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The proto-Lorenz system

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A quotient of the Lorenz dynamical system is constructed. This "proto-Lorenz" system has the Lorenz system as a double covering, and the double covering explains the two-eared nature of the strange attractor for the Lorenz system. Arbitrary coverings of the proto-Lorenz system are possible, leading to n-eared strange attractors.

1. The Lorenz equations and their order two symmetry

Fix parameters σ, r, and b. The system of ODEs given by

$$\dot{x}=\sigma y-\sigma x\,,\qquad \dot{y}=rx-y-xz\,,\qquad \dot{z}=xy-bz \tag{1.1}$$

is the *Lorenz system*, introduced by Lorenz in ref. [1]. It is useful to think of this system as a vector field on $\mathrm{X}=\mathbb{R}^3$, with coordinates (x, y, z).

The order two symmetry in the Lorenz system is the symmetry about the z-axis. Specifically, the Lorenz system is invariant under the involution s sending (x, y, z) to $(-x, -y, z)$. Note that two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are in the same orbit of this involution s if and only if $z_1=z_2$ and $(x_1, y_1)=\pm(x_2, y_2)$.

2. The orbit space of the involution

Let $\mathrm{Y}=\mathbb{R}^3$, with coordinates (u, v, z). Define a map π: $\mathrm{X}\to\mathrm{Y}$ by $\pi(x, y, z)=(x^2-y^2, 2xy, z)$ or, equivalently, by setting

$$u=x^2-y^2\,,\qquad v=2xy\,,\qquad z=z\,. \tag{2.1}$$

Theorem 2.2.
(1) The map π: $\mathrm{X}\to\mathrm{Y}$ is onto.
(2) The map π is a quotient map for the involution. In other words,

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$$\pi(x_1, y_1, z_1) = \pi(x_2, y_2, z_2) \quad \Leftrightarrow \quad z_1 = z_2 \text{ and } (x_1, y_1) = \pm(x_2, y_2)$$

and a subset of Y is open if and only if its inverse image in X is open. Thus the space Y is an orbit space for the involution on X, and its natural topology (as $\mathbb{R}^3$) is the identification topology for the map π.

(3) Away from the z-axis, π is a local diffeomorphism: the Jacobian of π is non-singular at any (x, y, z) with $(x, y) \neq (0, 0)$.

Proof. Fix a triple (u, v, z) in $\mathrm{Y}=\mathbb{R}^3$. Suppose first that $v \neq 0$; then any pre-image (x, y, z) of (u, v, z) under π must have x and y non-zero, so that $y=v/2x$. Solving the quadratic equation $x^2-y^2=u$, we find that $4x^4-4ux^2-v^2=0$, so that $x^2=\frac{1}{2}(u\pm\sqrt{u^2+v^2})$. Since $v\neq 0$, $\sqrt{u^2+v^2}$ is strictly larger than u in absolute value; thus the only solution to this equation is $x^2=\frac{1}{2}(u+\sqrt{u^2+v^2})$. This has exactly two non-zero solutions x_1 and x_2, with $x_2=-x_1$; the equation for y then gives the two corresponding solutions $y_1=v/2x_1$ and $y_2=v/2x_2$, so that also $y_2=-y_1$.

Suppose now that $v=0$; this forces either x or y to be zero, and which one is determined by the sign of u: if $u>0$ then $y=0$ and $x=\pm\sqrt{u}$, and if $u<0$ then $x=0$ and $y=\pm\sqrt{-u}$.

These computations prove both the first statement, and the stated criterion for when $\pi(x_1, y_1, z_1)=\pi(x_2, y_2, z_2)$. To see that π is a quotient map, it suffices to check that π is an open map, that is, it takes open sets to open sets (see ref. [2], section 2.11). Note that π can be thought of as the product of a self-map of $\mathbb{R}^2$ with the identity (on the third coordinate); since a product of open maps is open, and the identity is certainly an open map, it suffices to show that this map on $\mathbb{R}^2$ (sending (x, y) to $(x^2-y^2, 2xy)$) is open. Using complex coordinates $z=x+\mathrm{i}y$, this is simply the map sending z to z^2; since any analytic map is open, we are done.

To check the third statement, simply compute the Jacobian of π at a point (x, y, z); we obtain the matrix

$$\begin{pmatrix} 2x & -2y & 0 \\ 2y & 2x & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which has non-zero determinant away from the locus $x=y=0$. ■

3. The proto-Lorenz system on the orbit space

Away from the z-axes, there is a 1–1 correspondence between vector fields on the orbit space Y and vector fields on X which are invariant under the involution. Since the Lorenz system (1.1) on X is invariant under the involution, it descends to a vector field $\mathscr{L}$ on the quotient Y, at least away from the z-axes. I.e., if we denote by U the vector field on X, then the map π satisfies $\pi s=\pi$ and therefore the induced vector field $\mathscr{L}=\mathrm{D}\pi(\mathrm{U})$ on Y given by $\mathscr{L}(\pi(x))=\mathrm{D}\pi(x)\mathrm{U}(x)$ is well-defined. Our next task is to write down this vector field $\mathscr{L}$ on Y.

It is useful to note that any quadratic monomial in x and y can be written in terms of the variables u and v. To see this, let N denote the norm of the (u, v)-vector, $N=\sqrt{u^2+v^2}=x^2+y^2$. Then

$$x^2=\tfrac{1}{2}(u+N), \qquad xy=\tfrac{1}{2}v, \qquad y^2=\tfrac{1}{2}(-u+N).$$

For the vector field $\mathscr{L}$ on Y, we have

$$\dot{u}=2x\dot{x}-2y\dot{y}=2x(\sigma y-\sigma x)-2y(rx-y-xz)=-2\sigma x^2+2y^2+2(\sigma-r)xy+2xyz$$

$$=-\sigma(u+N)-u+N+(\sigma-r)v+vz,$$

$$\dot{v}=2y\dot{x}+2x\dot{y}=2y(\sigma y-\sigma x)+2x(rx-y-xz)=2rx^2+2\sigma y^2-2(\sigma+1)xy-2x^2z$$

$$=r(u+N)+\sigma(-u+N)-(\sigma+1)v-(u+N)z.$$

Re-writing this a bit we see that the descended vector field $\mathscr{L}$ on the orbit space Y can be written as

$$\dot{u}=(-\sigma-1)u+(\sigma-r)v+(1-\sigma)N+vz\,,\qquad \dot{v}=(r-\sigma)u-(\sigma+1)v+(r+\sigma)N-uz-Nz\,,$$

$$\dot{z}=\tfrac{1}{2}v-bz\,. \tag{3.1}$$

We call this descended vector field $\mathscr{L}$ the *proto-Lorenz* system on $Y=\mathbb{R}^3$. Note that although the map π is ramified along the z-axis in X, so that there is no a priori knowledge about $\mathscr{L}$ on the image of the z-axis (which is the z-axis again on Y), the proto-Lorenz system $\mathscr{L}$ is in fact a continuous vector field on all of Y. It is not, however, differentiable, because of the presence of the norm N in the formulae; it is differentiable away from the z-axis, of course.

4. The dynamics of the proto-Lorenz system $\mathscr{L}$

Figures 1 and 2 show phase space projections of the original Lorenz and the proto-Lorenz at $r=28.0$; $\sigma=10.0$; $b=\frac{8}{3}$, within the strange attractor regime. The two flows are related by the transformation

$$u=x^2-y^2\,,\qquad v=2xy\,,\qquad z=z\,,$$

taking the dynamics of the original Lorenz (x, y, z) to the proto-Lorenz (u, v, z).

The original Lorenz has a well studied bifurcation sequence leading to the strange attractor at $r=28$ (see, for example, refs. [3,4]). For $\sigma=10.0$; $b=\frac{8}{3}$ and $r<1$ the origin is a hyperbolic sink and the only attractor. At $r=1$ the origin becomes unstable in one direction and simultaneously two new attracting fixed points are born: a pitchfork bifurcation. The origin is a saddle point with a one-dimensional unstable manifold for all $r>1$. At $r\approx 24.74$ the new two fixed points undergo a subcritical Hopf bifurcation: two unstable periodic orbits absorb the two stable fixed points which then become unstable themselves. Lorenz [1] has shown that a closed simply connected region D containing the origin can be found such that the vector field is everywhere directed inward on the boundary. For $r>24.74$ there are no attracting fixed points, but because of the previous statement the motion must remain bounded. Where do the trajectories go? The answer is seen in the numerical integration

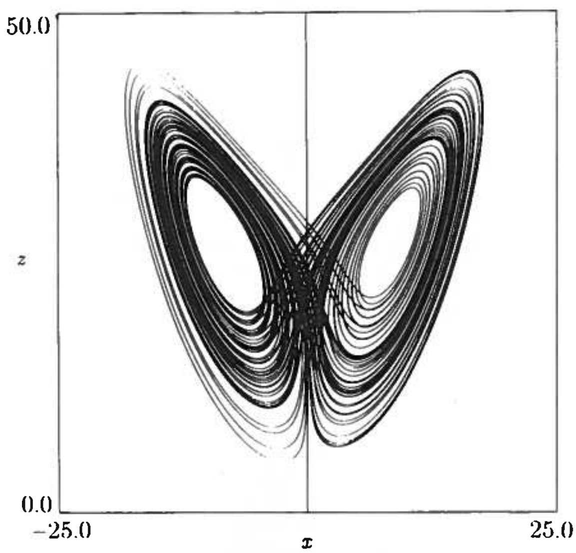

Fig. 1. The original Lorenz system, at $r=28.0$; $\sigma=10.0$; $b=\frac{8}{3}$.

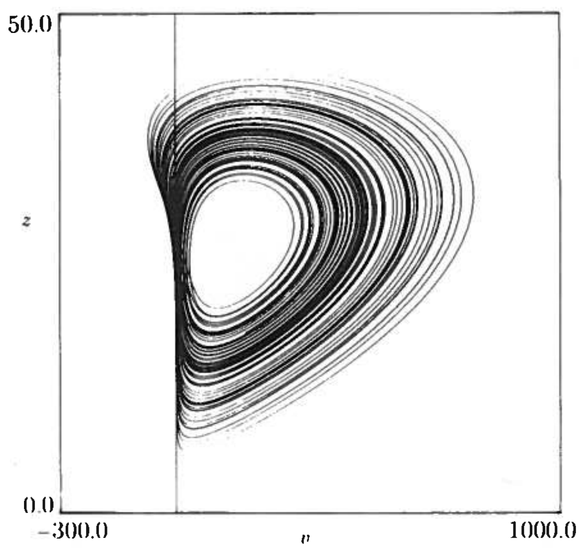

Fig. 2. The proto-Lorenz system, at $r=28.0$; $\sigma=10.0$; $b=\frac{8}{3}$.

in fig. 2. This strange invariant set is shown to come into existence with a homoclinic bifurcation at $r \approx 13.96$, but while the symmetric pair of fixed points remain attracting this is not seen in simulations, with the exception that if $24.06 < r < 24.74$, the strange attractor is stable and co-exists with the stable fixed points.

In the proto-Lorenz system some of the above analysis cannot be performed directly because the vector field is not differentiable at the origin. But we can argue that under the transformation above fixed points must map to fixed points, periodic orbits to periodic orbits. By similar arguments we can show that the stability of each will be preserved. Hence the entire bifurcation sequence of the Lorenz is reproduced in the proto-Lorenz, except that the symmetry pair of fixed points/pair of periodic orbits, are mapped to one fixed point/periodic orbit. This fixed point of the proto-Lorenz system is easily calculated to be at

$$u=0\,, \qquad v=2b(r-1)\,, \qquad z=r-1$$

away from the z-axis. The pitchfork bifurcation of the Lorenz becomes a transcritical bifurcation of the proto-Lorenz. This is to be expected; even in two dimensions the image of a pitchfork bifurcation (with equation $r=\mu^2$, say; μ is the magnitude of the fixed point) under the quotient of the involution (sending $\mu \to -\mu$) is a transcritical bifurcation (with corresponding equation $r=d$; $d=\mu^2$ is the quotient map).

We can also argue that the strange invariant set formed by the homoclinic bifurcation at $r \approx 13.96$ in the Lorenz system is preserved under the transformation to the proto-Lorenz. Indeed, since the strange invariant set is a subset of the "eventual image" $\bigcap_{t\geqslant 0} \phi_t(\mathrm{D})$ of the closed simply connected region D mentioned above (where ϕ_t denotes the flow) and since the quotient map π sends trajectories to trajectoriews and maps any compact set to a compact set, the "eventual image" of the corresponding region in the proto-Lorenz space is exactly the image of this "eventual image" in the original Lorenz system. Since the map π is a 2–1 map, the "strangeness" of the invariant set in the Lorenz system is therefore preserved in the proto-Lorenz: the existence of a dense orbit, the sensitive dependence of trajectories on initial conditions, etc. are all phenomena which are reproduced for the invariant set in the proto-Lorenz. Figure 2 exhibits a phase space projection of the proto-Lorenz which illustrates some of the dynamical features of the strange invariant set. The reader will no doubt observe the qualitative similarity to the Rössler attractor [5]; trajectories which start on the inside of the attractor wind around with increasing radius, and then are folded up and over back to the inside of the attractor.

Further numerical evidence of the relationship between the two attractors is seen in the return maps for the two systems.

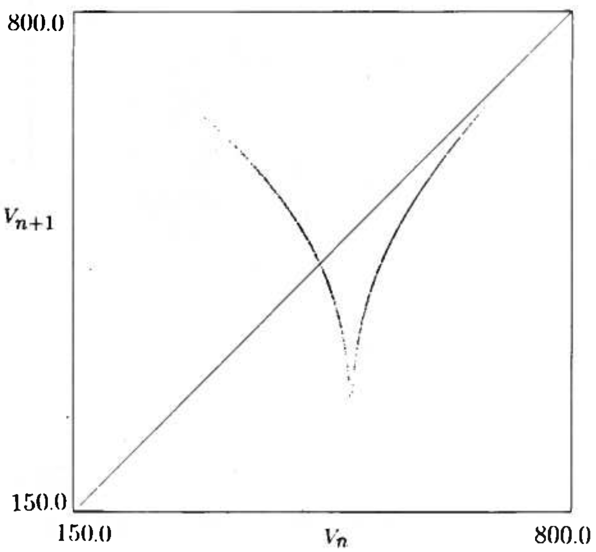

Fig. 3. The return map for the proto-Lorenz: successive v-values when u crosses zero (V_n) from negative to posiive.

Figure 3 is successive v-values when u crosses zero from negative to positive. This is the return map in v across the line $u=0$ for positive z. The map of successive maxima in z for the Lorenz is a return map across a similar vertical line in each lobe. Since no differentiation is made between the left and right lobe the map performs the same $2\to 1$ identification across the axis of symmetry as the transformation to the proto-Lorenz. The maps therefore have the same form.

A common method for reducing the complexity of analyzing the dynamics of the strange attractor in the Lorenz system is to consider the two-symbol sequence generated by successive left/right loops around either lobe. This sequence encodes which of the two lobes of the Lorenz system a trajectory is traversing through. The phenomenon of switching lobes is explained rather cleanly by examining the dynamics in the proto-Lorenz system. Away from the z-axes, the map π from the Lorenz space to the proto-Lorenz space is a *covering map* of degree 2, and above any small enough open set in the proto-Lorenz space there lies two identical open sets in the Lorenz space. These represent, locally at least, the two sheets of the covering space. Although these sheets are not globally well defined, it is possible to speak of *changing sheets*, by moving along a path in the Lorenz space from one point to the corresponding other point under the involution which is then on the other sheet.

If γ is such a path in the Lorenz space from a point p to its corresponding point $\tau(p)$ under the involution τ sending (x, y, z) to $(-x, -y, z)$, then $\pi(\gamma)$ is a closed path, or loop, in the proto-Lorenz space starting and ending at $\pi(p)$. Moreover such a loop will wind around the z-axis an odd number of times; its class in the homotopy group of $\mathbb{R}^3$-z-axis (which is an infinite cyclic group) will be an odd class. (See ref. [2] for details on covering spaces.)

Conversely, any loop in the proto-Lorenz space which winds around the z-axis an odd number of times will lift to a path in the Lorenz space which changes sheets. Thus, given a trajectory T_1 in the proto-Lorenz space, it will lift to a trajectory T_2 in the Lorenz space, and T_2 changes sheets whenever T_1 winds around the z-axis.

Relative to the attractor for the proto-Lorenz system, the z-axis pierces the attractor exactly at the point where the trajectories are folded up and over onto the main body of the attractor. Therefore trajectories wind around the z-axis exactly when they are folded up and over in the proto-Lorenz system. Thus trajectories in the original Lorenz system switch lobes (or sheets) exactly when the image trajectory in the proto-Lorenz is folded up and over.

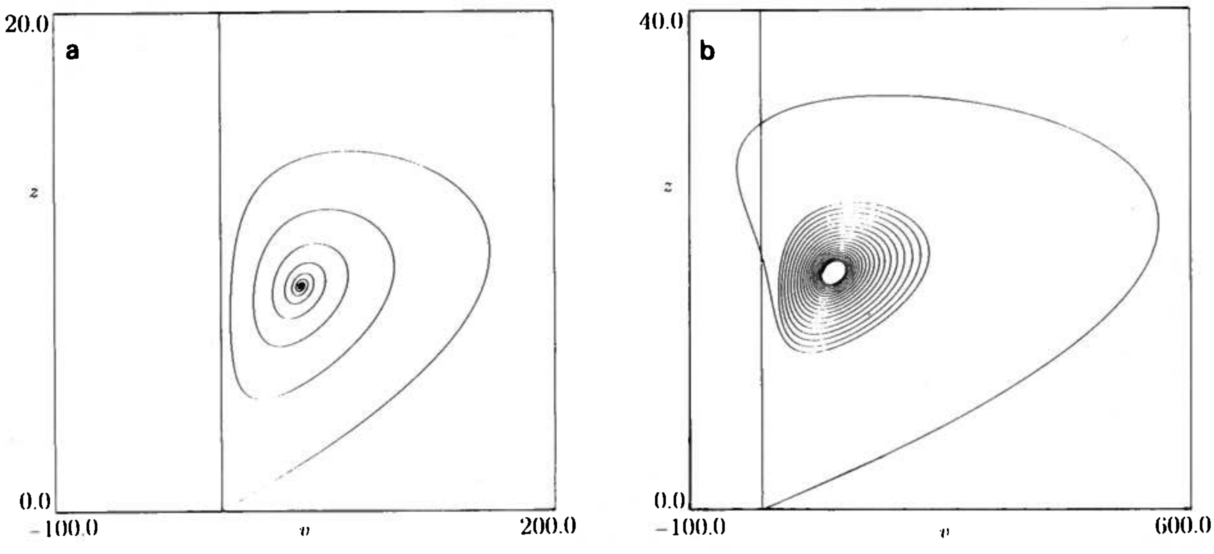

Fig. 4. (a) Solution with initial conditions near the unstable manifold of the origin in the proto-Lorenz system, with $r=10.0$. (b) Solution with initial conditions near the unstable manifold of the origin in the proto-Lorenz system, with $r=20.0$.

The evidence for the homoclinic bifurcation at $r=13.96$ in the Lorenz system is the switching of the basin of attraction of the left (right) fixed point from the left half space (right half space) to the right half space (left half space). This then translates to whether the unstable manifold of the origin crosses the z-axis as it spirals into the fixed point in the proto-Lorenz. As a numerical illustration of this fact, figs. 4a and 4b show the behavior of a trajectory started near the oriogin of the proto-Lorenz at $r=13.96$.

As will be seen in the following sections, a transformation to any number of lobes in the strange attractor is possible. We specifically document 1, 2 (the original Lorenz), 3 and 4 lobe systems. One might be led to conclude that an N-symbol sequence is necessary to describe the dynamics of an N-lobed attractor. However, because of the nature of the transformation, passing from one lobe to another can only occur in a cyclic order. Thus the N symbols are reduced back to 2: 1 or 0, for instance, if the trajectory changes to the next lobe or makes another loop around the same fixed point.

With the dynamics so reduced to a discrete system of two symbols on the N lobes we can construct a measure of the information stored by the system (after ref. [6]). This is an invariant quantity that tends to zero when the past and future state of the system are statistically independent and diverges in the case of pure determinism. Using this measure one can show that while the dynamics governing the lobe switching are chaotic and hence imply a loss of initial state information, as N increases the amount of information that the N-lobe Lorenz can store increases like $\log N$ (the entropy of a set of N equipartitioned symbols) (see refs. [6,7]). The N-lobe Lorenz attractor then becomes increasingly more deterministic with increasing N, while locally the system still displays sensitive dependence on initial conditions.

5. Covers $\mathscr{L}_n$ of the proto-Lorenz system

Recall that the orbit space Y carrying the proto-Lorenz system $\mathscr{L}$ is simply the space $\mathbb{R}^3$, with coordinates (u, v, z). It is convenient to view the (u, v)-factor $\mathbb{R}^2$ as the space $\mathbb{C}$ of complex numbers. (The point (u, v) corresponds to the complex number $w=u+iv$.)

For $n\geqslant 1$, let Y_n be a copy of $\mathbb{R}^3=\mathbb{C}\times\mathbb{R}$, with three real coordinates (u_n, v_n, z), or one complex and one real coordinate (w_n, z), where as above $w_n=u_n+iv_n$. Identify the coordinates u_1 and v_1 on Y_1 with the coordinates u and v respectively on the orbit space Y, effectively identifying Y_1 with Y in the obvious way. Denote by Y_n^1 the complement of the z-axis in Y_n.

For each $n\geqslant 1$, let γ_n be the nth-power map from $\mathbb{C}$ to itself, crossed with the identity on $\mathbb{R}$; we view the map γ_n as a map

$$\gamma_n\colon Y_n \to Y = Y_1 ,$$

with the formula

$$\gamma_n(w_n, z) = (w_n^n, z) , \qquad \text{or} \quad w = w_1 = w_n^n .$$

Note that γ_n maps the z-axis of Y_n onto the z-axis of Y, and maps the complement Y_n^1 of the z-axis onto the complement Y^1. Since the nth-power map is a local diffeomorphism away from the origin, this map γ_n is a local diffeomorphism away from the z-axis; i.e., γ_n is a local diffeomorphism on Y_n^1, mapping it to Y^1.

Hence any vector field on Y^1 will lift to a unique vector field on Y_n^1. Since the proto-Lorenz system $\mathscr{L}$ on Y can be restricted to Y^1, it can be lifted to a vector field $\mathscr{L}_n$ on Y_n^1. We call these vector fields *covers* of the proto-Lorenz system $\mathscr{L}$.

Our next task will be to write down these covers $\mathscr{L}_n$ of the proto-Lorenz system. Since by definition, the map γ_n sends $\mathscr{L}_n$ on Y_n^1 to $\mathscr{L}$ on Y^1, the chain rule gives the equation

$$(D\gamma_n)\mathscr{L}_n(w_n, z) = \mathscr{L}(\gamma_n(w_n, z)) = \mathscr{L}(w_n^n, z) , \tag{5.1}$$

where $D\gamma_n$ is the 3×3 real Jacobian for the map γ_n. (Here we are writing the vector fields in column notation, so that both $\mathscr{L}_n(w_n, z)$ and $\mathscr{L}(w_n^n\, z)$ are columns in $\mathbb{R}^3$.) We need to simply solve this equation for $\mathscr{L}_n(w_n, z)$.

Since the map γ_n is described most easily using complex numbers, but the proto-Lorenz system's description (3.1) is not, it is useful to be able to transfer easily between the two with some notation. For a complex number $c=a+ib$, define $\mathbf{A}(c)$ to be the 2×2 real matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

and define $\mathbf{A}_1(c)$ to be the first column of $\mathbf{A}(c)$:

$$\mathbf{A}(a+ib)=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad \mathbf{A}_1(a+ib)=\begin{pmatrix} a \\ b \end{pmatrix}.$$

The function $\mathbf{A}$ is a ring homomorphism of $\mathbb{C}$ into the ring of 2×2 real matrices.

The Jacobian $(D\gamma_n)$ of γ_n can now be written as

$$(D\gamma_n)(w_n, z)=\begin{pmatrix} \mathbf{A}(nw_n^{n-1}) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}.$$

Since $\mathbf{A}$ is a ring homomorphism, we see that the inverse of this Jacobian is

$$(D\gamma_n)^{-1}(w_n, z)=\begin{pmatrix} \frac{1}{n}\mathbf{A}(w_n^{1-n}) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}.$$

Therefore, solving (5.1), we have the formula

$$\mathscr{L}_n(w_n, z)=\begin{pmatrix} \frac{1}{n}\mathbf{A}(w_n^{1-n}) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}\mathscr{L}(w_n^n, z), \tag{5.2}$$

where the proto-Lorenz system is written in column vector form,

$$\mathscr{L}(u+iv, z)=\begin{pmatrix} (-\sigma-1)u+(\sigma-r+z)v+(1-\sigma)N \\ (r-\sigma-z)u-(\sigma+1)v+(r+\sigma-z)N \\ \frac{1}{2}v-bz \end{pmatrix},$$

with $N=\sqrt{u^2+v^2}$ as above.

For computational purposes it is useful to recall that for a complex number c, $c^{1-n}=(\bar{c})^{n-1}/|c|^{2n-2}$. Thus

$$\mathbf{A}(w_n^{1-n})=\frac{1}{|w_n|^{2n-2}}\mathbf{A}((\bar{w}_n)^{n-1}).$$

In addition, note that $N=\sqrt{u^2+v^2}=|w|=|w_n|^n=(u_n^2+v_n^2)^{n/2}$ in the variables of the cover Y_n.

By the construction of the proto-Lorenz system, the following is tautology.

Theorem 5.3. The double cover $\mathscr{L}_2$ of the proto-Lorenz system is the original Lorenz system.

6. The triple cover $\mathscr{L}_3$ of the proto-Lorenz system

The covers of the proto-Lorenz system each have the well-known strange attractor, with an interesting feature: the attractor of the nth cover $\mathscr{L}_n$ has n "ears". This generalizes the two-eared attractor of the original Lorenz system. In this section we will illustrate this by writing the equations of the triple cover $\mathscr{L}_3$ and presenting some numerical simulations of trajectories for this system.

The variables on Y_3 are u_3, v_3 and z, which we will rename p, q, and z respectively for convenience. Therefore $w_3=p+\mathrm{i}q$; since $w_3^3=(p^3-3pq^2)+\mathrm{i}(3p^2q-q^3)$, the equations for the map γ_3 are then

$$u=p^3-3pq^2\,,\qquad v=3p^2q-q^3\,,\qquad z=z\,.$$

The inverse of the Jacobian matrix for γ_3 is

$$(\mathrm{D}\gamma_3)^{-1}(w_3,z)=\begin{pmatrix}\dfrac{1}{3|w_3|^4}\mathbf{A}((\bar{w}_3)^2) & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & 1\end{pmatrix}=\begin{pmatrix}\dfrac{\frac{1}{3}(p^2-q^2)}{(p^2+q^2)^2} & \dfrac{\frac{2}{3}pq}{(p^2+q^2)^2} & 0\\ -\dfrac{\frac{2}{3}pq}{(p^2+q^2)^2} & \dfrac{\frac{1}{3}(p^2-q^2)}{(p^2+q^2)^2} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The proto-Lorenz system at the image $(u=p^3-3pq^2,\ v=3p^2q-q^3,\ z)$ of $(p,\ q,\ z)$ is

$$\begin{pmatrix}(-\sigma-1)(p^3-3pq^2)+(\sigma-r+z)(3p^2q-q^3)+(1-\sigma)N\\ (r-\sigma-z)(p^3-3pq^2)-(\sigma+1)(3p^2q-q^3)+(r+\sigma-z)N\\ \frac{1}{2}(3p^2q-q^3)-bz\end{pmatrix},$$

with $N=\sqrt{u^2+v^2}=(p^2+q^2)^{3/2}=M^3$, where we are writing $M=\sqrt{p^2+q^2}$. By (5.2), the product of these matrices is the expression for

$$\mathscr{L}_3(p,q,z)=\begin{pmatrix}\dot{p}\\ \dot{q}\\ \dot{z}\end{pmatrix}.$$

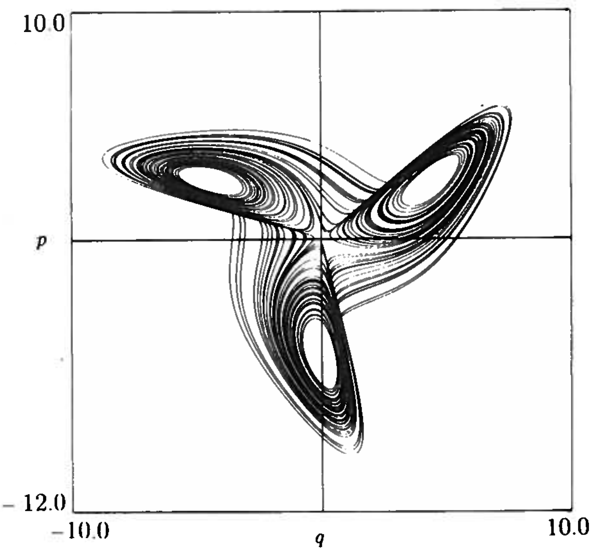

Fig. 5. The triple cover of the proto-Lorenz system, with $r=28.0$: p versus q.

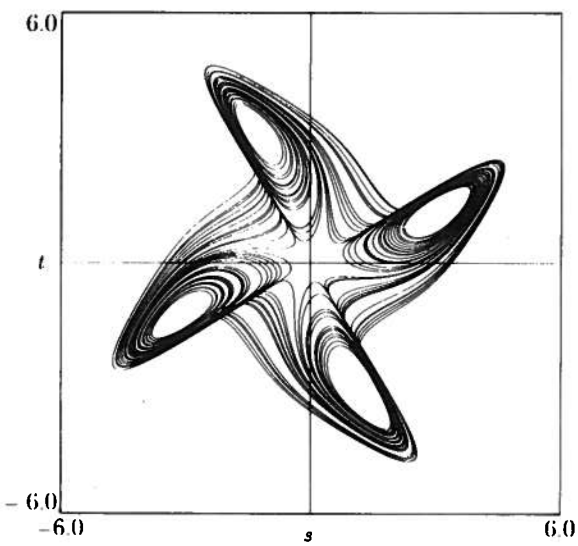

Fig. 6. The quartic cover of the proto-Lorenz system, with $r=28.0$: t versus s.

After some computation we obtain the formula for the triple cover $\mathscr{L}_3$ as

$$\begin{aligned}
\dot{p}&=\tfrac{1}{3}[-(\sigma+1)p+(\sigma-r+z)q]+[(1-\sigma)(p^2-q^2)+2(r+\sigma-z)pq]/3M\,,\\
\dot{q}&=\tfrac{1}{3}[(r-\sigma-z)p-(\sigma+1)q]+[2(\sigma-1)pq+(r+\sigma-z)(p^2-q^2)]/3M\,,\\
\dot{z}&=\tfrac{1}{2}(3p^2q-q^3)-bz\,.
\end{aligned} \tag{6.1}$$

Figure 5 shows a projection of the triple-cover system in the chaotic regime, $r=28$.

7. The quartic cover $\mathscr{L}_4$ of the proto-Lorenz system

In this section we will write down the equations for the vector field of the system $\mathscr{L}_4$, which is a 4-to-1 covering of the proto-Lorenz system $\mathscr{L}_1$. However it is more pleasant to think of $\mathscr{L}_4$ as a double cover of the original Lorenz system $\mathscr{L}_2$. This is possible, since the (complex) coordinate w_4 on the ambient space Y_4 satisfies $w_4^4=w_2^2=w_1$, so that in fact $w_2=w_4^2$. Indeed, for any m and n, if m divides n there is a natural map from Y_n to Y_m whose composition with γ_m is exactly γ_n.

Thus the coordinate w_4 stands in the same relation to w_2 as w_2 does to w_1. This simplifies the computation somewhat, and if we set $s=u_4$ and $t=v_4$ we obtain the equations

$$\dot{s}=[-\sigma s^3+(2\sigma+r-z)s^2t+(\sigma-2)st^2-(r-z)t^3]/2(s^2+t^2)\,,$$

$$\dot{t}=[(r-z)s^3+(\sigma-2)s^2t+(-2\sigma-r+z)st^2-\sigma t^3]/2(s^2+t^2)\,,\qquad \dot{z}=2s^3t-2st^3-bz$$

for the system $\mathscr{L}_4$.

Figure 6 shows a projection of the quartic-cover system in the chaotic regime, $r=28$.

References

[1] E.N. Lorenz, J. Atmos. Sci. 20 (1963) 130.
[2] J. Munkres, Topology: a first course (Prentice-Hall, Englewood Cliffs, NJ, 1975).
[3] C. Sparrow, The Lorenz equations (Springer, Berlin, 1982).
[4] J. Guckenheimer and P. Holmes, Applied mathematical sciences, Vol. 42. Nonlinear oscillations, dynamical systems, and bifurcations of vector fields (Springer, Berlin, 1983).
[5] O. Rössler, Phys. Lett. A 57 (1976) 397.
[6] R. Shaw, Z. Naturforsch. 36a (1981) 80.
[7] C.E. Shannon and W. Weaver, The mathematical theory of communication (University of Illinoid Press, Champaigne, IL, 1962).