Matching Conditions for Degenerating Plane Curves and Applications

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1 Introduction

The computation of the dimension of the space of plane curves of degree d with prescribed multiple base points has drawn great interest for some time. Several conjectures, notably the Segre Conjecture [17] and [5], the Nagata Conjecture [16], and the Harbourne-Hirschowitz Conjecture [13], [9], all deal with aspects of this problem. A number of methods have been developed, each having some success. For an arbitrary number of points, there are several results dealing with small multiplicities (at most M): Hirschowitz (via the Horace Method) for $M = 3$ [12], Mignon (also via the Horace Method) for $M = 4$ [15], and lately Yang (using a variety of methods) for $M = 7$. The authors have previously developed a technique based on a degeneration of the plane [3],[4] which treated the case of uniform multiplicities $m \leq 20$.

Less is known about larger multiplicities, even in the case of a fixed number $n > 9$ of points. Up to now there are only three approaches, each having some limited success. One is that of Evain [6], [7]; another is that of Harbourne-Roé [10]; and a third is that of Buckley and Zompatori [2], which is based on our original method.

This degeneration technique involved a reduction to studying two related systems on two components of a reducible degeneration of the plane. In order to prove that the original linear system in question had the expected dimension, it proved to be sufficient for the linear systems on the components to have the expected dimension. This required that these systems had reduced members in general. The analysis failed in case either one of the systems had a multiple base component. In this article we attempt to recover from this by introducing a more refined analysis of the situation.

We are able to apply this more refined analysis to make several computations which were previously impossible, both with our prior methods, or with any other of the other methods; examples are presented in the final section.

The analysis permits an understanding of the limit scheme when fat points come together at a single point in various cases; this we present in Section 3. These ideas are not unrelated to those of Evain, see [6] and [7]. When fat points approach a line, the method also provides in certain situations an analysis of the limiting system of curves, and we develop this for three, four, and five fat points in Sections 5 and 6. These lemmas have been exploited in [18] to prove the Harbourne-Hirschowitz Conjecture for multiplicities at most 7. These two threads (fat points approaching a point and approaching a line) are illustrated in the final section where we present examples utilizing both techniques.

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2 Matching Conditions

Let $p : \mathcal{X} \to \Delta$ be a proper flat family of surfaces with smooth total space $\mathcal{X}$. For $t \neq 0$, we assume that the fiber $X_t = p^{-1}(t)$ is smooth and connected. We assume that the central fiber X_0 is reduced but reducible, consisting of a smooth surface V and a union of smooth surfaces $W = \cup W_i$ meeting transversally along a (possibly reducible) curve $R = \cup R_i$, which has all smooth components $R_i = V \cap W_i$.

Fix a smooth irreducible curve $E \subset V$ which meets R transversally in $\tau = (E \cdot R)_V$ points. Let $\mathcal{C} \subset \mathcal{X}$ be an irreducible surface which is, via p, a proper flat family of curves $C_t \subset X_t$. We further assume that

$$\sigma = -(\mathcal{C} \cdot E) > 0.$$

The central fiber C_0 of the curve family $\mathcal{C}$ consists of a curve $C_V \subset V$ and $C_W \subset W$. Note that $(\mathcal{C} \cdot E) = (C_V \cdot E)$, and therefore the above assumption implies that $E \subset C_V$; hence since C_0 is a divisor on X_0, the curve C_W must pass through the points $r_1, \ldots, r_\tau$ where E meets R and W. It is our intention to understand better how the curve C_W behaves at these τ points.

Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowup of $\mathcal{X}$ along the curve E. This will create the exceptional divisor $T \subset \mathcal{X}'$ which will be a ruled surface; E becomes a section of the ruling on T. For $t \neq 0$, $X'_t = X_t$, but the central fiber X'_0 consists now of three parts, namely T and the proper transforms V' and W' of V and W respectively. Note that $V' \cong V$, but W' is isomorphic to the blowup of W at the τ points on R where E intersects R. We will denote by $F_1, \ldots, F_\tau$ the exceptional curves on W', all of which are fibers of the ruling of T.

Denote by $\mathcal{C}'$ the strict transform of $\mathcal{C}$. There is an integer α such that

$$\mathcal{C}' = \pi^*(\mathcal{C}) - \alpha T;$$

α is the multiplicity with which $\mathcal{C}$ contains E. The assumption that $\sigma > 0$ implies that $E \subset \mathcal{C}$, and so $\alpha > 0$. Note that by restricting to V, we see that C_V must contain E as a component with multiplicity $\mu \geq \alpha$; by restricting to W, we conclude that C_W passes through the points $r_1, \ldots, r_\tau$ with multiplicity at least α.

This condition, that C_W must have multiplicity at these points (or in some cases as noted later an even more complicated singularity) will be referred to as a *matching condition*, justifying the title of the article.

Proposition 2.1 *With the above notation,* *(a) if* $\tau + (E^2)_V \geq 0$*, then* $\alpha \geq \sigma/\tau$*.*

(b) if $\tau + (E^2)_V \leq 0$*, then* $\alpha \geq -\sigma/(E^2)_V$*.*

Proof: We first note that

$$T|_T = -E - \sum_i F_i \sim -E - \tau F$$

(where $\sim$ denotes numerical equivalence) and that

$$\pi^*(\mathcal{C})|_T = \pi^*(\mathcal{C})|_{\pi^*(E)} = \pi^*(\mathcal{C}|_E) \sim -\sigma F$$

where F is the class of a fiber of T. Hence

$$\mathcal{C}'|_T \sim -\sigma F - \alpha(-E - \tau F) = \alpha E + (\alpha\tau - \sigma)F. \tag{2.2}$$

Also note that by the triple point formula [14],

$$(E^2)_T = -\tau - (E^2)_V.$$

Note that $\mathcal{C}'|_T$ is effective.

To prove (a), we then have that $(E^2)_T \leq 0$; in this case if $\alpha\tau - \sigma < 0$, we could not have $\mathcal{C}'|_T$ being effective, and so $\alpha\tau - \sigma \geq 0$ as claimed.

To prove (b), we then have $(E^2)_T \geq 0$; then E is nef on T, and so $(E \cdot \mathcal{C}'|_T)_T \geq 0$. But

$$(E \cdot \mathcal{C}'|_T)_T = \alpha(E^2)_T + \alpha\tau - \sigma =$$
$$= \alpha(-\tau - (E^2)_V) + \alpha\tau - \sigma = -\sigma - \alpha(E^2)_V$$

which, being nonnegative, gives the result.

Q.E.D.

Corollary 2.3 *With the above notation, suppose that* E *is a* (-1)*-curve on* V*, i.e.,* E *is smooth, rational, and* $E^2 = -1$*. Then* $\alpha \geq \sigma/\tau$*.*

Proof: In this case $\tau + (E^2)_V = \tau - 1 \geq 0$ so case (a) of the Proposition applies.

Q.E.D.

When E is a (-1)-curve on V, the relation $(C_V \cdot E) = (\mathcal{C} \cdot E) = -\sigma$ implies that the linear system $|C_V|$ on V in which the restricted curve must lie must contain E as a base curve with multiplicity σ: $|C_V| = \sigma E + |M|$, where $M = C_V - \sigma E$ is the residual system.

The above corollary is most commonly used in the situation when the (-1)-curve E meets R in one point (so that $\tau = 1$). We conclude in this case that $\alpha \geq \sigma$, which means that the curve C_W on the other surface must have a point of multiplicity at least σ where it meets E. We note for reference that this imposes $\sigma(\sigma+1)/2$ conditions on the curve C_W.

The second most common use of the above considerations is when the (-1)-curve E meets the R in two points ($\tau = 2$). If we write $\sigma = 2\ell - e$, where $e = 0, 1$, the above result implies that at each of the two points where E meets R, the corresponding curve C_W must have multiplicity ℓ.

However this is not the whole story in this case. Note that since E is a (-1)-curve, and $\tau = 2$, the ruled surface T obtained by blowing E up is a ruled surface isomorphic to the minimal ruled surface $\mathbb{F}_1$, and by the triple point formula the curve E (where T meets V) is the (-1)-curve on T also. By (2.2), the restriction of the proper transform of the family of curves to T is

$$\mathcal{C}'|_T \equiv \alpha E + (\alpha\tau - \sigma)F$$

(where $\equiv$ denotes linear equivalence). Since $\alpha \geq \ell$, we see that $|\mathcal{C}'|_T|$ is contained in the linear system $|\ell E + (2\ell - (2\ell - e))F| = |\ell E + eF|$. If $e = 0$, we see that E is still a base curve for the family $\mathcal{C}'$, with multiplicity ℓ on T. If $e = 1$, the multiplicity of E in this system on T is $\ell - 1$. If $\ell = e = 1$ (the $\sigma = 1$ case), there is in fact no more to do, and the matching condition on C_W is simply that the curve pass through the two points. We will therefore assume in what follows that $\sigma \geq 2$.

In this case we blow up E again, creating a new ruled surface S, which is in fact a product surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (this follows from using the triple point formula to conclude that the self-intersection of E on S is zero). The same type of analysis as before shows that if $\mathcal{C}''$ is the proper transform of the family $\mathcal{C}'$ of curves on this second blowup, then

$$\mathcal{C}''|_S \equiv (\ell - e)E.$$

Indeed, using similar notation, we see that $\mathcal{C}'' = \pi^*\mathcal{C}' - (\ell - e)S$ and since $S|_S \equiv -E - F$,

$$\mathcal{C}''|_S = (\pi^*\mathcal{C}' - (\ell - e)S)|_S \equiv -(\ell - e)F - (\ell - e)(-E - F) = (\ell - e)E.$$

From this we conclude two things. First, the original curve C_W on W not only has to meet R in the two points of multiplicity ℓ, but these are in fact tacnodal, in the sense that there are infinitely near points of multiplicity $\ell - e$ required at both of these points. We see this because the proper transform as noted above meets the second exceptional curve at $\ell - e$ points. Secondly, we note that the divisor (of degree $\ell - e$) cut out on the two infinitely near exceptional curves on W must correspond, via the isomorphism afforded by the horizontal sections on the surface S: these two exceptional curves are both fibers of S, and the linear system on S consists of $\ell - e$ sections. The zero-sections of S give a natural correspondence between these two exceptional curves, and under this correspondence the two divisors of degree $\ell - e$ must agree. We will refer to this pair of tacnodal singularities as *a pair of tacnodal $(\ell, \ell - e)$-points, with corresponding second-order tangents*. We note finally that these two singularities impose, a priori,

$$\begin{aligned}
\ell(\ell+1) + (\ell - e)(\ell - e + 1) + \ell - e &= 2\ell^2 + (3 - 2e)\ell + e^2 - 2e \\
&= 2\ell^2 + (3 - 2e)\ell - e \\
&= \begin{cases} \ell(2\ell + 3) & \text{if } e = 0 \\ 2\ell^2 + \ell - 1 & \text{if } e = 1 \end{cases}
\end{aligned}$$

conditions on the matching curve C_W, if $\sigma \geq 2$.

We have proved the following.

Corollary 2.4 *With the above notation, suppose that E is a (-1)-curve on V.*

(a) If E meets the curve R in one point p, then the matching condition for the curve C_W is that it must have a point of multiplicity at least σ at p.

(b) If E meets the curve R in two points p_1, p_2, and $\sigma = 2\ell - e$ with $e = 0, 1$, then the matching condition for the curve C_W is that it must have a pair of tacnodal $(\ell, \ell - e)$-points, with corresponding second-order tangents, at the two points p_i.

Of course, the (-1)-curve E may meet the curve R in more than two points. We will not analyze this situation here, since these cases do not occur in the applications presented below.

3 Collisions of fat points

As a first application of the above analyses, let us use the above considerations to analyze degenerations of schemes defined by a collection of *fat point*s, i.e., points with multiplicity, on a surface X. The analysis is valid for arbitrary collections of multiplicities, but we will assume for simplicity that all the multiplicities are the same, equal to m.

Suppose that we have an ample divisor H on X, and we want to analyze the divisors in the linear system $|dH|$ for large d when the fat points come together.

Our approach will be the following. Suppose that k distinct fat points of multiplicity m degenerate in a general way to a point p on a surface X. Let us explain what we mean here by this.

Construct the trivial family $X \times \Delta$, where Δ is a disc, and in the central fiber blow up the point $(p, \{0\})$, to a plane $P \cong \mathbb{P}^2$. The new central fiber then consists of two surfaces, the blowup X' of X at p, and the plane P, meeting along a smooth rational curve R (which is the exceptional curve in X' and is a line in P). The assumption that the points are degenerating generally means that the k limit points are k general points on P.

If we denote by H the pullback bundle on X', one limiting line bundle on the central fiber is the bundle which is dH on X' and is trivial on P. Other limiting line bundles to dH on the general fiber are obtained by twisting this basic limiting line bundle by the divisor $-tP$, for any integer t; this gives the limiting line bundle which is $dH - tR$ on X', and $\mathcal{O}(t)$ on the plane P. Any general curve in dH on the general X with multiplicity m at the k points will degenerate to a curve in the central fiber, which must be a union of two curves, one in X' and one in P, satisfying the matching conditions as given in Section 2; these two curves must be divisors in the two limiting linear systems for some t.

One can make an analysis for every t, obtaining different limiting schemes; the constraint is that we must have both line bundles effective, so that the limiting curves in each of the surfaces exist. Note that for such a t, the curve in X' must be a divisor in $|dH - tR|$, and so the limiting curve on X must have a point of multiplicity t. We choose t so that it is minimal with respect to the property that $\mathcal{O}(t)$ has nonzero sections with multiplicity m at the k general limit points on P; this is in some sense the most general limit, since it gives rise to the lowest possible multiplicity of the limit curve on X'.

The extra conditions on the limiting curve, which define the limit scheme of the k fat points, are obtained by analyzing the matching conditions as given in Section 2.

One must make a separate analysis for each k. This we will execute for $k \leq 5$ in the following.

Proposition 3.1 *Suppose that k fat points of multiplicity m degenerate together on a surface X, to a point $p \in X$. Then the general limiting scheme (for $k \leq 5$) is given as follows.*

(a) If $k = 2$, the limit is a point of multiplicity m with an infinitely near point of multiplicity m.

(b) If $k = 3$ and $m = 2\ell - e$ with $e = 0, 1$, the limit is a point of multiplicity $3\ell - e$, with three infinitely near points of multiplicity $\ell - e$.

(c) If $k = 4$, there is an involution ι on the $\mathbb{P}^1$ which is the first-order neighborhood of the point p, and the limit is a point of multiplicity $2m$, with the extra condition that the tangent cone divisor D of degree $2m$ on the $\mathbb{P}^1$ is of the form $D = D_1 + D_2$ with $D_2 = \iota(D_1)$.

(d) If $k = 5$, and $m = 2\ell - e$ with $e = 0, 1$, the limit is a point of multiplicity $2m$ with a pair of infinitely near tacnodal $(\ell, \ell - e)$-points, with corresponding second-order tangents.

Proof: If $k = 2$, we have that the minimal degree on the plane P is $t = m$, and the only divisor in the linear system on P is m times the line L joining the two points of multiplicity m. Indeed, when we blow up the two points, L becomes a (-1)-curve on the blow up of P, with $\sigma = m$ (using the notation of Section 2), and $\tau = 1$. Hence the matching condition on the curves of X' are that they must have a point of multiplicity m at the point of intersection $L \cap R$, by Corollary 2.4(a). This means that on X we have a point of multiplicity m with an infinitely near point of multiplicity m, as stated in part (a).

Suppose that $k = 3$ and $m = 2\ell - e$ with $e = 0, 1$. Then the minimal degree t on P is $t = 3\ell - e$. The three lines joining pairs of the multiple points on P become (-1)-curves on the blow up of P at the three points, and their intersection with the system is $(3\ell - e) - 2(2\ell - e) = e - \ell$, so that $\sigma = \ell - e$ and $\tau = 1$ for each of these three lines. Again, by Corollary 2.4(a), this implies that the matching curves on X' must have three points of multiplicity $\ell - e$. Hence on X' we have a point of multiplicity $3\ell - e$, with three infinitely near points of multiplicity $\ell - e$ as claimed in (b).

Suppose that $k = 4$; then the minimal degree on P is $t = 2m$, and the linear system is composed with the pencil of conics through the four points. This pencil gives an involution on the curve R, by sending a point on R to the second intersection with R of the unique conic in the pencil through that point. The matching condition on X' is simply that the divisor of degree $2m$ on the curve R must be matched by one of the divisors obtained by restricting m members of the pencil of conics to R. This is the involution condition as stated in (c).

Suppose that $k = 5$; then the minimal degree on P is again $t = 2m$, and this time the unique member of the linear system on P is the conic C with multiplicity m. This conic C becomes a (-1)-curve on the blow up of P at the five points, and this time $\sigma = m = 2\ell - e$ and $\tau = 2$; hence Corollary 2.4(b) applies, and the matching condition on X' is that claimed in part (d).

Q.E.D.

It is instructive to compare the lengths of these limit schemes to the length of the scheme consisting of the k fat points of multiplicity m (which is $km(m+1)/2$).

For $k=2$, the limit scheme consists of one point of multiplicity m and one infinitely near point of multiplicity m (an (m,m) *tacnode*) and so the length is equal to $2 \cdot m(m+1)/2$ as required.

For $k=3$, the limit scheme consists of one point of multiplicity $3\ell - e$ and three infinitely near points of multiplicity $\ell - e$; thus the total length is

$$(3\ell - e)(3\ell - e + 1)/2 + 3(\ell - e)(\ell - e + 1)/2 = 6\ell^2 + (3 - 6e)\ell$$

which is the same as $3m(m+1)/2$.

For $k=4$, the limit scheme has length equal to the length of a fat point of multiplicity $2m$, plus the length of the involution condition (which is m linear conditions): this is therefore

$$(2m)(2m+1)/2 + m = 2m^2 + 2m$$

which is the same as $4 \cdot m(m+1)/2$ as needed.

Finally for $k=5$, the limit scheme has length equal to the length of the fat point of multiplicity $2m$, plus the length of the pair of tacnodal singularities with the correspondence condition. If $m=1$, we have a point of multiplicity 2 and two fixed tangents, which has length 5 as required. If $m \geq 2$, and $m = 2\ell - e$ as above, then the computation of the number of conditions for the tacnodal pair of singularities was made just before Corollary 2.4; it is $2\ell^2 + (3-2e)\ell - e$. Thus we have a total length of

$$(4\ell - 2e)(4\ell - 2e + 1)/2 + 2\ell^2 + (3 - 2e)\ell - e = 10\ell^2 + (5 - 10e)\ell$$

which is the same as $5 \cdot m(m+1)/2$ as it should be.

We remark that these limit schemes (for $k=4$ and $k=5$) are not simply defined by their base loci.

One could continue in the same spirit and make this type of analysis for larger k. However this would require the knowledge of the minimal t such that the system on P is nonempty, and also some detailed information about that system. As it is classically known, this is available for k up to 9, (see for instance [16]), but it becomes significantly more involved. For larger k even the minimal t is not known in general (although there are precise conjectures, see [9], [13]).

4 Proving linear systems are empty via degeneration

In this section we want to explain how to use a degeneration technique, combined with the matching conditions explained above, to prove that a complete linear system of curves on a surface is empty under suitable hypotheses. The degeneration technique is modeled after that used in [3], and will be useful in proving that certain linear systems of plane curves with prescribed multiple base points is empty. The matching lemmas developed in Section 2 permit a more refined analysis.

To be specific, suppose that one wants to prove that the linear system L on a surface $\mathbb{P}'$ is empty. Take a smooth flat family $\mathcal{P} \to \Delta$, where Δ is a disc, whose general fiber is $\mathbb{P}'$, and whose special fiber is the union of two smooth surfaces $\mathbb{P}$ and $\mathbb{F}$, meeting transversally along the smooth curve R.

Denote by $\mathcal{L}$ a line bundle on the threefold $\mathcal{P}$ whose restriction to the general fiber $\mathbb{P}'$ is the desired bundle L. $\mathcal{L}$ will restrict to line bundles $L_{\mathbb{P}}$ on $\mathbb{P}$ and to $L_{\mathbb{F}}$ on $\mathbb{F}$.

One can also take the bundle

$$\mathcal{L}(k) = \mathcal{L} \otimes \mathcal{O}_{\mathcal{P}}(k\mathbb{P})$$

to produce a limit of the line bundle L; note that these give the only limits of L on the central fiber.

The bundle $\mathcal{L}(k)$ restricts to the pairs of bundles $L_{\mathbb{F},k} = L_{\mathbb{F}}(kR)$ on $\mathbb{F}$ and $L_{\mathbb{P},k} = L_{\mathbb{P}}(-kR)$ on $\mathbb{P}$.

Suppose now that we have a family of curves $\mathcal{C}$ on the general fiber as considered above. Closing this family up to the special fiber, one obtains two curves $C_{\mathbb{F}}$ in $\mathbb{F}$ and $C_{\mathbb{P}}$ in $\mathbb{P}$. There will be a unique k such that $C_{\mathbb{P}}$ is a divisor in the system $L_{\mathbb{P},k}$, and $C_{\mathbb{F}}$ is a divisor in the system $L_{\mathbb{F},k}$.

Suppose that one has a set of (-1)-curves $E \subset \mathbb{F}$, meeting R transversally in at most two points, such that $E \cdot L_{\mathbb{F},k} < 0$; each of these curves gives a matching condition on the sections of $L_{\mathbb{P},k}$ according to the analysis of Section 2.

Similarly, there may be (-1)-curves $E \subset \mathbb{P}$, meeting the double curve transversally in at most two points, with $E \cdot L_{\mathbb{P},k} < 0$; these give matching conditions on the sections of $L_{\mathbb{F},k}$ in the same way.

Define $L^m_{\mathbb{P},k}$ to be the system on $\mathbb{P}$ defined by these matching conditions; this is a subsystem of $L_{\mathbb{P},k}$. Similarly define the subsystem $L^m_{\mathbb{F},k}$. We refer to these subsystems as the "matching systems".

Define $\hat{L}_{\mathbb{P},k}$ to be the subsystem of $L_{\mathbb{P},k}$ consisting of those elements vanishing on R; similarly define $\hat{L}_{\mathbb{F},k}$. These form the kernels of the natural restriction map to the systems on the double curve R, and we will therefore refer to them as the *kernel systems*. We also have the intersections $\hat{L}^m_{\mathbb{P},k}$ and $\hat{L}^m_{\mathbb{F},k}$ of the matching systems with the kernel systems.

The content of the matching lemmas developed in the previous section is exactly the following. If the system L on the general fiber $\mathbb{P}'$ is not empty, we can choose a family $\mathcal{C}$ of curves in this system, flat over Δ. For this family there will be a unique k such that the limit is a curve $\mathcal{C}_0 = C_{\mathbb{P}} \cup C_{\mathbb{F}}$ with $C_{\mathbb{P}}$ a member of $L^m_{\mathbb{P},k}$ and $C_{\mathbb{F}}$ a member of $L^m_{\mathbb{F},k}$; $C_{\mathbb{P}} = \mathcal{C} \cap \mathbb{P}$ and $C_{\mathbb{F}} = \mathcal{C} \cap \mathbb{F}$ scheme-theoretically. Furthermore, the restriction of $C_{\mathbb{P}}$ and $C_{\mathbb{F}}$ to the curve R must be the same.

Note that the curve $\mathcal{C}_0$ contains the double curve R if and only if both $C_{\mathbb{P}}$ and $C_{\mathbb{F}}$ are members of the kernel systems.

With these observations, the following Proposition is clear.

Proposition 4.1 *With the above notation and assumptions, suppose that for every k, there are no pairs of curves $(C_{\mathbb{P}}, C_{\mathbb{F}})$ in the corresponding matching systems $L^m_{\mathbb{P},k}$ and $L^m_{\mathbb{F},k}$ whose restrictions to the curve R agree. Then the system L on the general surface $\mathbb{P}'$ is empty.*

Note that the assumption implies that, for every k, at least one of the two kernel matching systems $\hat{L}^m_{\mathbb{P},k}$ or $\hat{L}^m_{\mathbb{F},k}$ is empty. Indeed, if neither is empty, then a limit curve containing the curve R is possible.

We can refine this Proposition by making the following observation:

Lemma 4.2 *With the above notations and assumptions, we have*

$$L^m_{\mathbb{P},k+1} = \hat{L}^m_{\mathbb{P},k} \text{ and}$$
$$L^m_{\mathbb{F},k-1} = \hat{L}^m_{\mathbb{F},k}$$

Proof: The content of the lemma is that, in these situations, the matching conditions with one less twist by the double curve R gives the same result as considering the kernel system for the matching system. The considerations are the same for the two surfaces; let us just consider the $\mathbb{P}$ system.

Suppose that E is a (-1)-curve on $\mathbb{F}$ with $(E \cdot L_{\mathbb{F},k}) = -\sigma$, and $(E \cdot R) = 1$. The matching condition in this case is that the divisors in $L_{\mathbb{P},k}$ must have a point of multiplicity σ at $p = E \cap R$. The kernel system consists of divisors which contain R, and so the residual system consists of divisors in $L_{\mathbb{P},k+1}$ with a point of multiplicity $\sigma - 1$ at p. Since

$$(E \cdot L_{\mathbb{P},k+1}) = (E \cdot L_{\mathbb{P},k}) - (E \cdot R) = \sigma - 1$$

this is exactly the same as the matching conditions imposed by E on $\mathcal{L}_{\mathbb{P},k+1}$ as claimed.

If instead $(E \cdot R) = 2$, the corresponding σ will decrease by two, but the multiplicity of the matching conditions (in this case given by the pair of tacnodal points with corresponding second-order tangents) goes down by one, exactly the contribution of the double curve R.

This proves the result.

Q.E.D.

This allows us to refine Proposition 4.1 as follows:

Proposition 4.3 *Suppose that there exists a k_0, such the system $L^m_{\mathbb{P},k_0}$ on $\mathbb{P}$ is empty and the kernel system $\hat{L}^m_{\mathbb{F},k_0}$ on $\mathbb{F}$ is also empty. Then the system L on the general surface $\mathbb{P}'$ is empty.*

Proof: Suppose on the contrary that the system L on the general surface is non-empty. Then there will exist a twist k and two curves in the matching systems $L^m_{\mathbb{P},k}$ on $\mathbb{P}$ and $L^m_{\mathbb{F},k}$ on $\mathbb{F}$ which is a limit of the general curve. This k cannot be greater than or equal to k_0, since the matching system $L^m_{\mathbb{P},k_0}$ is empty, and this implies that $L^m_{\mathbb{P},k}$ is also empty for $k \geq k_0$, since they are subsystems.

By Proposition 4.2, $L^m_{\mathbb{F},k_0-1} = \hat{L}^m_{\mathbb{F},k_0}$ is empty, and therefore as above all subsystems $L^m_{\mathbb{F},k}$ with $k < k_0$ are empty.

We conclude that for every k, one of the two systems $L^m_{\mathbb{P},k}$ or $L^m_{\mathbb{F},k}$ is empty, and hence by Proposition 4.1, L is empty.

Q.E.D.

In the next sections we will apply the results above to make some interesting reductions for analyzing linear systems of plane curves with prescribed multiple base points.

5 The Three Point Lemma

We begin with a preliminary computation which is necessary to execute the first of our reductions. We denote by $L_d(n_0, \ldots, n_r)$ the linear system of plane curves of degree d with $r+1$ general base points p_i having multiplicities n_i.

Consider $L = L_d(m_0, m_1, m_2, m_3)$ and denote by ℓ_i the line joining p_0 to p_i. Note that each ℓ_i becomes a (-1)-curve on the blowup of the p_i's.

Lemma 5.1 *Fix a degree d and non-negative integers m_1, m_2, m_3. Let $s = m_1 + m_2 + m_3$.*

(a) If $s \leq d$ then the smallest m_0 such that L is empty is $m_0 = d+1$. Moreover $L_d(d, m_1, m_2, m_3) = \sum_i m_i \ell_i + L_{d-s}(d-s)$ and has dimension $d-s$.

(b) If $s > d$ then L is empty if $m_0 \geq 2d - s + 1$.

(c) Assume $s > d$ and $m_i \geq s - d$ for each $i = 1, 2, 3$. Then $L_d(2d-s, m_1, m_2, m_3) = \sum_i (d - s + m_i)\ell_i + L_{2s-2d}(s-d, s-d, s-d, s-d)$ and has dimension $s-d$ (and the general member consists of $s-d$ conics in the pencil through the four points).

(d) Assume $s > d$, $r \geq 0$, and $2d \geq s + r \geq d$. Then ℓ_i splits off $L_d(2d-s-r, m_1, m_2, m_3)$ exactly $\max(0, d-s+m_i-r)$ times.

Proof: Statements (a), (c), and (d) are trivial, and (b) is just the fact that the excess intersection of L with a smooth conic through the four points cannot be negative.

Q.E.D.

Let's try to systematically apply the degeneration method as in [3]. This method uses a degeneration of the plane to a plane $\mathbb{P}$ union a ruled surface $\mathbb{F} \cong \mathbb{F}_1$, meeting along a line R in the plane attached to the (-1)-curve in $\mathbb{F}$. This is achieved by blowing up a line in the central fiber of the trivial family of planes.

We divide the base points by putting three points on the $\mathbb{F}$ side as follows. Suppose that we are considering the original system $L = L_d(m_1, m_2, m_3, \ldots, m_n)$. Let $s = m_1 + m_2 + m_3$, and assume that $s > d$ and $m_i \geq s - d$ for each $i = 1, 2, 3$.

Apply the degeneration method putting the first three points on $\mathbb{F}$ and twist by $k_0 = s-d$, i.e., subtract $(s-d)\mathbb{P}$, so that the relevant four systems are:

$$\begin{aligned}
\hat{L}_{\mathbb{F},k_0} &: L_d(2d-s+1, m_1, m_2, m_3) \\
L_{\mathbb{F},k_0} &: L_d(2d-s, m_1, m_2, m_3) \\
L_{\mathbb{P},k_0} &: L_{2d-s}(m_4, m_5, \ldots, m_n) \\
\hat{L}_{\mathbb{P},k_0} &: L_{2d-s-1}(m_4, m_5, \ldots, m_n)
\end{aligned}$$

(We are identifying $\mathbb{F}$ implicitly with a plane blown up at one point, and therefore systems on $\mathbb{F}$ with r base points correspond in an obvious way to systems on the plane with $r+1$ base points.)

By the previous Lemma, the kernel system $\hat{L}_{\mathbb{F},k_0}$ is empty, and $\dim(L_{\mathbb{F},k_0}) = s - d$.

The matching conditions give three extra points on the $\mathbb{P}$ surface, lying on the line R, having multiplicities $d - s + m_i$, $i = 1, 2, 3$ (using (c) of the previous Lemma). In other words, we have

$$\begin{aligned}
&\hat{L}_{\mathbb{F},k_0} : L_d(2d - s + 1, m_1, m_2, m_3) = \emptyset \\
&L^m_{\mathbb{F},k_0} : L_d(2d - s, m_1, m_2, m_3) \\
&L^m_{\mathbb{P},k_0} : L_{2d-s}([d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2], m_4, \dots, m_n) \\
&\hat{L}^m_{\mathbb{P},k_0} : L_{2d-s-1}([d - m_2 - m_3 - 1, d - m_1 - m_3 - 1, d - m_1 - m_2 - 1], \\
&\qquad m_4, \dots, m_n)
\end{aligned}$$

where the brackets indicate that these points are collinear, and otherwise general.

Note that the virtual dimension of $\mathcal{L}^m_{\mathbb{P}}$ is the same as the virtual dimension of the original system $\mathcal{L}$ on the plane. Suppose that this virtual dimension is negative, so that we are trying to use the method to show that $\mathcal{L}$ is empty.

Since we have that the kernel system $\hat{L}_{\mathbb{F},k_0}$ is certainly empty, the following is immediate by Proposition 4.3:

Proposition 5.2 *If $L^m_{\mathbb{P},k_0} = L_{2d-s}([d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2], m_4, \dots, m_n)$ is empty, then L is empty.*

This is a significant reduction, modulo the hypothesis that the extra three multiplicities created by the matching condition all are with points that lie on a line. We note that the multiplicity numbers that arise with the three-point lemma are exactly those that would arise from performing a quadratic Cremona transformation centered at the three points.

Remark 5.3 The above Proposition was formulated to consider the linear system L of degree d with general multiple base points. The result is a reduction to a system for which three of the points are collinear. It is not necessary in fact for the original system to have all *general* base points: one can apply the same construction in case there are constraints to the positions of the points. The only condition that is needed to apply the construction is that the three base points that one wants to make collinear are general enough to permit them to become collinear, and that this specialization does not affect whatever constraints there may be on the other points.

In any case, the reduction algorithm to prove in this way that a system L as above is empty is:

1. Choose three multiplicities m_1, m_2, m_3 so that $s = m_1 + m_2 + m_3 \geq d + 1$ and $m_i \geq s - d$ for each $i = 1, 2, 3$.
2. Reduce the degree of the system from d to $2d - s$.
3. Reduce these three multiplicities to $d - m_2 - m_3$, $d - m_1 - m_3$, $d - m_1 - m_2$ and put these points on a line.
4. Show that the resulting system is empty.

6 The Four- and Five-Point Lemmas

Let $L = L_d(m_0, m_1, m_2, m_3, m_4)$ denote the linear system of plane curves of degree d with five general multiple points $p_0, \ldots, p_4$ with multiplicities $m_0, \ldots, m_4$. Denote by ℓ_i the line joining p_0 to p_i, and by C the conic through the five points. These all become (-1)-curves on the blowup of the points.

Lemma 6.1 *Fix a degree d and non-negative integers $m_1, \ldots, m_4$. Let $s = m_1 + m_2 + m_3 + m_4$.*

(a) If $s \leq d$ then the smallest m_0 such that L is empty is $m_0 = d + 1$. One has $L_d(d, m_1, m_2, m_3, m_4) = \sum_i m_i\ell_i + L_{d-s}(d - s)$ and has dimension $d - s$.

(b) If $s > d$ write $s - d = 2t + e$ with $t \geq 0$ and $0 \leq e \leq 1$. Then L is empty if $m_0 = d + 1 - t - e$.

(c) Assume $s > d$, $s - d = 2t + e$ as above, and $t + e \leq d$ and $t + e \leq m_i$ for each i. Then $L_d(d - t - e, m_1, m_2, m_3, m_4) = \sum_i(m_i - t - e)\ell_i + tC + L_{3e}(2e, e^4)$ and has dimension $2e$.

(d) Assume $s > d$, $s - d = 2t + e$ as above, and $r \geq 0$. If $L_d(d - t - e - r, m_1, m_2, m_3, m_4)$ is not empty, then ℓ_i splits off $L_d(d - t - e - r, m_1, m_2, m_3, m_4)$ exactly $\max(0, m_i - t - e - r)$ times, and the conic C splits off exactly $\max(0, t - r)$ times.

Proof: Statement (a) is obvious; since $L_d(d, m_1, m_2, m_3, m_4) \cdot \ell_i = -m_i$, the line ℓ_i splits off m_i times leaving the residual system $L_{d-s}(d - s)$. Since this residual system is non-empty, the smallest m_0 such that L is empty is at least $d + 1$. Since $m_0 = d + 1$ gives an empty system, we are done.

To prove (b), consider a general cubic C passing through the five points, with a double point at the first point. Its proper transform on the five-fold blowup of the plane is nef. However

$$(C \cdot L) = 3d - 2(d + 1 - t - e) - s = d - s + 2t + 2e - 2 = e - 2 < 0$$

which is a contradiction if L is not empty.

To prove (c), let $L' = L_d(d - t - e, m_1, m_2, m_3, m_4)$, and note that $L' \cdot \ell_i = t + e - m_i$ which is nonpositive by assumption; therefore ℓ_i splits off the system $m_i - t - e$ times. Since $\sum_i(m_i - t - e) = s - 4t - 4e$, the residual system has degree $d - (s - 4t - 4e) = 4t + 4e - (2t + e) = 2t + 3e$, and the multiplicity of the residual system at p_0 is $d - t - e - (s - 4t - 4e) = 3t + 3e - (2t + e) = t + 2e$. The multiplicity of the residual system at p_i for $i > 0$ is $t + e$; therefore the residual system is $L_d(d, m_1, m_2, m_3, m_4) = \sum_i m_i\ell_i + L_{d-s}(d - s)$ and has dimension $d - s$.

The intersection of this residual system with the conic C is $2(2t + 3e) - (t + 2e) - 4(t + e) = 4t + 6e - t - 2e - 4t - 4e = -t$ so that C splits off the system t times. After splitting off C, this further residual system is $L_{3e}(2e, e^4)$ which has dimension 0 if $e = 0$ and has dimension 2 if $e = 1$.

Statement (d) follows by a similar analysis.

QED

Let's try to systematically apply the method outlined above using four points on the $\mathbb{F}$ side as follows. Suppose that we are considering the original system $L = L_d(m_1, \dots, m_n)$ (with $n \geq 4$). Let $s = m_1 + m_2 + m_3 + m_4$, assume that $s > d$, and write $s - d = 2t + e$ as above. Suppose further that $t + e \leq d$ and $t + e \leq m_i$ for $1 \leq i \leq 4$.

Apply the degeneration method putting the first four points on $\mathbb{F}$ and twist by $k_0 = t+e$, so that the relevant four systems are:

$$\begin{aligned} \hat{L}_{\mathbb{F},k_0} &: L_d(d - t - e + 1, m_1, m_2, m_3, m_4) \\ L_{\mathbb{F},k_0} &: L_d(d - t - e, m_1, m_2, m_3, m_4) \\ L_{\mathbb{P},k_0} &: L_{d-t-e}(m_5, m_6, \dots, m_n) \\ \hat{L}_{\mathbb{P},k_0} &: L_{d-t-e-1}(m_5, m_6, \dots, m_n) \end{aligned}$$

By the previous Lemma, the kernel system $\hat{L}_{\mathbb{F},k_0}$ is empty, and $\dim(L_{\mathbb{F},k_0}) = 2e$.

The matching conditions give five extra points on the $\mathbb{P}$ surface, lying on a line, having multiplicities $m_1 - t - e$, $m_2 - t - e$, $m_3 - t - e$, $m_4 - t - e$, and t (using (c) of the Lemma). The sum of these five extra multiplicities is $s - 3t - 4e$, which is equal to $d - (3t + 4e - s + d) = d - t - 3e$ and is therefore at most the degree $d - t - e$ of the $\mathbb{P}$ system. In particular the matching system $L^m_{\mathbb{P},k_0}$ has a moving divisor of degree $2e$ restricted to the line. This matches perfectly with the dimension of the system on $\mathbb{F}$.

We note that a small miracle now occurs: the virtual dimension of the $\mathbb{P}$ system $L^m_{\mathbb{P},k_0}$ is identical to the virtual dimension of the original system L: in fact if one computes the difference algebraically and simplifies, one obtains

$$v(L) - v(L_{\mathbb{P},k_0}) = e(e-1)/2$$

and since $e = 0$ or 1, this is zero.

Proposition 6.2 *If* $L^m_{\mathbb{P},k_0} = L_{d-t-e}([m_1 - t - e, m_2 - t - e, m_3 - t - 3m_4 - t - e, t], m_5, \dots, m_n)$ *is empty, then* L *is empty.*

As in the Three-Point Lemma, the proof is simply to apply Proposition 4.3: both $\hat{L}^m_{\mathbb{F},k_0}$ and $L^m_{\mathbb{P},k_0}$ are empty.

Again, this is a significant reduction, modulo the hypothesis that the extra five multiplicities created by the matching condition all are with points that lie on a line. In addition, the same considerations as for the Three-Point Lemma apply (see Remark (5.3)) with regard to having any constraints on the points: as long as the four points are general enough to become collinear, the set of points may enjoy some other constraints and this does not affect the reduction process.

In any case, the reduction algorithm to prove L is empty is:

1. Choose four multiplicities $m_1, \dots, m_4$ so that $s = m_1 + m_2 + m_3 + m_4 \geq d + 1$.

2. Write $s - d = 2t + e$ as above and check that $t + e \leq d$ and $t + e \leq m_i$ for $1 \leq i \leq 4$.

3. Reduce the degree of the system by $t + e$, from d to $d - t - e$.

4. Reduce these four multiplicities by $t + e$, from m_i to $m_i - t - e$.

5. Add one additional multiplicity of t.
6. Assume that the five 'new' points having multiplicities $m_i - t - e$ $(i = 1, 2, 3, 4)$ and t are collinear, and that the points are otherwise general.
7. Prove that the resulting system is empty.

We have developed also a Five-Point Lemma in a similar fashion, which we present below without proof. The details are quite parallel to those of the Three- and Four-Point Lemmas. We will not be applying the Five-Point Lemma in the applications presented below.

Proposition 6.3 *Let L be the linear system $L = L_d(m_1, \ldots, m_n)$. Suppose that the virtual dimension $v(L) < 0$. If $L^m_{\mathbb{P}} = L_{d-t-e}([m_1 - t - e, m_2 - t - e, m_3 - t - e, m_4 - t - e, m_5 - t - e], m_6, \ldots, m_n)$ has the expected dimension, then L is empty.*

Again, Remark (5.3) applies in this case also: constraints on the position of the points in the original system L are permitted, as long as they do not forbid the five points to become collinear.

7 Applications

We begin by presenting two computations which prove that certain linear systems are empty. Both computations use the ideas of Section 3 on collisions of fat points. The first case we consider is the famous linear system $L_{38}(12^{10})$ of virtual dimension -1 (see [8] for the first proof of its emptyness).

7.1 The System $L_{38}(12^{10})$ via collisions

Take five of the ten 12-tuple points and bring them together. The limit is, according to Proposition 3.1(d), a point of multiplicity 24, with two infinitely near tacnodal $(6, 6)$-point singularities having corresponding second-order tangents. The resulting system therefore has this one singularity, and five other general 12-tuple points.

We now perform four Cremona transformations. The first, centered at the point of multiplicity 24 and two of the 12-tuple points, results in a system of degree 28, with one point of multiplicity 14, three points of multiplicity 12, two points of multiplicity 2, and two tacnodal $(6, 6)$-points having corresponding second-order tangents. The two double points and the two tacnodal points lie on a line (the common tangent to the tacnodes).

The second, centered at the point of multiplicity 14 and two of the 12-tuple points, results in a system of degree 18, with one point of multiplicity 12, one point of multiplicity 4, four points of multiplicity 2, and we still have the two tacnodal $(6, 6)$-points having corresponding second-order tangents. Now all four of the double points, the multiplicity 4 point, and the two tacnodes all lie on a conic (the Cremona image of the tangent line noted above); this conic is tangent to the tacnodes.

The third Cremona transformation is centered at the point of multiplicity 12 and the two tacnodes: the result is a system of degree 12, with three points of multiplicity 6, one 4-tuple point, and four double points; two of the points of multiplicity 6 have corresponding tangents. These two points of multiplicity 6, the 4-tuple point, and the four double points all lie on a conic.

The fourth Cremona transformation is centered at the three points of multiplicity 6, and results in a system of degree 6, with one 4-tuple point and four double points. The given system of sextics has dimension five (it is in fact the Cremona image of a complete system of conics).

There are now also two lines which the sextics must meet in a corresponding way. This correspondence between the two lines is in fact given by projection from a point Q in this plane, not on either of the two lines. The two lines L and M and the five multiple points are now general; there is a conic containing the five multiple points, but there is always a conic through five general points.

It is not difficult to see that, under these assumptions, no sextic in this five-dimensional subsystem V satisfies the correspondence given by projection from Q; this is a priori six conditions on the five-dimensional system, and we must check that this gives an empty system.

Indeed, the techniques of the collision method presented in Section 3 show that we can bring the five points together: the result is a point P of multiplicity six, whose tangent cone must lie in a specific five-dimensional subspace of the six-dimensional ambient space. Upon doing this, the system of sextics becomes a system of six lines through the multiplicity six point, with a specific codimension one condition. Moreover this codimension one subspace V is general, which can be achieved by acting with projective transformations with fixed point P.

To see that the system is now empty, note that projection from P is exactly what gives the correspondence relation between the divisors cut out by the sextics on the two lines. Consider then the automorphism ϕ on one of the lines, say L, given by projection from P (from L to M) followed by projection from Q (from M back to L). This automorphism can also be considered to be general, modulo the condition that it fixes the point of intersection $L \cap M$. In order for us to have a sextic C satisfying the correspondence condition on the two lines, the divisor that C cuts on L must be fixed by this automorphism.

The reader can easily become convinced that, since ϕ is general, there are only finitely many fixed divisors of degree six for ϕ. By the generality of V, we can assume that none of these fixed divisors are cut out by curves C in V. This finishes the proof.

7.2 The System $L_{158}(50^{10})$ via collisions

We will now consider in a similar way the system of curves of degree 158 with ten multiple points of order 50. The interest in this system was communicated to us by B. Harbourne and J. Roé (see [11]); it is in some sense an extremal case that appears in their analysis of the Nagata Conjecture. The system has virtual dimension equal to -31, and so should be empty. This we will show.

Again collide five of the singularities together, and make an identical set of four Cremona transformations as above. We obtain a system of degree 48, with points of multiplicity 25, 25, 24, 16, 8, 8, 8, and 8; the two points of multiplicity 25 have corresponding tangents.

At this point the line joining the two 25-tuple points splits twice from the system, and the two lines joining the 25-tuple points to the 24-tuple point splits once, resulting in a system of degree 44, with three points of multiplicity 22, one point of multiplicity 16, and four points of multiplicity 8; two of the 22-tuple points have corresponding tangents.

Now a Cremona transformation at the three 22-tuple points results in a system of degree 22, with one point of multiplicity 16 and four points of multiplicity 8; in addition (as above), there are two lines L and M for which the intersection with the curves must correspond.

The four lines joining the 16-tuple point to the four 8-tuple points each split twice, and the conic through all five of the points splits four times, giving a residual system of degree 6, with one 4-tuple point and four double points. The correspondence condition with the two lines is still to be satisfied. This is the exact same situation as in the previous analysis! We conclude again that the system is empty.

We note that this immediately implies that the system $L_{79}(25^{10})$ is also empty, since it is exactly half of the above system. It is in fact this system that was found to be of interest in [11]. According to this article, the emptyness of this system implies that if d and m are positive integers with $d/m < 177/56 \approx 3.160714$, then there are no curves of degree d with 10 general m-tuple points. It is our understanding that this constant is the best known at this time. Nagata's Conjecture for this case would improve this constant to $\sqrt{10} \approx 3.162278$.

7.3 The System $L_{158}(50^{10})$ via Four-Point Lemmas

The above proof that the system $L_{158}(50^{10})$ is empty relied on the results of Section 3 on collisions of fat points. Let us remake the proof, this time exploiting the results of Section 6 where four points move onto a line.

The first step is to use the four-point lemma with four of the multiplicity 50 points. In this case (using the notation of Proposition 6.2), $d = 158$, $s = 200$, so that $s - d = 42$, and $t = 21$, with $e = 0$. We reduce to a system of degree 137, with six points of multiplicity 50, four points of multiplicity 29, and one of multiplicity 21; the four 29-tuple points and the 21-tuple point are collinear.

We will now perform several Cremona transformations. In the course of our analysis, the Cremona transformations carry the lines that become part of the constraints on the positions of the points to other rational curves of higher degree. In order to keep track of this, we introduce a matrix notation which encodes the linear system in question, and the constraints, as follows.

For a linear system having n fat points, and involving k constraint curves, we use a matrix with $k+2$ rows and $n+1$ columns. The columns (after the first) are indexed by the points. The first row is simply an indexing row that gives indexes to the n fat points. The second row indicates the degree (in the first column) and the multiplicities (in the succeeding colums) of the linear system under analysis. The third and following rows indicate the degree and multiplicity of the constraint curves (in the same manner as above) that the points must satisfy.

For example, the result of applying one Four-Point Lemma to the first four points of the system $L_{158}(50^{10})$ is denoted by the matrix:

	1	2	3	4	5	6	7	8	9	10	11
137	29	29	29	29	50	50	50	50	50	50	21
1	1	1	1	1	0	0	0	0	0	0	1

This indicates that the linear system has degree 137; that points 1, 2, 3, and 4 now have multiplicity 29, while points $5 - -10$ remain with multiplicity 50, and that the extra point (numbered 11) added in the Four-Point Lemma has multiplicity 29. The last row indicates that points 1, 2, 3, 4, and 11 lie on a line (each having multiplicity one of course).

We now perform four quadratic Cremona transformations. The first is centered at points 5, 6, and 7. The second is centered at points 8, 9, and 10. The third is centered at points 5, 6, and 7 again. The fourth is centered at points 1, 2, and 3. The result is a linear system of degree 83, indicated by the matrix:

	1	2	3	4	5	6	7	8	9	10	11
83	27	27	27	29	24	24	24	24	24	24	21
7	3	3	3	1	2	2	2	2	2	2	1

The line has been transformed to a septic, triple at the first three points, passing through the fourth, double at the next six, and passing through the eleventh.

Now use the Four-Point Lemma again, with points 4, 5, 6, and 7. This reduces the system to one of degree 74, and adds an additional point on the new line of multiplicity 9:

	1	2	3	4	5	6	7	8	9	10	11	12
74	27	27	27	20	15	15	15	24	24	24	21	9
7	3	3	3	1	2	2	2	2	2	2	1	0
1	0	0	0	1	1	1	1	0	0	0	0	1

We now perform two quadratic Cremona transformations; The first is centered at points 1, 2, and 3, while the second is centered at points 8, 9, and 10. The result is a linear system of degree 62, indicated by the matrix:

	1	2	3	4	5	6	7	8	9	10	11	12
62	20	20	20	20	15	15	15	19	19	19	21	9
4	1	1	1	1	2	2	2	1	1	1	1	0
4	1	1	1	1	1	1	1	2	2	2	0	1

Both constraint curves are now quartics. At this point use another Four-Point Lemma, with points 1, 2, 3, and 11, leading to:

	1	2	3	4	5	6	7	8	9	10	11	12	13
52	10	10	10	20	15	15	15	19	19	19	11	9	9
4	1	1	1	1	2	2	2	1	1	1	1	0	0
4	1	1	1	1	1	1	1	2	2	2	0	1	0
1	1	1	1	0	0	0	0	0	0	0	1	0	1

At this point the second of the two quartics meets the system negatively (the intersection number is -10), and must be in the fixed part of the system. In fact it splits three times. After splitting the quartic three times, one may check that the line (represented by the last row) splits once; the residual system is:

	1	2	3	4	5	6	7	8	9	10	11	12	13
39	6	6	6	17	12	12	12	13	13	13	10	6	8
4	1	1	1	1	2	2	2	1	1	1	1	0	0
4	1	1	1	1	1	1	1	2	2	2	0	1	0
1	1	1	1	0	0	0	0	0	0	0	1	0	1

We now perform three quadratic Cremona transformations; the first is centered at points 4, 8, and 9, the second is centered at points 4, 5, and 10, and the third is centered at points 4, 6, and 7. The result is a linear system of degree 30, indicated by the matrix:

	1	2	3	4	5	6	7	8	9	10	11	12	13
30	6	6	6	8	9	10	10	9	9	10	10	6	8
4	1	1	1	1	2	1	1	2	2	1	1	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1

All three constraint curves have been transformed into quartics at this point. Now use a Four-Point Lemma with points 10, 11, 12, and 13:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
28	6	6	6	8	9	10	10	9	9	8	8	4	6	2
4	1	1	1	1	2	1	1	2	2	1	1	0	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1	0
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1

At this point the first quartic splits off the system, and then the second quartic splits; after this, the line splits. Now the first quartic splits again, and then the third quartic splits, and finally the line splits again. The residual system has degree ten:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
10	2	2	2	2	3	5	5	3	3	1	3	1	3	0
4	1	1	1	1	2	1	1	2	2	1	1	0	0	0
4	1	1	1	1	1	2	2	1	1	2	0	1	0	0
4	1	1	1	3	1	1	1	1	1	1	1	0	1	0
1	0	0	0	0	0	0	0	0	0	1	1	1	1	1

This system is easily seen to be empty, by applying Cremona transformations.

Several other applications of the Three-point Lemma and the Four-Point Lemma have been presented in [1]. In particular the system $L_{38}(12^{10})$ ([12],[8]) of virtual dimension -1 can also be proved to be empty rather directly using the reductions afforded by the Lemmas.

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