CLASSIFICATION OF VARIETIES WITH CANONICAL CURVE SECTION VIA GAUSSIAN MAPS ON CANONICAL CURVES

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1. INTRODUCTION

Let $C \subset \mathbb{P}^{g-1}$ be a smooth canonical curve of genus $g \geq 3$. The purpose of this article is to further develop a method to classify varieties having C as their curve section, following the techniques of [CLM1]. In that article a careful analysis of the degeneration to the cone over the hyperplane section was made for *prime* Fano threefolds, that is Fano threefolds whose Picard group is generated by the hyperplane bundle. In this article we extend this method and classify Fano threefolds of higher index (which still have Picard number one) (see Theorem (3.2)). We are also able to classify Mukai varieties, i.e. varieties of dimension four or more with canonical curve sections (see Theorem (3.15)).

The classification of the varieties in question has been executed before, mostly by Fano, with modern proofs given by Iskovskih, Mukai and others. These proofs rely on several deep theorems, in particular for the Fano threefolds, the existence of lines and of smooth sections in the primitive submultiple of the anticanonical line bundle.

Our approach to the classification is completely different. It is based on a general remark concerning the degeneration to the cone over the hyperplane section, as mentioned above. Let $V \subset \mathbb{P}^r$ be a smooth, irreducible, projectively Cohen-Macaulay variety, with general hyperplane section W. Note that V flatly degenerates to the cone X over W, and therefore $[X]$ is a point in the Hilbert scheme $\mathcal{H}$ of V. Suppose one has an upper bound

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for $h^0(X, N_X)$, which is the dimension of the Zariski tangent space to $\mathcal{H}$ at $[X]$. Suppose in addition that one has a known irreducible family $\mathcal{F}$ of varieties containing $[V]$ as a member, and that the dimension of $\mathcal{F}$ is equal to the upper bound for $h^0(X, N_X)$. Since the closure of $\mathcal{F}$ in the Hilbert scheme $\mathcal{H}$ must contain the point $[X]$, we conclude that the dimension of $\mathcal{H}$ at $[X]$ is equal to the dimension of its Zariski tangent space there, and hence $[X]$ is a smooth point of $\mathcal{H}$, and the closure of the known family $\mathcal{F}$ is the unique component of $\mathcal{H}$ containing $[X]$ (and $[V]$).

One more ingredient now comes into play. Suppose further that the Hilbert scheme $\mathcal{H}'$ of the hyperplane sections W of V is irreducible, and that one can prove that as V varies in any component of its Hilbert scheme, the possible hyperplane sections W vary to fill up $\mathcal{H}'$. This assumption allows us to conclude then that the Hilbert scheme $\mathcal{H}$ is also irreducible: if there were two components, each would contain varieties with general hyperplane section W, and therefore each would contain the cone point $[X]$ over a general W. This would force $[X]$ to be a singular point of the Hilbert scheme, which, given the first part of the argument as described above, is a contradiction. One concludes then that there is only one component of $\mathcal{H}$, which is the closure of the known family $\mathcal{F}$ as described above. In this sense one obtains a classification result for varieties V.

This scheme can be applied to classify Fano threefolds of the principal series with Picard number one. First of all from papers of Fano and Iskovskih one may easily extract lists of families of such threefolds, and the number of parameters on which they depend are easily available. (We stress that our classification argument does not rely on the classification theorems of Fano and Iskovskih, but simply on the existence of the families, taken as examples.) Secondly an easy deformation theoretic argument proves that the general hyperplane section of a general Fano threefold of the principal series (and with Picard number one) varying in any component of the Hilbert scheme, is a general K3 surface of the correct genus. Therefore to complete the program outlined above, we must give the sharp upper bound for $h^0(X, N_X)$. By now it is well known that this computation rests on the study of Gaussian maps for the general curve section C. This is done in section 2 (see Theorem (2.15)).

We are also able to execute the program in an iterative fashion, to obtain similar classification statements for the Mukai varieties. The list of examples is also available, together with the number of parameters on which they depend.

The reader will notice that the above scheme provides a classification statement only when there are actually examples of such varieties. To finish one has to prove a negative statement, that varieties do not exist when there are no examples. This is provided by a theorem of Zak and L'vovskii, which, under suitable conditions on the Gaussian map of a curve, says that this curve cannot be the curve section of a variety which is not a cone. The Gaussian map computations which we have to perform to prove the classification part, suffice also to demonstrate the required non existence statement for the Fano and Mukai varieties.

In section 2 we will recall the definition of the relevant Gaussian maps and we will perform the computations of the coranks of these maps which are necessary for our purposes, as indicated above. In section 3 we execute the classification program for the Fano threefolds and Mukai varieties. Since the case of prime Fano threefolds was already treated in [CLM1], we treat here only the cases of higher index.

2. THE CORANK THEOREM

We recall here the definition of the *Gaussian map*

$$\Phi_{\omega_C} : \bigwedge^2 H^0(C,\omega_C) \to H^0(C,\omega_C^{\otimes 3})$$

which sends $s \wedge t$ to $s \otimes dt - t \otimes ds$. The Corank One Theorem from [CLM1] (Theorem 4) states that for a general hyperplane section C_g of a general *prime* K3 surface S_g (that is one whose Picard group is generated by the hyperplane bundle) the corank of the Gaussian map $\Phi_{\omega_{C_g}}$ is one if $g = 11$ or $g \geq 13$.

In this section we will extend this result to *reembedded* K3 surfaces. Let $\mathcal{H}_g$ be the component of the Hilbert scheme of prime K3 surfaces of genus $g \geq 3$. Let $\mathcal{H}_2$ be the family of prime genus 2 K3 surfaces, that is double covers of $\mathbb{P}^2$ ramified along a sextic curve. For $r \geq 2$ let $\mathcal{H}_{r,g}$ be the component of the Hilbert scheme whose general elements are obtained by embedding prime K3 surfaces of genus g via the r-th multiple of the primitive class, except for $r = g = 2$ when we set $\mathcal{H}_{2,2} = \mathcal{H}_2$. Moreover set $\mathcal{H}_{1,g} = \mathcal{H}_g$. We denote by $S_{r,g}$ any smooth surface represented by a point in $\mathcal{H}_{r,g}$; note that, for $(r,g) \neq (2,2)$, $S_{r,g} = v_r(S_g) \subset I\!P^{N(r,g)}$, where $N(r,g) = 1 + r^2(g-1)$, $S_g \subset I\!P^g$ is a K3 surface and v_r is the r-th Veronese embedding. Let C_g be a general hyperplane section of S_g; we will say that $S_{r,g}$ is *non trigonal* (respectively that $Cliff S_{r,g} \geq 2$ / *non DP*) if C_g is not a trigonal curve (respectively $Cliff C_g \geq 2$ / C_g does not lie on a smooth Del Pezzo surface). We finally denote by $C_{r,g}$ any smooth curve section of $S_{r,g}$, if $(r,g) \neq (2,2)$, and by $C_{2,2}$ any smooth curve in the linear system on $S_{2,2}$ given by twice the primitive linear system.

The following general remark is the basis of our computation. Let C be a smooth hyperplane section of a smooth K3 surface S and consider the commutative diagram

$$\text{(2.1)} \qquad \begin{array}{ccc} \bigwedge^2 H^0(S, \mathcal{O}_S(1)) & \stackrel{\Phi_{\mathcal{O}_S(1)}}{\longrightarrow} & H^0(S, \Omega^1_S(2)) \\ & & \downarrow \phi \\ \downarrow \pi & & H^0(C, \Omega^1_S(2)_{|C}) \\ & & \downarrow \psi \\ \bigwedge^2 H^0(C, \mathcal{O}_C(1)) & \stackrel{\Phi_{\omega_C}}{\longrightarrow} & H^0(C, \omega_C^{\otimes 3}) \end{array}$$

where π and ϕ are restriction maps, $\Phi_{\mathcal{O}_S(1)}$ and Φ_{ω_C} are Gaussian maps and ψ is the map on differentials induced by the sheaf map $\Omega^1_{S|C} \to \Omega^1_C$. We have the following:

Lemma (2.2). *Let C be a smooth hyperplane section of a smooth K3 surface S, ϕ and ψ as in diagram (2.1). If $\Phi_{\mathcal{O}_S(1)}$ and ϕ are surjective then Φ_{ω_C} has corank one.*

Proof: From (2.1) we have $Im\ \psi = Im\ (\psi \circ \phi \circ \Phi_{\mathcal{O}_S(1)}) = Im\ (\Phi_{\omega_C} \circ \pi) = Im\ \Phi_{\omega_C}$ since $\phi \circ \Phi_{\mathcal{O}_S(1)}$ and π are surjective. On the other hand the exact sequence

$$0 \to N^*_{C/S}(2) \to \Omega^1_S(2)_{|C} \to \omega_C^{\otimes 3} \to 0$$

shows that $corank\ \psi \leq h^1(N^*_{C/S}(2)) = h^1(\mathcal{O}_C(1)) = h^1(\omega_C) = 1$, hence $corank\ \Phi_{\omega_C} = 1$ since Φ_{ω_C} is not surjective by Wahl's theorem ([W]) because C lies on a K3 surface. ∎

By (2.1) and Lemma (2.2) the computation of the corank of the Gaussian map $\Phi_{\omega_{C_{r,g}}}$ is then reduced to studying the maps $\Phi_{\mathcal{O}_{S_{r,g}}(1)}$ and $\phi_{r,g} : H^0(S_{r,g}, \Omega^1_{S_{r,g}}(2)) \to H^0(C_{r,g}, \Omega^1_{S_{r,g}}(2)_{|C_{r,g}})$. The corank of $\Phi_{\mathcal{O}_{S_{r,g}}(1)}$ was calculated in [CLM2, Theorem 1.1]. As for $\phi_{r,g}$ we have:

Lemma (2.3). *Suppose $k \geq 1$ and r, g are such that either $r = 1, g = 11$ or $g \geq 13$; or $r \geq 2, g \geq 2$. Then $H^1(S_{r,g}, \Omega^1_{S_{r,g}}(k)) = 0$ and therefore $\phi_{r,g,k} : H^0(S_{r,g}, \Omega^1_{S_{r,g}}(k+1)) \to H^0(C_{r,g}, \Omega^1_{S_{r,g}}(k+1)_{|C_{r,g}})$ is surjective if one of the following holds:*

(a) $k \geq 2$ and $r \geq 2, g \geq 3$ except $k = r = 2, g = 3$;

(b) $k = 1$ and $r = 3, g \geq 5$ or $r = 4, g \geq 4$ or $r \geq 5, g \geq 3$;

(c) $S_{r,g}$ represents a general point of $\mathcal{H}_{r,g}$, $k = 1$ and $r = 1, g = 11, g \geq 13$ or $r = 2, g \geq 7$;

(d) $S_{r,2}$ has smooth ramification divisor, $g = 2, k = 1$ and $r \geq 4$ or $k \geq 2$ and $r \geq 3$.

Moreover for $k = r = 2, g = 3$ or $k = 1$ and $r = 4, g = 3$ or $r = 2, g = 6$ and $Cliff S_{2,6} \geq 2$, we have instead corank $\phi_{r,g,k} = 1$.

Proof: By definition of $S_{r,g}$ we have

$$H^1(S_{r,g}, \Omega^1_{S_{r,g}}(k)) \cong H^1(S_g, \Omega^1_{S_g}(rk)) \cong H^1(S_g, T_{S_g}(-rk))^* \tag{2.4}$$

where S_g is a smooth K3 surface representing a point of $\mathcal{H}_g$.

Consider now for $g \geq 3$ the Euler sequence of $S_g \subset I\!\!P^g$ tensored by $\mathcal{O}_{S_g}(-j)$

$$0 \to \mathcal{O}_{S_g}(-j) \to H^0(\mathcal{O}_{S_g}(1))^* \otimes \mathcal{O}_{S_g}(1-j) \to T_{I\!\!P^g|S_g}(-j) \to 0. \tag{2.5}$$

Let us show that

$$h^0(T_{I\!\!P^g|S_g}(-j)) = \begin{cases} g+1 & \text{for } j = 1 \\ 0 & \text{for } j \geq 2 \end{cases} \tag{2.6}$$

and

$$H^1(T_{I\!\!P^g|S_g}(-j)) = 0 \quad \text{for } j \geq 1. \tag{2.7}$$

For $j \geq 1$, $H^0(\mathcal{O}_{S_g}(-j)) = 0$ and $H^1(\mathcal{O}_{S_g}(-j)) = 0$ by Kodaira vanishing, hence (2.5) gives $h^0(T_{I\!\!P^g|S_g}(-j)) = (g+1)h^0(\mathcal{O}_{S_g}(1-j))$, which is (2.6). Also $H^1(\mathcal{O}_{S_g}(1-j)) = 0$ by Kodaira vanishing for $j \geq 2$ and for $j = 1$ because S_g is a K3 surface, therefore by (2.5) we have

$$H^1(T_{I\!\!P^g|S_g}(-j)) = Ker\ \{H^2(\mathcal{O}_{S_g}(-j)) \to H^0(\mathcal{O}_{S_g}(1))^* \otimes H^2(\mathcal{O}_{S_g}(1-j))\} =$$

$$= (Coker\ \{H^0(\mathcal{O}_{S_g}(1)) \otimes H^0(\mathcal{O}_{S_g}(j-1)) \to H^0(\mathcal{O}_{S_g}(j))\})^* = 0$$

by the surjectivity of this multiplication map, hence (2.7). From the normal bundle sequence

$$0 \to T_{S_g}(-j) \to T_{I\!\!P^g|S_g}(-j) \to N_{S_g}(-j) \to 0$$

we will see that

$$h^1(T_{S_g}(-j)) = \begin{cases} h^0(N_{S_g}(-1)) - g - 1 & \text{for } j = 1 \\ h^0(N_{S_g}(-j)) & \text{for } j \geq 2 \end{cases} . \tag{2.8}$$

In fact since $h^2(\Omega^1_{S_g}) = h^{1,2}(S_g) = 0$ it follows a fortiori that $h^0(T_{S_g}(-1)) = h^2(\Omega^1_{S_g}(1)) = 0$, hence (2.8) follows by (2.6) and (2.7).

Whence, by (2.4) and (2.8), we reduced the proof of cases (a) through (c) to show that

$$h^0(N_{S_g}(-rk)) \leq \begin{cases} g+1 & \text{for } r = k = 1 \\ 0 & \text{for } rk \geq 2 \end{cases} . \tag{2.9}$$

Now let C_g be a general hyperplane section of S_g, X_g a general cone over C_g in $I\!P^g$. Then S_g flatly degenerates to X_g and $h^0(N_{S_g}(-rk))$ is upper semicontinuous, hence

$$h^0(N_{S_g}(-rk)) \leq h^0(N_{X_g}(-rk)) = \sum_{h \geq 0} h^0(N_{C_g/I\!P^{g-1}}(-rk-h)). \tag{2.10}$$

For S_g general and $k = r = 1, g = 11$ or $g \geq 13$ or $k = 1, r = 2, g \geq 7$ we have that

$$h^0(N_{C_g/I\!P^{g-1}}(-rk-h)) = \begin{cases} g+1 & \text{for } h = 0, k = r = 1 \\ 0 & \text{for } h \geq 1 \text{ and } h = 0, k = 1, r = 2 \end{cases}$$

by [CLM1, Theorem 4 and Lemma 4]. Hence we get (2.9) by (2.10). This proves the vanishing under hypothesis (c). Similarly, if we assume (a) or (b), we have that $h^0(N_{C_g/I\!P^{g-1}}(-rk-h)) = 0$ for $h \geq 0$ since the ideal of C_g is generated by hypersurfaces of degree $\leq rk - 1$ in all cases except when $k = 1, r = 3, g \geq 5$ and C_g is either isomorphic to a plane quintic or trigonal. But again in this case $h^0(N_{C_g/I\!P^{g-1}}(-3-h)) = 0$ for $h \geq 0$ by [CM2, Proposition 0.2 and Theorem 2.3] and by [T, Theorem 2.4]. Therefore (2.9) follows. In hypothesis (d) we have $g = 2, k = 1$ and $r \geq 4$ or $k \geq 2, r \geq 3$ and S_2 is a double cover $\pi : S_2 \to I\!P^2$ with smooth ramification divisor B. The Euler sequence

$$0 \to \mathcal{O}_{S_2}(-rk) \to H^0(\mathcal{O}_{S_2}(1))^* \otimes \mathcal{O}_{S_2}(1-rk) \to \pi^* T_{I\!P^2}(-rk) \to 0$$

gives, as above by Kodaira vanishing, that $H^0(\pi^* T_{I\!P^2}(-rk)) = H^1(\pi^* T_{I\!P^2}(-rk)) = 0$; the normal bundle sequence is

$$0 \to T_{S_2}(-rk) \to \pi^* T_{I\!P^2}(-rk) \to N_\pi(-rk) \to 0$$

and $N_\pi \cong \pi^*\mathcal{O}_B(3)$, hence $h^1(S_2, T_{S_2}(-rk)) = h^0(N_\pi(-rk)) = h^0(\pi^*\mathcal{O}_B(3-rk)) = 0$. By (2.4) this gives the required vanishing under hypothesis (d).

For $g = 3$ and $k = r = 2$ or $k = 1, r = 4$ we have that $H^1(S_{r,3}, \Omega^1_{S_{r,3}}(k+1)) = 0$ by part (a) and from (2.4) and (2.8) we get $h^1(S_{r,3}, \Omega^1_{S_{r,3}}(k)) = h^0(N_{S_3}(-4)) = h^0(\mathcal{O}_{S_3}) = 1$, hence *corank* $\phi_{r,3,k} = 1$. For $k = 1, r = 2, g = 6$ and $Cliff S_{2,6} \geq 2$, we will prove that $h^1(S_{2,6}, \Omega^1_{S_{2,6}}(1)) = h^1(S_6, \Omega^1_{S_6}(2)) = 1$, hence *corank* $\phi_{2,6,1} = 1$ because $H^1(S_{2,6}, \Omega^1_{S_{2,6}}(2)) = 0$ by part (a). Let $\mathbb{G} = \mathbb{G}(1,4)$ be the Grassmannian of lines in $I\!P^4$ and $\mathcal{O}_{\mathbb{G}}(1)$ the line bundle giving the Plücker embedding. Recall that the K3 surface S_6 is scheme-theoretically cut out by three sections of $\mathcal{O}_{\mathbb{G}}(1)$ and one section of $\mathcal{O}_{\mathbb{G}}(2)$. From the normal bundle sequence

$$0 \to N^*_{S_6/\mathbb{G}}(2) \to \Omega^1_{\mathbb{G}}(2)_{|S_6} \to \Omega^1_{S_6}(2) \to 0$$

and the fact that $h^2(N^*_{S_6/\mathbb{G}}(2)) = h^0(\mathcal{O}_{S_6}(-1)^{\oplus 3} \oplus \mathcal{O}_{S_6}) = 1$, we are reduced to prove

$$H^p(\Omega^1_{\mathbb{G}}(2)_{|S_6}) = 0 \quad \text{for } p = 1,2. \tag{2.11}$$

From the exact sequence

$$0 \to \Omega^1_{\mathbb{G}} \otimes \mathcal{I}_{S_6/\mathbb{G}}(2) \to \Omega^1_{\mathbb{G}}(2) \to \Omega^1_{\mathbb{G}}(2)_{|S_6} \to 0$$

we see that (2.11) follows once we prove

$$H^p(\Omega^1_{\mathbb{G}} \otimes \mathcal{I}_{S_6/\mathbb{G}}(2)) = 0 \quad \text{for } p = 2,3 \tag{2.12}$$

since $H^p(\Omega^1_{\mathbb{G}}(2)) = 0$ for $p = 1,2$ by Bott vanishing. But also (2.12) follows easily by Bott vanishing since the Koszul resolution of the ideal sheaf $\mathcal{I}_{S_6/\mathbb{G}}$ gives the exact sequence

$$0 \to \Omega^1_{\mathbb{G}}(-3) \to \Omega^1_{\mathbb{G}}(-2)^{\oplus 3} \oplus \Omega^1_{\mathbb{G}}(-1) \to {\Omega^1_{\mathbb{G}}}^{\oplus 3} \oplus \Omega^1_{\mathbb{G}}(-1)^{\oplus 3} \to$$

$$\to \Omega^1_{\mathbb{G}}(1)^{\oplus 3} \oplus \Omega^1_{\mathbb{G}} \to \Omega^1_{\mathbb{G}} \otimes \mathcal{I}_{S_6/\mathbb{G}}(2) \to 0. \quad \blacksquare$$

(2.13) *Remark.* Part of the above Lemma, namely the fact that $H^1(S_{1,g}, \Omega^1_{S_{1,g}}(1)) = H^1(S_g, T_{S_g}(-1)) = 0$ for $g = 11$ or $g \geq 13$ also follows by work of Mori and Mukai ([MM], [M1]).

Before coming to the statements and proofs of the theorems announced in the introduction, we will collect in one single table all the useful information about the K3 surfaces $S_{r,g} \subset \mathbb{P}^{N(r,g)}$, their hyperplane sections $C_{r,g}$ and the cones $\widehat{S}_{r,g} \subset \mathbb{P}^{N(r,g)+1}$ over $S_{r,g}$. This will be proved and used along the paper:

Table (2.14)

r	g	$corank\ \Phi_{\omega_{C_{r,g}}}$	$h^0(N_{C_{r,g}}(-2))$	bound on $h^0(N_{\widehat{S}_{r,g}})$	hypothesis on $S_{r,g}$	hypothesis on $C_{r,g}$
1	6	10	≤ 1	85	general	general
1	7	9	0	98	general	general
1	8	7	0	114	general	general
1	9	5	0	132	general	general
1	10	4	0	153	general	general
1	11	1	0		general	general
1	12	2	0	201	general	general
1	≥ 13	1	0		general	general
1	≥ 17	1	0		general	any smooth
2	2	13			any smooth	any smooth
2	3	10	≤ 1	139	any smooth	any smooth
2	4	7	0	234	any smooth	any smooth
2	5	4	0	363	non trigonal	any smooth
2	6	2	0	525	$Cliff \geq 2$	any smooth
2	≥ 7	1	0		general	any smooth
3	2	10	1		smooth ramif. div.	general
3	2	18	1		smooth ramif. div.	special
3	3	5	0		any smooth	any smooth
3	4	2	0	889	any smooth	any smooth
3	≥ 5	1	0		any smooth	any smooth
4	2	1	0		smooth ramif. div.	any smooth
4	3	2	0	1209	any smooth	any smooth
4	≥ 4	1	0		any smooth	any smooth
≥ 5	2	1	0		smooth ramif. div.	any smooth
≥ 5	≥ 3	1	0		any smooth	any smooth

Note that the empty spaces in column 5 of the above table correspond to the cases were there are no threefolds with the given r and g.

We now prove the main result of the section, which is the announced computation of the corank of the Gaussian map.

Theorem (2.15). *For $r \geq 1$ and $g \geq 2$ let $S_{r,g}$ be a smooth K3 surface representing a point of $\mathcal{H}_{r,g}$ and $C_{r,g}$ a smooth hyperplane section of $S_{r,g}$. Then under the hypotheses on $S_{r,g}$ and $C_{r,g}$ given in columns 6 and 7 of Table (2.14), the values of corank $\Phi_{\omega_{C_{r,g}}}$ are given in column 3.*

Proof: For $r = 1, g \geq 6$ and $C_{r,g}$ general the corank of the Gaussian map $\Phi_{\omega_{C_{r,g}}}$ was already computed in previous articles: for $6 \leq g \leq 9$ or $g = 11$ it was done in [CM1]; for $g = 10$ in [CU]; for $g \geq 12$ in Theorem 4 and Proposition 3 of [CLM1]. Now consider r and g as in the table below

Table (2.16)

r	g
1	≥ 17
2	≥ 7
3	≥ 5
4	2 or ≥ 4
≥ 5	≥ 2

and $C_{r,g}$ any smooth hyperplane section of $S_{r,g}$. In all these cases, using the main result of [CLM2], it easily follows that *corank* $\Phi_{\omega_{C_{r,g}}} = 1$: By Theorem 1.1 and Remark 2.7 of [CLM2] we have that the Gaussian map $\Phi_{\mathcal{O}_{S_{r,g}}(1)}$ is surjective for the values of r and g in Table (2.16). Under the same hypotheses from Lemma (2.3) we have that $\phi_{r,g} = \phi_{r,g,1}$ is surjective. Hence Lemma (2.2) gives that $\Phi_{\omega_{C_{r,g}}}$ has corank one.

It remains to compute the corank of $\Phi_{\omega_{C_{r,g}}}$ in the lower genera. For $g \geq 3$ we have that $S_{r,g} = v_r(S_g)$, where v_r is the r-th Veronese embedding, hence *corank* $\Phi_{\omega_{C_{r,g}}} =$ *corank* $\Phi_{\omega_{C'}}$ where C' is a smooth curve cut out on S_g by a hypersurface of degree r. When $3 \leq g \leq 5$ the K3 surface S_g is a complete intersection, hence so is C' and the corank of the Gaussian map of C' can be computed by a result of Wahl as follows. If $C' \subset \mathbb{P}^g$ is a complete intersection of type $(d_1, \ldots, d_{g-1})$, setting $k = \sum_{j=1}^{g-1} d_j - g - 1$ we have a diagram

$$
(2.17)\qquad
\begin{array}{ccc}
\bigwedge^2 H^0(I\!\!P^g, \mathcal{O}_{I\!\!P^g}(k)) & \overset{\Phi_{\mathcal{O}_{I\!\!P^g}(k)}}{\longrightarrow} & H^0(I\!\!P^g, \Omega^1_{I\!\!P^g}(2k)) \\
 & & \downarrow \phi_k \\
\downarrow \pi_k & & H^0(C', \Omega^1_{I\!\!P^g}(2k)_{|C'}) \\
 & & \downarrow \psi_k \\
\bigwedge^2 H^0(C', \omega_{C'}) & \overset{\Phi_{\omega_{C'}}}{\longrightarrow} & H^0(C', \omega_{C'}^{\otimes 3})
\end{array}
$$

and π_k is surjective since $H^1(\mathcal{I}_{C'}(k)) = 0$, $\Phi_{\mathcal{O}_{I\!\!P^g}(k)}$ is surjective by [W, Theorem 6.4], ϕ_k is surjective if $2k \neq d_j$, $\forall j = 1, \ldots, g-1$ ([W, Proposition 6.6]). In the cases at hand the $(g-1)$-tuples $(d_1, \ldots, d_{g-1})$ are given by the following

Table (2.18)

$(d_1, \ldots, d_{g-1})$	g	r	k
$(2,2,2,2)$	5	2	2
$(2,2,3)$	4	2	2
$(2,3,3)$	4	3	3
$(2,4)$	3	2	2
$(3,4)$	3	3	3
$(4,4)$	3	4	4

thus we see that $2k \neq d_j$ (and ϕ_k is surjective) except for the case $(2,4)$. For the map $\psi_k : H^0(\Omega^1_{I\!\!P^g|C'}(2k)) \to H^0(\Omega^1_{C'}(2k))$ we get *corank* $\psi_k = H^1(N^*_{C'/I\!\!P^g}(2k))$ as long as $H^1(\Omega^1_{I\!\!P^g|C'}(2k)) = 0$. From the Euler sequence

$$0 \to \Omega^1_{I\!\!P^g|C'}(2k) \to H^0(\mathcal{O}_{C'}(1)) \otimes \mathcal{O}_{C'}(2k-1) \to \mathcal{O}_{C'}(2k) \to 0$$

and the fact that $H^1(\mathcal{O}_{C'}(2k-1)) = H^0(\mathcal{O}_{C'}(1-k))^* = 0$ we see that $H^1(\Omega^1_{I\!\!P^g|C'}(2k)) = 0$ since the multiplication map $H^0(\mathcal{O}_{C'}(2k-1)) \otimes H^0(\mathcal{O}_{C'}(1)) \to H^0(\mathcal{O}_{C'}(2k))$ is surjective. Excluding the case $(2,4)$ for the moment, by (2.17) we have

$$corank\ \Phi_{\omega_{C'}} = corank\ \psi_k = h^1(N^*_{C'/I\!\!P^g}(2k)) = h^0(N_{C'/I\!\!P^g}(-k)) = \sum_{j=1}^{g-1} h^0(\mathcal{O}_{C'}(d_j - k))$$

and this gives the required values. In the case $(2,4)$, $C' = Q \cap S_3 \subset I\!\!P^3$ with Q a quadric surface, and

Claim (2.19). *In diagram (2.17) we have:*

(i) corank $\phi_2 = 1$*;*

(ii) corank $\psi_2 = 10$;

(iii) Ker $\psi_2 \not\subseteq Im\ \phi_2$.

By Claim (2.19) we deduce that $corank\ \Phi_{\omega_{C'}} = corank\ \Phi_{\omega_{C'}} \circ \pi_2 = corank\ \psi_2 \circ \phi_2 \circ \Phi_{\mathcal{O}_{I\!P^3}(2)} = corank\ \psi_2 \circ \phi_2 = corank\ \psi_2 = 10$.

Proof of Claim (2.19): By the above computation $corank\ \psi_2 = h^0(\mathcal{O}_{C'}) + h^0(\mathcal{O}_{C'}(2)) = 1 + g(C') = 10$, hence (ii). From the exact sequence

$$0 \to \Omega^1_{I\!P^3} \otimes \mathcal{I}_{C'}(4) \to \Omega^1_{I\!P^3}(4) \to \Omega^1_{I\!P^3}(4)_{|C'} \to 0$$

we see that $corank\ \phi_2 = h^1(\Omega^1_{I\!P^3} \otimes \mathcal{I}_{C'}(4))$ since $H^1(\Omega^1_{I\!P^3}(4)) = 0$. Tensoring by $\Omega^1_{I\!P^3}(4)$ the Koszul resolution of $\mathcal{I}_{C'}$ we have

$$0 \to \Omega^1_{I\!P^3}(-2) \to \Omega^1_{I\!P^3}(2) \oplus \Omega^1_{I\!P^3} \to \Omega^1_{I\!P^3} \otimes \mathcal{I}_{C'}(4) \to 0$$

and therefore $h^1(\Omega^1_{I\!P^3} \otimes \mathcal{I}_{C'}(4)) = 1$ by Bott vanishing, hence (i). To see (iii) consider the following diagram

$$\begin{array}{ccccc} 0 \to H^0(N^*_{S_3}(4)) & \stackrel{f}{\longrightarrow} & H^0(\Omega^1_{I\!P^3}{}_{|S_3}(4)) & & \\ & & & \nwarrow \phi_3 & \\ & \downarrow \pi & & & H^0(\Omega^1_{I\!P^3}(4)) \\ & & & \swarrow \phi_2 & \\ 0 \to H^0(N^*_{C'}(4)) & \stackrel{g}{\longrightarrow} & H^0(\Omega^1_{I\!P^3}{}_{|C'}(4)) & & \end{array}$$

where $Im\ f = Ker\ \{\psi_3 : H^0(\Omega^1_{I\!P^3}{}_{|S_3}(4)) \to H^0(\Omega^1_{S_3}(4))\} \cong \mathbb{C}$ and $Im\ g = Ker\ \psi_2$ and π is the restriction map. By [CLM2, proof of Theorem 1.1] we know that $corank\ \phi_3 = 1$ and that, if F is a generator of $Ker\ \psi_3$, we have that $f(F) \notin Im\ \phi_3$. If we let $F' \in H^0(N^*_{C'}(4))$ be such that $g(F') = \pi(f(F))$ it follows that $g(F') \notin Im\ \phi_2$ and hence that $Ker\ \psi_2 \not\subseteq Im\ \phi_2$. For otherwise, since $corank\ \phi_3 = 1$, we have that $Im\ \pi \subseteq Im\ \phi_2$ and this contradicts (i) since π is surjective, as it can be easily seen by Bott vanishing. This proves Claim (2.19).

In the case $g = 6, r = 2$ and $Cliff S_{2,6} \geq 2$, that is when $C' = S_6 \cap Q \subset I\!P^6$, with Q a quadric hypersurface and $Cliff S_6 \geq 2$, we have a diagram

$$\begin{array}{ccc} \bigwedge^2 H^0(S_6, \mathcal{O}_{S_6}(2)) & \stackrel{\Phi_{\mathcal{O}_{S_6}(2)}}{\longrightarrow} & H^0(S_6, \Omega^1_{S_6}(4)) \\ & & \downarrow \phi_{S_6} \\ \downarrow \pi_{S_6} & & H^0(C', \Omega^1_{S_6}(4)_{|C'}) \\ & & \downarrow \psi_{S_6} \\ \bigwedge^2 H^0(C', \omega_{C'}) & \stackrel{\Phi_{\omega_{C'}}}{\longrightarrow} & H^0(C', \omega_{C'}^{\otimes 3}) \end{array}$$

and

Claim (2.20). *In the above diagram we have:*

(i) π_{S_6} *and* $\Phi_{\mathcal{O}_{S_6}(2)}$ *are surjective;*

(ii) corank $\psi_{S_6} \leq 1$*;*

(iii) corank $\phi_{S_6} = 1$.

From Claim (2.20) it follows that *corank* $\Phi_{\omega_{C_{2,6}}} =$ *corank* $\Phi_{\omega_{C'}} =$ *corank* $\psi_{S_6} \circ \phi_{S_6} \leq 2$. On the other hand if *corank* $\Phi_{\omega_{C_{2,6}}} \leq 1$ then $H^0(N_{C_{2,6}}(-2)) = 0$ since $C_{2,6}$ is a canonical curve: this follows by [CM1, Lemma 1.10] in the case *corank* $\Phi_{\omega_{C_{2,6}}} = 0$ and by [CM2, Lemma 5.2] in the case *corank* $\Phi_{\omega_{C_{2,6}}} = 1$. By a theorem of Zak (see [Z], [B], [BEL] or [L]) we have that $C_{2,6} \subset \mathbb{P}^{20}$ is not 2-extendable, that is there is no threefold $V \subset \mathbb{P}^{22}$ different from a cone such that $C_{2,6} = V \cap \mathbb{P}^{20}$. But this contradicts the fact that the K3 surface $S_6 \subset \mathbb{P}^6$ is a quadric section of a threefold $V_3 \subset \mathbb{P}^6 : V_3 = \mathbb{G} \cap H_1 \cap H_2 \cap H_3$ where $\mathbb{G} = \mathbb{G}(1,4) \subset \mathbb{P}^9$ is the Grassmannian of lines in $\mathbb{P}^4$ in the Plücker embedding and the H_i's are hyperplanes. Therefore $C_{2,6} = v_2(S_6) \cap H' = v_2(V_3) \cap H' \cap H''$ (with H', H'' hyperplanes).

Proof of Claim (2.20): As above we have *dim Coker* $\psi_{S_6} \leq h^1(N^*_{C'/S_6}(4)) = h^1(\mathcal{O}_{C'}(2)) = h^1(\omega_{C'}) = 1$, that is (ii). For (iii) observe that *dim Coker* $\phi_{S_6} = h^1(\Omega^1_{S_6}(2)) = 1$ by Lemma (2.3). Also the map π_{S_6} is surjective since it is the restriction map and $H^1(\mathcal{I}_{C'/S_6}(2)) = H^1(\mathcal{O}_{S_6}) = 0$. The fact that $\Phi_{\mathcal{O}_{S_6}(2)}$ or equivalently $\Phi_{\mathcal{O}_{S_{2,6}}(1)}$ is surjective has been proved in [CLM2, proof of Theorem 1.1] and this gives (i). This ends the proof of the Claim (2.20).

To finish the proof of the theorem let us consider the cases $g = 2, r = 2, 3$. For $g = 2$ the K3 surface S_2 is a double cover of $\mathbb{P}^2$ given by a linear system $|H|$ with $H^2 = 2$ and the curve $C' \cong C_{r,2}$ is a smooth member of $|rH|$. For $r = 2$ we have that $C_{2,2}$ is a hyperelliptic curve of genus 5, hence *corank* $\Phi_{\omega_{C_{2,2}}} = 13$ by [CM2, Proposition 1.1] or [W, Remark 5.8.1]. When $r = 3$ a general member $C_{3,2}$ of $|3H|$ is isomorphic to a smooth plane sextic, hence *corank* $\Phi_{\omega_{C_{3,2}}} = 10$ by [CM2, section 2], while the special members have a 2:1 map onto a smooth plane cubic, hence *corank* $\Phi_{\omega_{C_{3,2}}} = 18$ by [CM2, Corollary 3.3].

This then concludes the proof of Theorem (2.15). ■

(2.21) *Remark.* Note that in the case $r = 3, g = 2$ of Theorem (2.15), the corank of the Gaussian map $\Phi_{\omega_{C_{3,2}}}$ *is not constant* in the linear system $|\mathcal{O}_{S_{3,2}}(1)|$. On the other hand using a diagram like (2.1) it is easy to deduce, as in Lemma (2.2), that if S is a smooth K3 surface, L is an effective line bundle on S and we have Φ_L surjective and $H^1(S, \Omega^1_S \otimes L) = 0$, then, for every smooth $C \in |L|$, the corank of Φ_{ω_C} is one. It is an interesting question to find better hypotheses that insure the constancy of the corank of Φ_{ω_C} in a given linear system on a smooth K3 surface. The other intriguing feature is that the example given above of non constancy of the corank of the Gaussian map is the same as Donagi-Morrison's (essentially unique, see [CP]) example of non constancy of the gonality of the smooth curves in a linear system on a K3 surface.

Before the end of this section we will also prove a result about the Gaussian maps $\Phi_{\omega_{C_{r,g}}, \omega^{\otimes k}_{C_{r,g}}}$ that will be useful in the next section.

Proposition (2.22). *Fix $r \geq 1$ and $g \geq 2$. Assume that $S_{r,g}$ and $C_{r,g}$ satisfy the hypotheses given in columns 6 and 7 of Table (2.14). Then the Gaussian map $\Phi_{\omega_{C_{r,g}}, \omega^{\otimes k}_{C_{r,g}}}$ is surjective for every $k \geq 2$ except for $k = 2$ and $(r, g) = (2, 3), (1, 6), (3, 2)$. Moreover if $N_{C_{r,g}}$ is the normal bundle of $C_{r,g}$ in $I\!PH^0(C_{r,g}, \mathcal{O}_{C_{r,g}}(1))^*$, then $h^0(N_{C_{r,g}}(-2))$ is given by column 4 of Table (2.14) and $h^0(N_{C_{r,g}}(-k)) = 0$ for every $k \geq 3$.*

Proof: Set $C = C_{r,g}$ and $S = S_{r,g}$. Since C is a canonical curve it is well known that, for $k \geq 2$, we have

$$Coker\ \Phi_{\omega_C, \omega_C^{\otimes k}} \cong H^0(N_C(-k))^*$$

(see for example [CM1, Proposition 1.2]), hence the assertion on surjectivity of $\Phi_{\omega_{C_{r,g}}, \omega^{\otimes k}_{C_{r,g}}}$ follows by proving the vanishing of $h^0(N_{C_{r,g}}(-k))$. For $r = 1, g \geq 6$ and C general the values of $h^0(N_C(-k))$ are given in Lemma 4 of [CLM1]. Suppose now $r \geq 2, g \geq 3$ or $r \geq 5, g = 2$ and C any smooth hyperplane section of S. Set $R(\mathcal{O}_S(1), \mathcal{O}_S(k)) = Ker\ \{H^0(\mathcal{O}_S(1)) \otimes H^0(\mathcal{O}_S(k)) \to H^0(\mathcal{O}_S(k+1))\}$ and consider the diagram

$$\begin{array}{ccc}
R(\mathcal{O}_S(1), \mathcal{O}_S(k)) & \stackrel{\Phi_{\mathcal{O}_S(1), \mathcal{O}_S(k)}}{\longrightarrow} & H^0(S, \Omega^1_S(k+1)) \\
 & & \downarrow \phi \\
\downarrow \pi & & H^0(C, \Omega^1_S(k+1)_{|C}) \\
 & & \downarrow \psi \\
R(\mathcal{O}_C(1), \mathcal{O}_C(k)) & \stackrel{\Phi_{\omega_C, \omega_C^{\otimes k}}}{\longrightarrow} & H^0(C, \omega_C(k+1)).
\end{array}$$

Since $\Phi_{\mathcal{O}_S(1), \mathcal{O}_S(k)}$ is surjective by [CLM2, Theorem 1.1 and Remark 2.7; note that for $r = 2, g = 5, 6$ the proof of Theorem 1.1 works just assuming $S_{2,5}$ non trigonal or $Cliff S_{2,6} \geq$

2], and ψ is surjective because *Coker* $\psi \subseteq H^1(N^*_{C/S}(k+1)) = H^1(\mathcal{O}_C(k)) = 0$ for $k \geq 2$, it follows that

$$corank\ \Phi_{\omega_C,\omega_C^{\otimes k}} \leq corank\ \phi = h^1(S,\Omega^1_S(k)) = \begin{cases} 0 & \text{for } (k,r,g) \neq (2,2,3) \\ 1 & \text{for } (k,r,g) = (2,2,3) \end{cases}$$

by Lemma (2.3).

For $g=2, r=3$ the values of $h^0(N_C(-k))$ are in [CM2, section 2] for C general and in [CM2, Theorem 3.2] for C special. For $r=1, g \geq 17$ or $r=4, g=2$ we have *corank* $\Phi_{\omega_C} = 1$ by Theorem (2.15), hence $h^0(N_C(-k)) = 0$ for $k \geq 2$ ($k=2$ by [CM2, Lemma 5.2]). ∎

(2.23) *Remark.* In the case $r=1, g \geq 17$ Theorem (2.15) is a generalization of the Corank One Theorem of [CLM1] (Theorem 4) since there the fact that *corank* $\Phi_{\omega_{C_g}} = 1$ was proved for a *general* hyperplane section C_g of a *general* prime K3 surface S_g, while here the same holds for *any* smooth hyperplane section $C_{1,g}$ of a general S_g. On the other hand it should be noticed that the proof of this fact is not independent of Theorem 4 of [CLM1] as the latter is used to deduce the corank one from the surjectivity of $\Phi_{\mathcal{O}_{S_g}(1)}$ (in Lemma (2.3)). Alternatively one can avoid using Theorem 4 of [CLM1] by invoking results of Mori and Mukai ([MM], [M1]); see remark (2.13).

3. VARIETIES WITH CANONICAL CURVE SECTION

We begin by recalling Zak's theorem mentioned in the introduction. A smooth nondegenerate variety $X \subset \mathbb{P}^m$ is said to be *k-extendable* if there exists a variety $Y \subset \mathbb{P}^{m+k}$ that is not a cone and such that $X = Y \cap \mathbb{P}^m$. Zak's theorem says that if *codim* $X \geq 2$, $h^0(N_X(-1)) \leq m+k$ and $h^0(N_X(-2)) = 0$, then X is not k-extendable. For a canonical curve $C \subset \mathbb{P}^{g-1}$ one has also $h^0(N_C(-1)) = g + \text{corank } \Phi_{\omega_C}$ ([CM1, Proposition 1.2]), hence the knowledge of *corank* Φ_{ω_C} (and of $h^0(N_C(-2))$ that is also given by a Gaussian map) gives information on the possibility of extending C.

Next we consider a smooth nondegenerate variety $X \subset \mathbb{P}^m$ of dimension $n \geq 3$ such that its general curve sections C are canonical curves. By a well-known equivalence criterion we must have $-K_X = (n-2)H$, where H is the hyperplane divisor. Hence a general surface section S is a K3 surface, and therefore $S = S_{r,g}, C = C_{r,g}$ for some r, g.

We denote such an X by $X^n_{r,g}$. When $n = 3$ we denote $V_{r,g} = X^3_{r,g}$. Note that $V_{r,g} \subset \mathbb{P}^{N(r,g)+1} = \mathbb{P}H^0(V_{r,g}, -K_{V_{r,g}})^*$ is anticanonically embedded and, by Riemann-Roch, $N(r,g) = h^0(V_{r,g}, -K_{V_{r,g}}) - 2 = -\frac{1}{2}K^3_{V_{r,g}} + 1 = 1 + r^2(g-1) = \frac{1}{2}dr^3 + 1$, where $H = r\Delta, d = \Delta^3$.

These varieties $X^n_{r,g}$ have been extensively studied, both in dimension three (Fano, Iskovskih [I1], [I2], etc.) and in dimension $n \geq 4$ (Mukai [M1], [M2]). Observe that by applying some basic adjunction theory (e.g. [KO]) we can divide the varieties $X^n_{r,g}$ in two categories (with two exceptions): (a) $n = 3$ and $r \geq 1$, that is Fano threefolds of index r; (b) $n \geq 4$ and $r = 1$, that is Fano manifolds of dimension $n \geq 4$ and index 1. In fact for $n \geq 4, r \geq 2$ the only possible values are $n = 4, 5, r = 2$ and $(X, \Delta) = (Q, \mathcal{O}_Q(1)), (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$, where Q is a smooth quadric in $\mathbb{P}^5$, since in all other cases we have $r(n-2) > n+1$. Even though we will not make use of adjunction theory in our proofs, the interesting cases are (a) and (b) above.

The case of Fano threefolds of index one was already treated in [CLM1], thus from now on we will assume $r \geq 2$ or $n \geq 4$.

Given $r \geq 2$ and $g \geq 2$, we will denote by $\mathcal{V}_{r,g}$ the Hilbert scheme of smooth Fano threefolds $V_{r,g} \subset \mathbb{P}^{N(r,g)+1}$ of index r, anticanonically embedded and by $\mathcal{V}_{r,g,1}$ the closure in $\mathcal{V}_{r,g}$ of the locus of Fano threefolds with Picard number one.

We now present a list of examples taken from the papers of Fano and Iskovskih of families of Fano threefolds $V_{r,g}$ of the principal series and of index greater than one (see e.g. [M]):

Table (3.1)

r	g	$d = \Delta^3$	$N(r,g)$	number of parameters	$V_{r,g}$
4	3	1	33	1209	$\mathbb{P}^3$
3	4	2	28	889	quadric in $\mathbb{P}^4$
2	3	2	9	139	double cover of $\mathbb{P}^3$ ramified along a quartic
2	4	3	13	234	cubic in $\mathbb{P}^4$
2	5	4	17	363	complete intersection of two quadrics in $\mathbb{P}^5$
2	6	5	21	525	$\mathbb{G}(1,4) \cap \mathbb{P}^6 \subset \mathbb{P}^9$

Our first classification result is the following:

Theorem (3.2). *Fix $r \geq 2$ and $g \geq 2$. Then we have:*

(i) $\mathcal{V}_{r,g} = \emptyset$ for (r,g) such that: $r = 2, g = 2$ or $g \geq 11$; $r = 3, g = 2, 3$ or $g \geq 5$; $r = 4, g = 2$ or $g \geq 4$; $r \geq 5, g \geq 2$; $r = 2, 7 \leq g \leq 10$ assuming furthermore that the hyperplane section $S_{2,g}$ of $V_{2,g}$ is non trigonal and non DP;

(ii) $\mathcal{V}_{r,g,1} = \emptyset$ for (r,g) not in Table (3.1);

(iii) For (r,g) as in Table (3.1) $\mathcal{V}_{r,g,1}$ is irreducible and the families of Fano-Iskovskih form a dense open subset of smooth points of $\mathcal{V}_{r,g,1}$.

Proof: Suppose first $g = 2$. Then $d = 2(g-1)/r = 2/r$ is an integer only if $r = 2$, but then for the K3 surface $S_{2,2}$ we have that $\mathcal{O}_{S_{2,2}}(1) = \mathcal{O}_{S_{2,2}}(2\Delta_{|S_{2,2}}) \cong \mathcal{O}_{S_2}(2D)$ where ϕ_D gives the 2:1 map onto $\mathbb{P}^2$. But this is a contradiction since on S_2 the line bundle $\mathcal{O}_{S_2}(2D)$ is not very ample. Similarly for $r = g = 3$ we have that d is not an integer. For $r = 3, g \geq 5$ or $r = 4, g \geq 4$ or $r \geq 5, g \geq 3$, we have by Theorem (2.15) that $corank\ \Phi_{\omega_{C_{r,g}}} = 1$, hence by [CM1, Proposition 1.2] $h^0(N_{C_{r,g}}(-1)) = N(r,g)+1$. Also $h^0(N_{C_{r,g}}(-2)) = 0$ by [CM2, Lemma 5.2], and therefore Zak's theorem implies that $C_{r,g}$ is not 2-extendable, hence that $\mathcal{V}_{r,g} = \emptyset$ for the above values of r and g. For $r = 2, g \geq 7$ observe that, as in the proof of Lemma (2.3), for the K3 surface $S_{2,g} = v_2(S_g) \subset \mathbb{P}^{N(2,g)}$ we have $h^1(T_{S_{2,g}}(-\alpha)) = h^1(T_{S_g}(-2\alpha)), \alpha = 1, 2$, hence

(3.3)
$$h^0(N_{S_{2,g}}(-\alpha)) = \begin{cases} N(2,g)+1+h^1(T_{S_g}(-2)) = N(2,g)+1+h^0(N_{S_g}(-2)) & \text{for } \alpha = 1 \\ h^1(T_{S_g}(-4)) = h^0(N_{S_g}(-4)) & \text{for } \alpha = 2 \end{cases}.$$

By Zak's theorem to show that $V_{2,g}$ does not exist under the hypotheses given in (i) is then enough to prove that $h^0(N_{S_g}(-2)) = 0$. Now let C_g be a general hyperplane section of S_g; as in (2.10) we have

$$h^0(N_{S_g}(-2\alpha)) \leq \sum_{h \geq 0} h^0(N_{C_g/\mathbb{P}^{g-1}}(-2\alpha - h)). \tag{3.4}$$

Since $h^0(N_{C_g/\mathbb{P}^{g-1}}(-2)) = corank\ \Phi_{\omega_{C_g}, \omega_{C_g}^{\otimes 2}}$ we only need to check that $\Phi_{\omega_{C_g}, \omega_{C_g}^{\otimes 2}}$ is surjective. But this follows by [BEL, Theorem 2] for $Cliff C_g \geq 3$ and by the results of [T, Theorems 2.10 and 2.6] for $Cliff C_g = 1, 2$ except when C_g is bielliptic. But the latter case cannot happen by [CP, Corollary 2.5 and Theorem 3.1]; it can also be excluded by [CvdG]. This proves (i). To see (ii) and (iii) we will use the following

Lemma (3.5). *Suppose $V_{r,g}$ has Picard number one. Then a general hyperplane section $S_{r,g}$ of a general deformation of $V_{r,g}$ is represented by a general point of $\mathcal{H}_{r,g}$.*

Proof of Lemma (3.5): Set $V = V_{r,g}, S = S_{r,g}$ and consider the usual deformation diagram

$$\begin{array}{ccccccccc} H^1(V,T_V) & \xrightarrow{\alpha_S} & H^1(S,T_{V_{|S}}) & \to H^2(V,T_V(-1)) & \xrightarrow{\gamma_S} & H^2(V,T_V) & \to H^2(S,T_{V_{|S}}) \\ & & \uparrow \beta_S & & & & \\ & & H^1(S,T_S) & & & & \end{array}$$

First we prove

(3.6) β_S is surjective;

(3.7) $h^2(V,T_V(-1)) = 1$;

(3.8) $H^2(V,T_V) = 0$.

In fact the map β_S comes from the exact sequence

$$0 \to T_S \to T_{V_{|S}} \to N_{S/V} \to 0 \tag{3.9}$$

hence *Coker* $\beta_S \subseteq H^1(N_{S/V}) = H^1(\mathcal{O}_S(1)) = 0$, that is (3.6). By Serre duality we get $h^2(V,T_V(-1)) = h^1(\Omega^1_V) = 1$ since $\mathcal{O}_V(1) = \mathcal{O}_V(-K_V)$ and V has Picard number one. This proves (3.7). To see (3.8) observe that by (3.9) we have $H^2(T_{V|S}) = 0$, hence if $H^2(T_V) \neq 0$ we necessarily have $h^2(T_V) = 1$ and γ_S is surjective. On the other hand γ_S is given by multiplication with the equation of the hyperplane defining S, hence the multiplication map $\mu : H^0(\mathcal{O}_V(1)) \otimes H^2(T_V(-1)) \to H^2(T_V)$ is surjective. But then, by Bertini's theorem, there is a smooth surface S' in the codimension one sublinear system $Ker\ \mu \subset H^0(\mathcal{O}_V(1))$. Replacing S with S' we get that $\gamma_{S'}$ is the zero map, and this contradicts the hypothesis that $H^2(T_V) \neq 0$. Alternatively (3.8) follows by Kodaira-Nakano vanishing since $H^2(T_V) \cong H^1(\Omega^1_V(K_V)) = H^1(\Omega^1_V(-1)) = 0$.

Now by (3.8) every infinitesimal deformation of V is unobstructed and by (3.7) we have that *dim Coker* $\alpha_S = 1$, hence every infinitesimal algebraic deformation of S comes from a deformation of V. This proves Lemma (3.5).

To see (ii) of Theorem (3.2) by Lemma (3.5) we can assume that $S_{r,g}$ is general. Moreover note that if (r,g) are not in Table (3.1) and the existence of $V_{r,g}$ is not excluded by (i), we have $r = 2, g \geq 7$. Therefore by Theorem (2.15) we have again $corank\ \Phi_{\omega_{C_{r,g}}} = 1$, hence we conclude as above that $\mathcal{V}_{r,g,1} = \emptyset$. This gives (ii).

To see (iii) observe that the Fano threefolds $V_{r,g}$ of the principal series are projectively

Cohen-Macaulay, hence they flatly deform to the cone $\widehat{S}_{r,g}$ over their hyperplane section $S_{r,g}$, and similarly for $S_{r,g}$ to the cone $\widehat{C}_{r,g}$ over $C_{r,g}$. Setting $V = V_{r,g}, S = S_{r,g}, C = C_{r,g}, N = N(r,g)$ and denoting by $T_{r,g}$ the tangent space to $\mathcal{V}_{r,g,1}$ at the point representing $V_{r,g}$, we have

$$(3.10) \qquad dim\ T_{r,g} = h^0(N_{V/\mathbb{P}^{N+1}}) \leq h^0(N_{\widehat{S}/\mathbb{P}^{N+1}}) = \sum_{h\geq 0} h^0(N_{S/\mathbb{P}^N}(-h)) \leq$$

$$\leq h^0(N_{S/\mathbb{P}^N}) + \sum_{h\geq 1} h^0(N_{\widehat{C}/\mathbb{P}^N}(-h)) = h^0(N_{S/\mathbb{P}^N}) + \sum_{h\geq 1, j\geq 0} h^0(N_{C/\mathbb{P}^{N-1}}(-h-j)) =$$

$$= h^0(N_{S/\mathbb{P}^N}) + h^0(N_{C/\mathbb{P}^{N-1}}(-1)) + 2h^0(N_{C/\mathbb{P}^{N-1}}(-2))$$

by Proposition (2.22). As we know $h^0(N_{S/\mathbb{P}^N}) = dim\ \mathcal{H}_{r,g} = 18 + (N+1)^2$ and $h^0(N_{C/\mathbb{P}^{N-1}}(-1)) = N + corank\ \Phi_{\omega_C}$ hence we deduce that

$$(3.11) \qquad h^0(N_{S/\mathbb{P}^N}) + h^0(N_{C/\mathbb{P}^{N-1}}(-1)) + 2h^0(N_{C/\mathbb{P}^{N-1}}(-2)) = f(r,g) :=$$

$$= N(r,g)^2 + 3N(r,g) + 19 + corank\ \Phi_{\omega_{C_{r,g}}} + 2h^0(N_{C_{r,g}}(-2)).$$

By Theorem (2.15), Proposition (2.22) and Table (3.1) we see that $f(r,g)$ is equal to the number of parameters of the examples of Fano-Iskovskih. By (3.10) and (3.11) we also have that $f(r,g)$ is an upper bound for $h^0(N_{\widehat{S}/\mathbb{P}^{N+1}})$, hence the argument presented in the Introduction applies and we conclude that $\mathcal{V}_{r,g,1}$ is irreducible and that the families of Fano-Iskovskih form a dense open subset of smooth points. ■

(3.12) *Remark.* The theorem just proved characterizes the set of integers (r,g) with $r \geq 2, g \geq 2$ for which $\mathcal{V}_{r,g,1}$ is non empty, that is when there exists a smooth Fano threefold $V_{r,g}$ with Picard number one. Moreover it gives, in (iii), a characterization of the *general* such $V_{r,g}$. In fact it is true that *any* $V_{r,g}$ (with Picard number one) is one of the examples of Fano-Iskovskih (see [I1], [I2], [M]).

(3.13) *Remark.* Part (i) of Theorem (3.2) can be summarized in the following diagram

≥ 5	x	x	x	x	x	x	x	x	x	x
4	x	•	x	x	x	x	x	x	x	x
3	x	x	•	x	x	x	x	x	x	x
2	x	•	•	•	•	*tdp*	*tdp*	*tdp*	*tdp*	x
r / g	2	3	4	5	6	7	8	9	10	≥ 11

where x means that there is no $V_{r,g}$, $\bullet$ means that there is one and *tdp* means that there is no $V_{r,g}$ with trigonal or DP hyperplane section $S_{r,g}$. In fact again from the classification of Iskovskih it is true that $\mathcal{V}_{r,g} = \emptyset$ also for $r = 2, g = 9, 10$. For $r = 2, g = 7, 8$ instead there are Fano threefolds, but with Picard number two or three (see [I1], [I2], [M]).

We now turn to the higher dimensional case and classify the Mukai varieties. For $n \geq 4, r \geq 1$ and $g \geq 2$ we let $\mathcal{X}_{n,r,g}$ be the Hilbert scheme of smooth Fano n-folds $X^n_{r,g} \subset \mathbb{P}^{N(r,g)+n-2}$ of index $r(n-2)$ and $\mathcal{X}_{n,r,g,1}$ the closure in $\mathcal{X}_{n,r,g}$ of the locus of Fano n-folds with Picard number one.

For Fano manifolds of dimension $n \geq 4$ and index $r(n-2)$, of the principal series and with Picard number one, we present the list given by Mukai ([M1], [M2]) of examples of families of such $X^n_{r,g}$. Of course, since the general hyperplane section of an $X^n_{r,g}$ is an $X^{n-1}_{r,g}$, we only present the list of the maximum dimensional varieties. There is one family for each genus $g \geq 6$ (for $r = 1, g \leq 5$ they are complete intersections) and we denote the maximum dimension of the examples in the list by $n(g)$:

Table (3.14)

r	g	$n(g)$	number of parameters	$X^{n(g)}_{r,g}$
1	6	6	145	$\widetilde{\mathbb{G}(1,4)} \cap Q \subset \mathbb{P}^{10}$
1	7	10	210	$SO(10, \mathbb{C})/P \subset \mathbb{P}^{15}$
1	8	8	189	$\mathbb{G}(1{,}5) \subset \mathbb{P}^{14}$
1	9	6	174	$Sp(6, \mathbb{C})/P \subset \mathbb{P}^{13}$
1	10	5	181	$G_2/P \subset \mathbb{P}^{13}$
2	5	5	405	$v_2(\mathbb{P}^5) \subset \mathbb{P}^{20}$

For $4 \leq n < n(g)$ Mukai shows that any $X^n_{1,g}$ is a linear section of the $X^{n(g)}_{1,g}$.

Our final result gives the classification as follows:

Theorem (3.15). *Fix $n \geq 4, r = 1$ and $g \geq 6$ or $r \geq 2$ and $g \geq 2$. Then we have:*

(i) $\mathcal{X}_{n,r,g} = \emptyset$ for (r, g) such that: $r \geq 3, g \geq 2$; $r = 2$ and $2 \leq g \leq 4$ or $g \geq 8$ or $g = 5, 6, n \geq 6$ or $g = 6, n \geq 4$ and $Cliff S_{2,6} \geq 2$ or $g = 7$ and $S_{2,7}$ non trigonal and non DP;

(ii) $\mathcal{X}_{n,r,g,1} = \emptyset$ for (r, g) not in Table (3.14) or for (r, g) in Table (3.14) and $n > n(g)$;

(iii) For (r, g) as in Table (3.14) and $4 \leq n \leq n(g)$ we have that $\mathcal{X}_{n,r,g,1}$ is irreducible and the examples of families of Mukai form a dense open subset of smooth points of $\mathcal{X}_{n,r,g,1}$.

Proof: By (i) of Theorem (3.2) to prove (i) of Theorem (3.15) we need to consider the cases $r = 2$ and $3 \leq g \leq 10, g \neq 7$, $r = 3$ and $g = 4$ or $r = 4$ and $g = 3$. For $(r, g) = (3, 4)$ and $(r, g) = (4, 3)$ we have by Theorem (2.15) that $corank\ \Phi_{\omega_{C_{r,g}}} = 2$ and $h^0(N_{C_{r,g}}(-2)) = 0$ by Proposition (2.22), hence $C_{r,g}$ is not 3-extendable by Zak's theorem. Similarly for $r = 2, g = 5$ (respectively $g = 6$) and $S_{2,5}$ non trigonal (respectively $CliffS_{2,6} \geq 2$), we have $corank\ \Phi_{\omega_{C_{2,5}}} = 4$ (respectively $corank\ \Phi_{\omega_{C_{2,6}}} = 2$) and $h^0(N_{C_{r,g}}(-2)) = 0$, hence $C_{2,5}$ is not 5-extendable (respectively $C_{2,6}$ is not 3-extendable). For $r = 2, g = 5, 6$ and $S_{2,5}$ trigonal or $CliffS_{2,6} = 1$ we have instead by [T, Theorems 2.6 and 2.4] and [CM2, Theorem 2.3] that $corank\ \Phi_{\omega_{C_g}, \omega_{C_g}^{\otimes 2}} \leq 3, h^0(N_{C_g}(-3)) = 0$, hence $S_{2,g}$ is not 4-extendable by (3.3) and (3.4). The same argument shows that, for $8 \leq g \leq 10$, $S_{2,g}$ is not 2-extendable since in this case $corank\ \Phi_{\omega_{C_g}, \omega_{C_g}^{\otimes 2}} \leq 1, h^0(N_{C_g}(-3)) = 0$ again by [BEL, Theorem 2], [T, Theorems 2.4, 2.6, 2.10 and 1.7] and [CM2, Theorems 3.2 and 3.4]. For $r = 2, g = 4$ we have $deg\ X^n_{2,4} = deg\ C_{2,4} = 2N(2, 4) - 2 = 24$ and $\mathcal{O}_{X^n_{2,4}}(1) = \mathcal{O}_{X^n_{2,4}}(2\Delta)$, hence $2^n\Delta^n = 24$ and therefore $n \leq 3$. For $r = 2, g = 3$ if there exists an $X^4_{2,3}$ we get $deg\ X^4_{2,3} = deg\ C_{2,3} = 2N(2, 3) - 2 = 16 = 2^4\Delta^4$, hence $\Delta^4 = 1$. By restricting to the surface section $S_{2,3}$ one easily checks that $|\Delta|$ is base point free, birational and $h^0(\mathcal{O}_{X^4_{2,3}}(\Delta)) = 4$ and this of course is a contradiction. Alternatively we can exclude the existence of a smooth $X^4_{2,3}$ using some basic adjunction theory: we have $-K_{X^4_{2,3}} = 4\Delta$, with $H = 2\Delta$, hence Δ is ample; by [KO] it must be $(X^4_{2,3}, \mathcal{O}_{X^4_{2,3}}(\Delta)) \cong (Q, \mathcal{O}_Q(1))$ where Q is a smooth quadric in $\mathbb{P}^5$, but this contradicts the fact that $h^0(\mathcal{O}_{X^4_{2,3}}(1)) = h^0(\mathcal{O}_{X^4_{2,3}}(2\Delta)) = 12$ while $h^0(\mathcal{O}_Q(2)) = 20$. This proves (i).

To see part (ii), by part (i) which was just proved and by (ii) of Theorem (3.2)(and by Lemma (3.5)), it remains to consider the cases $r = 1, g \geq 6$. Let $r = 1, 7 \leq g \leq 10$. We notice that by [CLM1, Lemma 4] one has $h^0(N_{C_{1,g}}(-2)) = 0$, hence Zak's theorem applies, and therefore if we set $\nu := corank\ \Phi_{\omega_{C_{1,g}}} + 1$, then $C_{1,g}$ is not ν-extendable. On the other hand ν is computed in [CLM1, Table 2] and is equal to $n(g)$. Hence we conclude that $\mathcal{X}_{n,r,g,1} = \emptyset$ for $r = 1, 7 \leq g \leq 10$ and $n > n(g)$. For $r = 1$ and $g \geq 11$ by [CLM1, Theorem 4, Lemma 4 and Table 2] we have $corank\ \Phi_{\omega_{C_{1,g}}} \leq 2$ and $h^0(N_{C_{1,g}}(-2)) = 0$, hence $C_{1,g}$ is not 3-extendable.

For $r = 1, g = 6$ we have $corank\ \Phi_{\omega_{C_{1,6}}} = 10$ by [CLM1, Table 2] but in fact there is no smooth $X^7_{1,6}$ in this case (note that $h^0(N_{C_{1,6}}(-2)) \neq 0$!). To see this observe first that, by (iii) below, a general smooth $X^6_{1,6}$ is the intersection of a quadric hypersurface $Q \subset I\!\!P^{10}$ and the cone $\widetilde{\mathbb{G}}$ over the Grassmannian $\mathbb{G} = \mathbb{G}(1,4) \subset I\!\!P^9$ in the Plücker embedding. Now suppose there is $X^7_{1,6} \subset I\!\!P^{11}$. Then an argument like Lemma (3.5) shows that a general hyperplane section of a general deformation of the $X^7_{1,6}$ is a general $X^6_{1,6}$, hence such that $X^6_{1,6} = \widetilde{\mathbb{G}} \cap Q = X^7_{1,6} \cap H$. Since the ideal of $\mathbb{G} \subset I\!\!P^9$ is generated by five quadrics we have $H^0(\mathcal{I}_{\mathbb{G}}(2)) =< Q_1, \ldots, Q_5 >$, and from this it follows that also the ideals of $X^6_{1,6}$ and $X^7_{1,6}$ are generated by quadrics. In fact if we set $I\!\!P^9 = \{x_{10} = x_{11} = 0\} \subset I\!\!P^{11}$ we have $H^0(\mathcal{I}_{X^6_{1,6}}(2)) =< Q_1, \ldots, Q_5, Q >$ and $H^0(\mathcal{I}_{X^7_{1,6}}(2)) =< \overline{Q}_1, \ldots, \overline{Q}_5, \overline{Q} >$, where $\overline{Q}_i$ and $\overline{Q}$ are quadrics in $I\!\!P^{11}$ that restrict to Q_i and Q respectively for $x_{11} = 0$. But then the variety $Y \subset I\!\!P^{11}$ of dimension 8 and degree 5 defined scheme-theoretically by the five quadrics $\overline{Q}_i, i = 1, \ldots, 5$, is a variety that extends twice the Grassmannian $\mathbb{G} \subset I\!\!P^9$ and hence Y must be a cone over $\mathbb{G}$ with vertex a line (this follows for example from [CLM3]), and then $X^7_{1,6}$ is singular. Therefore (ii) is proved.

As in the proof of part (iii) of Theorem (3.2), using successive degenerations to cones over hyperplane sections, to see part (iii) of Theorem (3.15) we need to show that the family of known examples of $X^n_{r,g}$ has the same dimension as the dimension of the tangent space $T^n_{r,g}$ to the Hilbert scheme $\mathcal{X}_{n,r,g,1}$ at the cone points. In this case we have

$$dim\ T^n_{r,g} = h^0(N_{X^n_{r,g}}) \leq h^0(N_{V_{r,g}}) + (n-3)h^0(N_{V_{r,g}}(-1)) + [n-3+\binom{n-3}{2}]h^0(N_{V_{r,g}}(-2))$$

since $h^0(N_{V_{r,g}}(-k)) = 0$ for $k \geq 3$ because $h^0(N_{C_{r,g}}(-k)) = 0$ by [CLM1, Lemma 4] and Proposition (2.22). Therefore we get

$$h^0(N_{V_{r,g}}) + (n-3)h^0(N_{V_{r,g}}(-1)) + [n-3+\binom{n-3}{2}]h^0(N_{V_{r,g}}(-2)) \leq$$

$$\leq dim\ \mathcal{V}_{r,g,1} + (n-3)h^0(N_{C_{r,g}}(-1)) + [3n-9+\binom{n-3}{2}]h^0(N_{C_{r,g}}(-2)) =$$

$$= N(r,g)^2 + nN(r,g) + 19 + (n-2)\ corank\ \Phi_{\omega_{C_{r,g}}} + [3n-7+\binom{n-3}{2}]h^0(N_{C_{r,g}}(-2)).$$

Now one easily sees that the values of the upper bound given above coincide with the number of parameters of the examples of Mukai in Table (3.14) for $n = n(g)$ or of their hyperplane sections for $n < n(g)$. ∎

(3.16) *Remark.* Since Pinkham's work on cones over elliptic curves it is well-known that, in general, the converse of Zak's theorem does not hold. We would like to note here that from the two theorems just proved we can deduce more examples in the case of *smooth extensions*, that is smooth nondegenerate varieties $X \subset \mathbb{P}^m$ such that $h^0(N_X(-2)) = 0$ and X is not smoothly k-*extendable*, but $h^0(N_X(-1)) > m+k$. Examples are given by the varieties $C_{2,4}, C_{3,3}, S_{3,3}$ and $V_{2,4}$. Also note that in the case $r = 1, g = 6$ we have that $C_{1,6}$ is *infinitely many times extendable* (just take quadric sections of cones over $\mathbb{G}(1,4) \subset \mathbb{P}^9$ with vertices linear spaces), but it is not smoothly 6-*extendable*. Of course in this case we have $h^0(N_{C_{1,6}}(-2)) \neq 0$.

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