VARIATIONS ON PASCAL'S THEOREM

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CONTENTS

ABSTRACT. In this paper we present a variety of statements that are in the spirit of the famous theorem of Pascal, often referred to as the Mystic Hexagon. We give explicit equations describing the conditions for $d + 4$ points to lie on rational normal curves. A collection of problems of Pascal type are considered for quadric surfaces in $\mathbb{P}^3$. Finally we reprove, using computer algebra methods, a remarkable theorem of Richmond, Segre, and Brown, for quadrics in $\mathbb{P}^4$ containing five general lines.

INTRODUCTION

We recall the famous theorem of Pascal, which states that for six points p_i ($1 \leqslant i \leqslant 6$) in general position in the plane, they lie on a conic if and only if the three derived points of intersection of the opposite lines of the hexagon with vertices equal to the p_i lie on a line. We give the precise statement in Theorem 1.

Various authors have considered generalizations of Pascal's theorem to prove statements which we will call "of Pascal type". The general form of such statements is the following. Let $\mathcal{F}$ and $\mathcal{G}$ be irreducible families of varieties in a projective space. Let $\mathcal{C}$ be a collection of subsets of this projective space, from which one can form a derived collection of subsets $\mathcal{D}$ (also in this space) using elementary projective geometry constructions (e.g., taking spans, intersections, etc.).

A statement of Pascal type would be one with the following form: *There is a member of $\mathcal{F}$ containing the collection $\mathcal{C}$ if and only if there is a member of $\mathcal{G}$ containing the derived collection $\mathcal{D}$.*

For example Pascal's original theorem has $\mathcal{F}$ being the space of conics in a given plane, $\mathcal{G}$ the space of lines, $\mathcal{C}$ a sextuple of points, and $\mathcal{D}$ triples of points, with the triples being defined by the sextuples in the way indicated in Pascal's Theorem.

In this paper we will consider some instances of theorems of Pascal type. When $\mathcal{F}$ is the family of rational normal curves of degree d in $\mathbb{P}^d$, Caminata and Schaffler in [CS, Theorem A] have generalized Pascal's Theorem when $\mathcal{C}$ is the space of $(d + 4)$-tuples of points in general position in $\mathbb{P}^d$.

When $\mathcal{F}$ is the family of quadric hypersurfaces in $\mathbb{P}^4$, and $\mathcal{C}$ is a collection of five lines in general position there, a remarkable theorem of Pascal type has been observed by Richmond [R] in 1899 (in one direction) and full proofs were given by B. Segre [S] and Brown [B] in 1945.

In this paper we first present a computer algebra approach to the original Pascal's Theorem, first executed (it seems) by Stefanovi'c and Milo'sevi'c in [SM]. Then we consider the problem of characterizing points on rational normal curves, and give a condition for $d + 4$ points to lie on one, which simplifies the Caminata-Schaffler statement.

Secondly we turn our attention to the case of ten points on a quadric surface in $\mathbb{P}^3$; we have partial results which are adjacent to but not exactly of Pascal type.

Finally we take up the case of five lines on quadrics in $\mathbb{P}^4$, and re-prove the Richmond-Segre-Brown result via computer algebra methods, which is in the same spirit as the approach in [SM]. The proof follows Segre's approach, but is purely algebraic, and avoids the long complicated geometric arguments that Segre and Brown use.

1. POINTS ON RATIONAL NORMAL CURVES

Any set of $d + 3$ points in general position on $\mathbb{P}^d$ lie on a unique rational normal curve.

One can ask: given $d + 4$ points in general position in $\mathbb{P}^d$, how can we detect whether they lie on a rational normal curve?

For $d = 2$, the question is answered by Pascal's Theorem for conics in the plane; we remind the reader how this goes in Subsection 1.1.

The case $n = 3$ has been considered by Chasles [Chasles]; he proposes a rather complicated condition which we do not see how to extend to higher dimensions.

We present an iterative approach that works for all n, and builds on Pascal's Theorem for $n = 2$. This is illustrated for $d = 3$ in subsection 1.2 before turning our attention to the general case in subsection 1.3.

1.1. **The case $d = 2$.** We recall the famous "Mystic Hexagon" Theorem of Pascal:

Theorem 1. *Let $p_1, \ldots, p_6$ be six points in the plane, no three collinear. Let L_i be the line joining p_i to p_{i+1} (indices taken modulo 6), for $1 \leqslant i \leqslant 6$. Let q_j be the point $L_j \cap L_{j+3}$ for $1 \leqslant i \leqslant 3$. Then the six points $\{p_i\}_{1 \leqslant i \leqslant 6}$ lie on an irreducible conic if and only if the three points $\{q_j\}_{1 \leqslant j \leqslant 3}$ are collinear.*

There are a number of proofs of this theorem; it may be that one of the most recent and modern proof (using symbolic algebra packages) was given in [SM] and goes as follows.

Since no three of the points are collinear, the p_i's lie on an irreducible conic if and only if they lie on a conic, i.e., they satisfy a quadratic polynomial in the coordinates of $\mathbb{P}^2$. If we choose such coordinates $[x_i : y_i : z_i]$ for p_i, then a quadratic polynomial $ax^2 + by^2 + cz^2 + dxy + exz + fyz$ has p_i as a root if and only if the row vector $v_i =$

$(x_i^2, y_i^2, z_i^2, x_i y_i, x_i z_i, y_i z_i)$ is orthogonal to the column vector $(a, b, c, d, e, f)^\top$. Hence to find such a quadratic polynomial defining the conic, we must find such a nonzero column vector that is orthogonal to the six row vectors v_i given by each of the points. This is possible if and only if the determinant of the 6×6 matrix whose rows are the v_i's is zero. This determinant is a polynomial F in the 18 variables which define the coordinates of the points. It is quadratic in each of the sets of six variables for each point, and has therefore total degree 12.

On the other hand, each point q_j has projective coordinates given by quadratic polynomials in the six variables defining the two points p_j and p_{j+3}, given by the 2×2 minors of the corresponding 2×3 matrix formed by the coordinates of p_j and p_{j+3}. These expressions for the coordinates of the q_j are quartic polynomials in the p coordinates. Then the three q_j's are collinear if and only if the determinant of the 3×3 matrix whose rows are their coordinates is zero. This is a polynomial G, again in the 18 variables. It is cubic in each of the three sets of q variables, hence of total degree 12 in the p variables also.

Any modern symbolic algebra package (we verified this with SageMath) can deduce that $F = G$ as polynomials in the 18 variables, which was first noted in [SM].

We observe that the identity $F = G$ proves a bit more, namely Pappus' Theorem, which deals with the case when the points may lie three each on two lines.

In [TW1], Traves and Whelau give a very nice constructive analogue to Pascal's Theorem, namely a criterion for when ten points in the plane lie on a cubic curve. There are very many extensions of Pascal's Theorem via the perspective of more general Cayley-Bacharach theorems; the reader may consult [EGH] for a history.

1.2. **The case** $n = 3$**.** Our analysis of the general case is illustrated by the case $n = 3$, which we now take up as an example. Let $p_1, \ldots, p_7$ be seven points in general position in $\mathbb{P}^3$. We note that if we project from any one of the points, say p_i, the remaining six go to a set S_i of six points in $\mathbb{P}^2$, which are also in general position.

Proposition 2. *The points* $\{p_i\}_{1 \leqslant i \leqslant 7}$ *lie on a twisted cubic curve in* $\mathbb{P}^3$ *if and only if two of the sets* S_i *and* S_j*, with* $1 \leqslant i < j \leqslant 7$*, lie on a conic in* $\mathbb{P}^2$*.*

Proof. If the points lie on a twisted cubic, then it is clear that all of the sets S_i lie on the image of the cubic, which is a conic.

Conversely, suppose that S_i and S_j lie conics C_i and C_j respectively. Consider the two cones T_i and T_j formed by the two joins of the conics and the corresponding projection points p_i and p_j. These are quadric surfaces that meet in the line L joining p_i to p_j. The residual intersection is the desired twisted cubic. $\qquad\square$

To implement this criterion with 7 specific points, we make two explicit projections, and then apply the two conditions of Pascal's Theorem, which involve checking two determinants being zero. This of course gives two conditions on the coordinates of the seven points.

This is the correct number: the first six points lie on a unique twisted cubic, and it is two conditions for the seventh point in $\mathbb{P}^3$ to lie on this curve.

We can make another count of parameters as follows. The space of twisted cubics depend on 12 parameters; such a curve is parametrized by four cubic polynomials, which have then 16 coefficients; the automorphisms of the parametrization are the three dimensions of automorphisms of $\mathbb{P}^1$ and the scaling in $\mathbb{P}^3$. Alternatively, they are all projectively equivalent in $\mathbb{P}^3$, and the 15 dimensions of automorphisms of $\mathbb{P}^3$ give the 12 parameters, since there is a 3-dimensional stabilizer for any one of them.

Then seven points on these cubics depend on $12 + 7 = 19$ parameters. But seven points in $\mathbb{P}^3$ depend on 21 parameters, and $21 - 91 = 2$.

1.3. **The general case.** Now we fix $d + 4$ points $\{p_i\}_{1 \leqslant i \leqslant d+4}$ in general position in $\mathbb{P}^d$. Note first that if we choose a point p_i, we may project the others to $\mathbb{P}^{d-1}$, obtaining a set S_i of $d + 3$ points in general position in $\mathbb{P}^{d-1}$.

Note secondly that if we choose a set R of $d - 2$ of the points, where $p_i \notin R$, we may project (from R) the remaining six points to a set T_R in $\mathbb{P}^2$.

Theorem 3. *The set of $d + 4$ points in $\mathbb{P}^d$ lie on a rational normal curve of degree d if and only if S_i lies on a rational normal curve of degree $d - 1$ in $\mathbb{P}^{d-1}$ and T_R lies on a conic in $\mathbb{P}^2$.*

Proof. If the $d + 4$ points lie on a rational normal curve, the assertion is true in that direction. Conversely, we have two cones at hand: one is the cone C_i over the rational normal curve in $\mathbb{P}^{d-1}$ with vertex p_i; the other is a quadric cone Q_R over the conic in $\mathbb{P}^2$ with vertex the span of R.

These two cones intersect in the set of $d - 2$ lines joining p_i to the points of R. The residual intersection has degree $2(d-1) - (d-2) = d$, and is the desired rational normal curve. $\square$

Applying this iteratively, we have:

Corollary 4. *Given $d + 4$ points in general position in $\mathbb{P}^d$, they lie on a rational normal curve of degree d if and only if the following holds. Fix $d - 1$ of the points to form a subset U. For each point p in U, we may project from the span of the complement of p in U to a plane. This gives six points in the plane, namely the projections of the complement of U and the point p. For each such point in U, these six points must lie on a conic.*

Note that in the Corollary, this gives exactly $d - 1$ conditions on the points.

Remark 5. We see that to apply the above criterion for $d + 4$ points in $\mathbb{P}^d$ to lie on a rational normal curve, we must ultimately make $d - 1$ applications of Pascal's Theorem, which, as we have observed in Section 1.1, is one polynomial in the coordinates of the points.

Hence the variety of solutions here is at most codimension $d - 1$ in the relevant parameters. It is easily seen to be exactly $d - 1$, by counting dimensions as in the $d = 3$ case.

The family of rational normal curves in $\mathbb{P}^d$ depends on $d^2 + 2d - 3$ parameters. (They are all projectively equivalent, and the stabilizer is 3-dimensional.) Therefore the $d + 4$-tuples of points on such a curve depend on $d^2 + 3d + 1$ parameters.

The parameters for $d + 4$ points in general are $d^2 + 4d$; the difference is $d - 1$ as claimed.

Remark 6. The criteria presented by Caminata and Schaffler in [CS, Theorem A] basically reduces to asking that however one projects from a subset of $d - 2$ points to a plane, the remaining six points lie on a conic, and this is then rephrased via Pascal's Theorem to a linear condition on the derived intersection points. Their argument is algebraic in nature, using the Grassmann-Cayley algebra; the proof above seems more geometric.

1.4. **Equations.** In [CGMS], the authors, among other things, determine equations for $(d + 4)$-tuples of points to lie on a rational normal curve of degree d in $\mathbb{P}^d$. The above considerations also enable us to give similar explicit equations for this problem. We assume $d \geqslant 3$ here.

Given $d + 4$ points in $\mathbb{P}^d$, using projective transformations we may fix the first $d + 2$ of them to be the coordinate points and the point $[1 : 1 : \cdots : 1]$. The coordinates of the remaining two points give our parameter space for the problem. We set these coordinates to be $[a_0 : \cdots : a_d]$ and $[b_0 : \cdots : b_d]$. We restrict ourselves to the open subset where the points are in general position.

We now apply Corollary 4, taking U to be the first $d - 1$ of the coordinate points:

$$U = \{[1 : 0 : \cdots : 0], [0 : 1 : 0 : \cdots : 0], \ldots, [0 : \cdots : 0 : 1 : 0 : 0]\}$$

Using homogeneous coordinates $[z_0 : z_1 : \cdots : z_d]$, the projection from the complement of the i-point ($0 \leqslant i \leqslant d - 2$ here) is exactly the projection to the plane with coordinate $[z_i : z_{d-1} : z_d]$.

The six points of the construction in this plane are then given by $[1 : 0 : 0]$ (the image of the i-th point) $[0 : 1 : 0]$, $[0 : 0 : 1]$, and $[1 : 1 : 1]$ (the projections of the other coordinate points not in U) and $[a_i : a_{d-1} : a_d]$, $[b_i : b_{d-1} : b_d]$ (the projections of the two final points). We then take the double Veronese images of these six points to produce the following six points in $\mathbb{P}^5$:

$$
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
a_i^2 & a_{d-1}^2 & a_d^2 & a_i a_{d-1} & a_i a_d & a_{d-1} a_d \\
b_i^2 & b_{d-1}^2 & b_d^2 & b_i b_{d-1} & b_i b_d & b_{d-1} b_d
\end{array}
$$

We must impose that the determinant of the corresponding 6×6 matrix is zero, which simplifies to the equation

$$a_i a_d b_{d-1} b_d - a_{d-1} a_d b_i b_d - a_i a_{d-1} b_{d-1} b_d + a_{d-1} a_d b_i b_{d-1} + a_i a_{d-1} b_i b_d - a_i a_d b_i b_{d-1} = 0.$$

These equations, as i ranges from 0 to $d - 2$, give the required equations describing the family of points lying on a rational normal curve, using this parametrization. This holds for the points $[\underline{a}]$ and $[\underline{b}]$ satisfying some open conditions that ensure the general position of the $d + 4$ points. They are exactly the equations coming from the collinearity of the three derived points in the plane, via the assertions of Pascal's Theorem.

We note that one may interpret the vanishing of the above determinant differently. It is clear that the determinant is equal to the lower right 3×3 minor, and the vanishing of this minor says exactly that the three points

$$(a_i a_{d-1}, b_i b_{d-1}), (a_i a_d, b_i b_d), (a_{d-1} a_d, b_{d-1} b_d)$$

are collinear in the affine plane.

These consideration additionally show that our equations define the variety of points on a rational normal curve as a set-theoretic complete intersection.

2. Quadrics in $\mathbb{P}^3$

One may imagine that an extension of Pascal's Theorem to quadrics in $\mathbb{P}^3$ would be available, but there does not seem to be an exact analogue available. In [Chasles, Note 32] there are presented several statements for tetrahedra related to quadric surfaces, but they do not seem to present the same type of phenomena as a Pascal-type theorem.

2.1. **Ten points.** Quadrics in $\mathbb{P}^3$ form a 9-dimensional complete linear system, and there is in general a unique quadric through 9 general points; there is no quadric through 10 general points. The condition that there is a quadric through 10 points is easy to describe in terms of the coordinates of the points $\{p_i\}_{1 \leqslant i \leqslant 10}$.

If $p_i = [a_{i0} : a_{i1} : a_{i2} : a_{i3}]$, we re-embed these points into $\mathbb{P}^9$ via the double Veronese map, obtaining 10 points in $\mathbb{P}^9$. They will be dependent in $\mathbb{P}^9$ if and only if there is a quadric through the ten points in $\mathbb{P}^3$. This gives a straightforward polynomial condition to check; that condition is the determinant of the 10×10 matrix whose rows are the quadratic monomials in the a_{ij}'s, and hence has degree 20.

There does not seem to be an alternative construction, a la Pascal's Theorem for conics in the plane, to re-write this polynomial as arriving from a different geometric condition. However recently Traves and Whelau [TW2] translated the condition for ten points to be on a quadric surface in $\mathbb{P}^3$ in a condition for four points in $\mathbb{P}^3$ to be coplanar.

There are some elementary statements one can make, taking into account the geometry of a quadric surface. An example is the following.

Proposition 7. *Let there be ten points in general position in $\mathbb{P}^3$. Take the ten points, divide them into two sets of five points. If the ten points lie on a quadric, then there are twisted cubic curves in $\mathbb{P}^3$ passing through each of the two sets of five points, that meet each other in five additional points. The converse holds as well: if two such twisted cubics exist, the ten points lie on a quadric.*

The proof is easily seen by considering the quadric as $\mathbb{P}^1 \times \mathbb{P}^1$, and noticing that there are curves of bidegree $(1, 2)$ and $(2, 1)$ on the quadric through general five points; they will meet in five points.

One may extend the set of conditions and include requiring that the quadric contains various low degree curves. We take up several examples of these in the following subsections.

2.2. **Seven Points and One Line.** It is three conditions for a quadric to contain a line, and therefore if we fix seven general points and one general line, there should not be a quadric containing them; the condition that there is one should be one condition on the parameters describing the points and the line.

We again have a geometric condition in the spirit of the statement for ten points, but it is not exactly of Pascal type.

Proposition 8. *Let there be given a general line and seven general points in $\mathbb{P}^3$. If this collection lies on a quadric surface, there is a rational quartic curve passing through the points and meeting the line in three points. The converse is also true.*

Again the proof is immediate upon considering a curve of bidegree $(1, 3)$ on the quadric, which will be the rational quartic.

2.3. **Four Points and Two Lines.** Given four points and two lines in general position in $\mathbb{P}^3$, we again should have one condition that they lie on a quadric. Call the points $p_0, \ldots, p_3$. There is a unique line L_i containing p_i and meeting the two given lines, for $1 \leqslant i \leqslant 3$. There is a quadric through the four given points and the two given lines if and only if there is a quadric through p_0 and L_1, L_2, L_3. Hence this case reduces to the next case.

2.4. **One Point and Three Lines.** In this case there is an elementary Pascal-type condition to state.

Proposition 9. *Given a general point and general three lines in $\mathbb{P}^3$, they lie on a quadric surface if and only if there is a line through the point meeting each of the three lines. Equivalently, this happens if the projection from the point maps the three lines to three concurrent lines in the plane.*

Again, this is elementary: the three lines (if on a quadric) come from one of the two rulings, and the line through the point is the unique line from the other ruling through the point.

In this case we also offer an algebraic analysis. Let us choose coordinates so that the given point P is at $[1 : 0 : 0 : 0]$ and the three given lines each contain the other three coordinate points: L_1 contains $[0 : 1 : 0 : 0]$, L_2 contains $[0 : 0 : 1 : 0]$, and L_3 contains $[0 : 0 : 0 : 1]$. Then each line is determined by an additional general point $R_i = [a_{i0} : a_{i1} : a_{i2} : a_{i3}]$ on L_i. We seek the conditions on the a_{ij}'s so that P and the L_i lie on a quadric.

That the quadric contains the four coordinate points means that the quadratic equation has no square terms, and we are left with the remaining six monomial terms. The conditions are that the three points R_i are zeroes of the polynomial, and that the tangent plane to the quadric at the coordinate point passes through R_i as well.

The first three conditions are quadratic in the a_{ij}'s, and the second three are linear. The corresponding matrix condition can be easily seen to be that the determinant of the following 6×6 matrix must be zero:

$$\begin{pmatrix} a_{10}a_{11} & a_{10}a_{12} & a_{10}a_{13} & a_{11}a_{12} & a_{11}a_{13} & a_{12}a_{13} \\ a_{20}a_{21} & a_{20}a_{22} & a_{20}a_{23} & a_{21}a_{22} & a_{21}a_{23} & a_{22}a_{23} \\ a_{30}a_{31} & a_{30}a_{32} & a_{30}a_{33} & a_{31}a_{32} & a_{31}a_{33} & a_{32}a_{33} \\ a_{10} & 0 & 0 & a_{12} & a_{13} & 0 \\ 0 & a_{20} & 0 & a_{21} & 0 & a_{23} \\ 0 & 0 & a_{30} & 0 & a_{31} & a_{32} \end{pmatrix}$$

The determinant of this matrix has degree 9, tridegree 3 in each of the three sets of variables for the three points R_i. This determinant is:

$$(a_{21}a_{30} - a_{20}a_{31})(a_{12}a_{30} - a_{10}a_{32})(-a_{13}a_{20} + a_{10}a_{23})(a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}).$$

The first three terms express degenerate cases in which two of the lines meet, and are therefore coplanar. This geometric condition obviously guarantees the existence of the quadric as the plane containing the two lines that meet, together with the plane through the third line and P. Factoring these out, we obtain the relatively simple algebraic condition that

$$a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} = 0$$

and this expresses exactly the condition that the projection of the three lines from P to the plane $z_0 = 0$ are concurrent.

2.5. **One Conic and Five Points.** In this case we again have a geometric condition.

Proposition 10. *Let there be given a general conic and five general points in $\mathbb{P}^3$. Then they lie on a quadric surface if and only if there is a twisted cubic passing through the five points and meeting the conicat three points.*

The conic is of bidegree $(1,1)$ on the quadric, and the required twisted cubic is of bidegree $(1,2)$. We note that the twisted cubic is not unique: there is a second one, having bidegree $(2,1)$.

2.6. **Twisted Cubic and Three Points.** In this case there is a Pascal-type proposition to state, which again follows from relatively elementary considerations.

Proposition 11. *Given a twisted cubic and three general points in $\mathbb{P}^3$, they lie on a quadric surface if and only if the twisted cubic meets the plane in three additional points which, together with the three given points, lie on a conic in that plane.*

In this case the twisted cubic is of bidegree $(1,2)$, the conic has bidegree $(1,1)$, and the statement follows directly from the geometry of $\mathbb{P}^1 \times \mathbb{P}^1$.

3. THE RICHMOND-SEGRE-BROWN THEOREM FOR LINES IN $\mathbb{P}^4$

There is a remarkable extension of Pascal's Theorem to quadrics in $\mathbb{P}^4$, which is the following construction.

For a quadric in $\mathbb{P}^4$ to contain a line is 3 conditions. The space of quadrics in $\mathbb{P}^4$ is of dimension 14. Therefore it is expected that there are no quadrics containing five general lines, and this is indeed the case.

The Grassmann variety of lines in $\mathbb{P}^4$ is 6-dimensional, so the number of parameters for quintuples of lines is 30. Let $\mathcal{I}$ be the incidence correspondence consisting of pairs (Q, C) where Q is a quadric, C is a collection of five skew lines, and $C \subset Q$. The family of lines on a quadric threefold has dimension three, so that the fiber of $\mathcal{I}$ to the projective space of quadrics in $\mathbb{P}^4$ has dimension 15. Hence $\mathcal{I}$ has dimension 29.

Now given five general lines on a general quadric, we claim that there is only that one quadric containing them. The reason is that the quadrics containing them is a linear system, and if it is positive dimensional, the intersection of the given quadric with a general other member of the system would be a Del Pezzo surface of degree 4 in $\mathbb{P}^4$ containing the five lines. However on any such Del Pezzo, there is a line meeting all five skew lines. This is not possible by the generality of the five lines on the quadric.

Hence the second projection from $\mathcal{I}$ to the space of five general lines is birational, and so the image has dimension 29 as well, hence it has codimension one.

Therefore it is one condition for a quintuple of lines to lie on a quadric.

What is that condition? This is the content of a remarkable theorem first addressed by Richmond at the end of the 19th century [R]. It was rediscovered by B. Segre in a paper in 1945 who does not reference Richmond; he may not have been aware. In the next year Brown [B] gave a different proof, and Brown references both Richmond and Segre.

It is convenient to make a definition:

Definition 12. Fix an ordered set of five lines L_i, $1 \leqslant i \leqslant 5$. For each i, consider the point R_i of intersection of L_i with the span of L_{i-1} and L_{i+1} (indices taken modulo five). We say that the quintuple of lines is *of RSB type* (after Richmond, Segre, and Brown) if those five points $\{R_i\}_{1 \leqslant i \leqslant 5}$ are dependent in $\mathbb{P}^4$.

The theorem is the following.

Theorem 13. *Five lines in general position in $\mathbb{P}^4$ lie on a quadric if and only if they are of RSB type.*

The proofs given in the last centuries by Richmond, Segre, and Brown are a tour-de-force of four-dimensional synthetic geometry, involving rather complicated constructions.

Nowadays a computer algebra proof is available, once we formulate the problem in explicit linear algebra terms. We will explain now the approach.

We choose two points P_i and Q_i on the line L_i. Using projective transformations we may assume that the P_i are the five coordinates points of $\mathbb{P}^4$.

We describe the conditions on the quadric to contain the five lines as follows. We write the general quadric as the zeroes of a homogeneous quadratic polynomial in the homogeneous coordinate $[z_0 : \cdots : z_4]$. That the quadric contains the coordinate points P_i is equivalent to having no pure square terms in the quadratic polynomial. This leaves 10 terms with the monomials $z_i z_j$, $i \neq j$.

Now imposing that the quadric contain the Q_i's give five linear conditions on the coefficients. These linear conditions are quadratic in the coordinates of the Q_i's.

Finally imposing that the quadric contains the line L_i, is now equivalent to imposing that the tangent hyperplane to the quadric at P_i contains Q_i. This is five more linear conditions on the coefficients, which are each linear in the coordinates of the Q_i's.

Hence the existence of the quadric amounts to the vanishing of the determinant F of the 10×10 matrix described by the above ten conditions on the ten coeffients. This matrix has five rows with entries of degree two in the coordinates and five rows with linear entries in the coordinates. Hence the degree of the polynomial F in these coordinates is 15.

Let us now describe the condition for the five lines to be of RSB type. The calculation of the points R_i can be made explicitly as follows. The hyperplane spanned by L_{i-1} and L_{i+1} is exactly the hyperplane spanned by the four points $P_{i-1}, Q_{i-1}, P_{i+1}, Q_{i+1}$. Hence the coefficients of this plane are given the 4×4 minors of the 4×5 matrix whose rows are the coordinates of those four points. Each of these minors is a quadratic polynomial in the coordinates of the Q_i's.

The intersection of this hyperplane with the line L_i is obtained by taking the parametric equation for L_i and solving for the intersection point; this gives a point R_i whose coordinate are now cubic polynomials in the coordinates of the Q_i's.

Now the condition that the R_i's are dependent in $\mathbb{P}^4$ is the determinant G of the corresponding 5×5 matrix whose rows are the coordinates of the R_i's. Since the R_i are cubic, we see that G is also of degree 15.

In the same spirit as was discussed in section 1.1 for Pascal's Theorem in the plane, these two polynomials can be computed explicitly with any modern symbolic algbra package. We used Sagemath, and verified that $F = G$ as polynomials of degree 15 in the variable coordinates of the Q_i's. This proves the theorem.

Remark 14. In Segre's paper, he also considered the two relevant determinants above. He was unable to prove algebraically that these two determinant were equal. However he argued via synthetic methods that their ratio must be a constant (and he computed the constant!). Hence our proof follows his overall line of argument, but the use of the computer simplifies things greatly.

Remark 15. One can give a different criterion for five lines to lie on a quadric hypersurface in $\mathbb{P}^4$ by following the line of argument of Traves and Whelau [TW2] for ten points on a quadric surface in $\mathbb{P}^3$, as follows. Choose four of the five lines, say L_1, L_2, L_3, L_4. The quadrics containing them form a 2-dimensional family, and this family has three reducible members, obtained by splitting the four lines into two pairs, and taking the union of the hyperplanes through the pairs. To be specific, if we denote the hyperplane containing L_i and L_j by H_{ij}, then we obtain three (reducible) quadrics by setting $Q_1 = H_{12} + H_{34}$, $Q_2 = H_{13} + H_{24}$, and $Q_3 = H_{14} + H_{23}$. It is easy to see that these three are independent.

Denote the equations of the quadric Q_i by F_i. Then the map ϕ given by $[F_1 : F_2 : F_3]$ sends $\mathbb{P}^4$ rationally to $\mathbb{P}^2$. Since the F_i's are quadratics, the image under ϕ of a general line in $\mathbb{P}^4$ will be a conic.

Proposition 16. *The five lines in $\mathbb{P}^4$ lie on a quadric hypersurface if and only if the image of the fifth line L_5 under this rational map is a line, not a conic. This is equivalent to asking that the images of any three distinct points of L_5 are collinear.*

This is not very deep, actually: the map ϕ is simply the composition of the Veronese map of degree two with a projection onto $\mathbb{P}^2$ given by the three F_i's. The existence of the desired quadric hypersurface is (tautologically) equivalent to the existence of a hyperplane in the Veronese space containing the image of the five lines. Since the center of projection for ϕ (the zeroes of the F_i's) already contains four of the lines, the criterion is that the fifth line must be cut out by a linear equation in the $\mathbb{P}^2$.

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