{L}_9(3^8, 1^9)$ or of $\mathcal{L}_6(2^7, 1^8)$. By applying (if necessary) Lemma 6, we can assume that $e = 0$. Then the hypotheses of Lemma 9 are met and we can make a basic move which does not arrive at a forbidden system. If the result of the move is a multiple of $\mathcal{L}_6(2^8, 1^4)$ or of $\mathcal{L}_9(3^8, 1^9)$ or of $\mathcal{L}_6(2^7, 1^8)$, that are empty by Propositions 2, 7, and Corollary 3, we apply Lemma 6 again and conclude that the original system is empty. If not we have reduced to a linear system with lower multiplicities. So, repeated applications of a basic move either results in an empty system or eventually decreases N, so that we can finish by induction. □

We notice that the theorem is certainly false for the forbidden systems (a) and (b) of Theorem 1.

REFERENCES

[1] C. Ciliberto and R. Miranda, *Degenerations of planar linear systems*, J. Reine Ang. Math. **501** (1998), 191–220.

[2] C. Ciliberto and R. Miranda, *Matching Conditions for Degenerating Plane Curves and Applications*, in "Projective Varieties with Unexpected Properties", Proceedings of the Siena Conference, C. Ciliberto, A. V. Geramita, B. Harbourne, R. M. Miró-Roig, K. Ranestad ed., W. de Gruyter, 2005, 177–198.

[3] C. Ciliberto and R. Miranda, *Examples of non–effective rays at the boundary of the Mori cone of blow–ups of the plane*, pre–print 2021.

[4] C. Ciliberto, B. Harbourne, R. Miranda, J. Roé, *Variations on Nagata's conjecture*, Clay Mathematics Proceedings **18** (2013), 185–203.

[5] M. Nagata, *On the fourteenth problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA TOR VERGATA, VIA O. RAIMONDO 00173 ROMA, ITALIA

E-mail address: cilibert@axp.mat.uniroma2.it

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS (CO), 80523,USA

E-mail address: rick.miranda@colostate.edu