# TRIPLE COVERS IN ALGEBRAIC GEOMETRY

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**1. Introduction.** The main purpose of this article is to develop the foundations of the theory of triple coverings in algebraic geometry, i.e. flat finite maps of degree 3 between irreducible varieties over an algebraically closed field $k$. The approach is "from the bottom up," i.e., given a variety $Y$, what data is required to construct a triple cover of $Y$? Theorem (3.6) is essentially the following.

Theorem 1.1. *A triple cover of $Y$ is determined by a locally free rank 2 $\mathcal{O}_Y$-module $E$ and a map $\Phi: S^3E \to \Lambda^2E$, and conversely.*

The above result is in the spirit of the corresponding statement for double covers, namely that a double cover of $Y$ is determined by a line bundle $L$ on $Y$ and a divisor $D \in |L^{-2}|$ ($D$ is the branch locus of the cover). Double covers have been used in a variety of ways to understand and construct varieties and it is my belief that triple covers will play an increasingly important role in this area as their properties become better understood.

The following results are obtained as corollaries to the general theory.

Theorem 1.2. *The general triple cover in dimension $\geq 2$ has a singular branch locus.*

Theorem 1.3. *The general triple cover in dimension $\geq 4$ is singular.*

Theorem 1.4. *The moduli space of trigonal curves of genus $g$ is connected, of dimension $2g + 1$, and is unirational.*

The final sections of the article are devoted mainly to the computation of the standard invariants of varieties which are triple covers, especially curves and surfaces. As one application, I propose a method for constructing surfaces of general type with $K^2$ arbitrarily close to $3e(X)$.

**2. The local analysis.** Let $k$ be an algebraically closed field of characteristic unequal to 2 or 3. Let $\mathcal{O}_Y$ be a local $k$-algebra which is an integral domain. Let $\mathcal{O}_X$ be a flat $\mathcal{O}_Y$-algebra which is integral over $\mathcal{O}_Y$ of rank 3.

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Assume that $\mathcal{O}_X$ is also an integral domain. $\mathcal{O}_Y$ naturally sits inside $\mathcal{O}_X$, and every $a \in \mathcal{O}_X - \mathcal{O}_Y$ will satisfy a unique irreducible monic cubic polynomial $p_a(x)$ over $\mathcal{O}_Y$.

*Definition* 2.1. The *minimal cubic polynomial* of $a \in \mathcal{O}_X$ is the polynomial $p_a(x)$ if $a \notin \mathcal{O}_Y$ and is $(x - a)^3$ if $a \in \mathcal{O}_Y$.

LEMMA 2.2. *$\mathcal{O}_X$ naturally splits as $\mathcal{O}_X = \mathcal{O}_Y \oplus E$, where $E$ is the submodule of $\mathcal{O}_X$ consisting of those elements whose minimal cubic polynomial has no square term.*

*Proof.* For any $a \in \mathcal{O}_X$, write its minimal cubic polynomial as $x^3 + c_2(a)x^2 + c_1(a)x + c_0(a)$; define a map $\alpha$ from $\mathcal{O}_X$ to $\mathcal{O}_Y$ by sending $a$ to $-c_2(a)/3$. It is clear that $\alpha$ is the identity on $\mathcal{O}_Y$ and the kernel of $\alpha$ is precisely $E$. It remains to show that $\alpha$ is an $\mathcal{O}_Y$-linear homomorphism. It suffices to prove this after passing to the fraction fields $K_Y$ and $K_X$ of $\mathcal{O}_Y$ and $\mathcal{O}_X$, respectively, where we make the analogous definitions. In this case $K_X$ is an extension of $K_Y$ of degree 3 and so $K_X = K_Y(e)$ for some $e \in K_X - K_Y$. By replacing $e$ by $e + c_2(e)/3$, we may assume $e \in E$. Let $e' = e^2 + 2c_1(e)/3$. A computation shows that $e' \in E: (e')^3 - (c_1(e)^3/3)e' - (c_0(e)^2 + 2c_1(e)^3/27) = 0$. Note that $K_X$ is generated as a vector space over $K_Y$ by 1, $e$ and $e'$. A further calculation shows that for $y_i \in K_Y$,

$$y_0 + y_1 e + y_2 e' \in E \Leftrightarrow y_0 = 0,$$

and so the projection map is the map sending $y_0 + y_1 e + y_2 e'$ to $y_0$ and is therefore $K_Y$-linear. Q.E.D.

The projection of $\mathcal{O}_X$ onto $\mathcal{O}_Y$ is 1/3 of the trace map, and $E$ is the submodule of "trace-zero" elements of $\mathcal{O}_X$. The projection of $\mathcal{O}_X$ onto $E$ is given by the well-known Tschirnhausen transformation, or "completing the cube"; if the minimal cubic polynomial for $a \in \mathcal{O}_X$ is $x^3 + c_2x^2 + c_1x + c_0$, send $a$ to $a + c_2/3$. For this reason I will call $E$ the *Tschirnhausen module* of $\mathcal{O}_X$ over $\mathcal{O}_Y$; it is a free $\mathcal{O}_Y$-module of rank 2.

The multiplication in $\mathcal{O}_X$ is given by an $\mathcal{O}_Y$-linear map $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{O}_X$; since $\mathcal{O}_X \cong \mathcal{O}_Y \oplus E$, this map can be rewritten as a map $(\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y) \oplus (\mathcal{O}_Y \otimes_{\mathcal{O}_Y} E) \oplus (E \otimes_{\mathcal{O}_Y} \mathcal{O}_Y) \oplus (E \otimes_{\mathcal{O}_Y} E) \to \mathcal{O}_Y \oplus E$. The first three coordinates of this map define the multiplication in $\mathcal{O}_Y$ and the left and right $\mathcal{O}_Y$-module structure on $E$, respectively, and so are already determined. The fourth coordinate is a map $E \otimes_{\mathcal{O}_Y} E \to \mathcal{O}_Y \oplus E$, which must factor through the second symmetric power $S^2E$ of $E$ if the multiplication in $\mathcal{O}_X$ is to be commutative. Conversely, any $\mathcal{O}_Y$-linear map $\phi: S^2 E \to \mathcal{O}_Y \oplus E$ will define a commutative multiplication on $\mathcal{O}_X$ which is $\mathcal{O}_Y$-linear. However, there *are* conditions on $\phi$ in order that the multiplication be associative and that the submodule $E$ of $\mathcal{O}_X$ be the Tschirnhausen module of $\mathcal{O}_X$ over $\mathcal{O}_Y$. To analyze these conditions, choose a basis $\{z, w\}$ for $E$ as a free $\mathcal{O}_Y$-module. The induced basis of $S^2E$ is $\{z^2, zw, w^2\}$, and so the map $\phi$ takes the form

$$\begin{aligned} \phi(z^2) &= g + az + bw \\ \phi(zw) &= h + ez + fw \\ \phi(w^2) &= i + cz + dw \end{aligned} \tag{2.3}$$

where $a, b, \ldots, i$ are in $\mathcal{O}_Y$.

LEMMA 2.4. *The map $\phi$ induces an associative multiplication on $\mathcal{O}_X$ if and only if*

$$g = be + f^2 - af - bd, \qquad h = bc - ef,$$

and

$$i = e^2 + cf - ac - de.$$

*Proof.* Associativity will follow from the associativity of the triple products in $S^3E$, which is generated by $z^3$, $z^2w$, $zw^2$ and $w^3$. Since the multiplication is already commutative and there is only one way to write $z^3$ and $w^3$ as products, associativity is equivalent to the satisfaction of the equations $z \cdot zw = w \cdot z^2$ and $z \cdot w^2 = w \cdot zw$ in $\mathcal{O}_X$. Using (2.3) to compute these products via $\phi$, one finds that

$$z \cdot zw = (eg + fh) + (h + ae + ef)z + (be + f^2)w,$$

$$w \cdot z^2 = (ah + bi) + (ae + bc)z + (g + af + bd)w,$$

$$z \cdot w^2 = (cg + dh) + (i + ac + de)z + (bc + df)w,$$

and

$$w \cdot zw = (eh + fi) + (e^2 + cf)z + (h + ef + df)w.$$

Equating the corresponding coefficients of $z$ and $w$ in the above equations leads to the conditions of the lemma. Moreover, these values for $g$, $h$ and $i$ imply the equality of the $\mathcal{O}_Y$ coordinates above automatically, so no further conditions are necessary. Q.E.D.

COROLLARY 2.5. *A commutative and associative $\mathcal{O}_Y$-algebra structure on $\mathcal{O}_X = \mathcal{O}_Y \oplus E$ is equivalent to an $\mathcal{O}_Y$-linear map $\phi_2 : S^2E \to E$.*

*Proof.* The equations of the previous lemma imply that the map $\phi$ is determined by its second coordinate $\phi_2$. Q.E.D.

LEMMA 2.6. *A map $\phi : S^2E \to E$ in the form of (2.3) induces a multiplication on $\mathcal{O}_X$ for which $E$ is the Tschirnhausen module for $\mathcal{O}_X$ over $\mathcal{O}_Y$ if and only if $f = -a$ and $e = -d$.*

*Proof.* The submodule $E$ of $\mathcal{O}_Y \oplus E$ will be the Tschirnhausen module if and only if the minimal cubic polynomials of the generators $z$ and $w$ of $E$ have no square term; this means that $z^3$ must be in the $\mathcal{O}_Y$-span of 1 and $z$, and $w^3$ must be in the $\mathcal{O}_Y$-span of 1 and $w$. Using (2.3) to compute $z^3$ and $w^3$, one finds that

$$z^3 = (ag + bh) + (g + a^2 + be)z + (ab + bf)w,$$

and

$$w^3 = (ch + di) + (ce + cd)z + (i + cf + d^2)w.$$

Hence, $b(a + f) = c(e + d) = 0$ in $\mathcal{O}_Y$. If $b = 0$, then $z^2 = g + az$ in $\mathcal{O}_X$ and so $z$ would satisfy a monic quadratic polynomial over $\mathcal{O}_Y$; after passing to the fraction fields, $z$ would generate an intermediate quadratic extension of $K_Y$ under the cubic extension $K_X$, a contradiction. Hence, $b \neq 0$, and a similar argument with $w$ shows that $c \neq 0$. Since $\mathcal{O}_Y$ is an integral domain, we must have $a + f = e + d = 0$. Q.E.D.

The previous lemmas give all the conditions on the multiplication in $\mathcal{O}_X$.

THEOREM 2.7. *Assume that $\mathcal{O}_Y$ is a local k-algebra which is an integral domain.*

2.7.1. *Let $\mathcal{O}_X$ be a flat $\mathcal{O}_Y$-algebra, which is an integral domain, and integral over $\mathcal{O}_Y$ of rank 3. Then*

(a) $\mathcal{O}_X \cong \mathcal{O}_Y \oplus E$ *where $E$ is the free rank 2 $\mathcal{O}_Y$-submodule of $\mathcal{O}_X$*

*consisting of elements whose minimal cubic polynomial has no square term.*

(b) *The multiplication in* $\mathcal{O}_X$ *is determined by an* $\mathcal{O}_Y$*-linear map* $\phi: S^2E \to \mathcal{O}_Y \oplus E$. If $\{z, w\}$ *are a basis of* $E$ *over* $\mathcal{O}_Y$, *then* $\phi$ *is of the form*

$$\phi(z^2) = 2(a^2 - bd) + az + bw$$

$$\phi(zw) = -(ad - bc) - dz - aw$$

$$\phi(w^2) = 2(d^2 - ac) + cz + dw$$

*where* $a$, $b$, $c$ *and* $d$ *are in* $\mathcal{O}_Y$ *and* $bc \neq 0$. *The map* $\phi$ *is determined by its second coordinate* $\phi_2: S^2E \to E$.

2.7.2. *Conversely, given a free rank* 2 $\mathcal{O}_Y$*-module* $E$ *with basis* $\{z, w\}$, *let* $\phi_2: S^2E \to E$ *be of the form*

$$\phi_2(z^2) = az + bw$$

$$\phi_2(zw) = -dz - aw$$

$$\phi_2(w^2) = cz + dw$$

*for* $a$, $b$, $c$ *and* $d$ *in* $\mathcal{O}_Y$ *with* $bc \neq 0$. *Then the map* $\phi: S^2E \to \mathcal{O}_Y \oplus E$ *defined as in* (2.7.1(b)), *whose second coordinate is* $\phi_2$, *induces a commutative and associative* $\mathcal{O}_Y$*-algebra structure on* $\mathcal{O}_X = \mathcal{O}_Y \oplus E$ *making* $\mathcal{O}_X$ *into a flat and integral* $\mathcal{O}_Y$*-algebra of rank* 3, *for which* $E$ *is the Tschirnhausen module.*

*Proof.* The first part now follows by writing $g$, $h$ and $i$ in terms of $a$, $b$, $c$ and $d$ using Lemmas 2.4 and 2.6. The converse also follows from the lemmas; in fact, one need not have $bc \neq 0$ for the converse, but it follows from the arguments in the proof of Lemma 2.6 that if either $b$ or $c$ is zero, then $\mathcal{O}_X$ is not an integral domain. Q.E.D.

*Remarks.*

2.8.1. It does not seem easy to give a simple criterion in terms of $a$, $b$, $c$ and $d$ for $\mathcal{O}_X$ to be an integral domain. If $b \neq 0$, we may tensor with $K_Y$ and solve for $w$ in terms of $z$ using the equation for $\phi(z^2)$. Then $\mathcal{O}_X$ is an integral domain if and only if the minimal cubic polynomial

$$z^3 + 3(bd - a^2)z + (3abd - 2a^3 - b^2c)$$

is irreducible over $K_Y$. If $c = 0$, it is not; $z + a$ is a factor. However, there may be other values of $c$ for which it is also reducible.

2.8.2. If we think of $z$ and $w$ as variables over $\mathcal{O}_Y$ and let

$$F(z, w) = z^2 - az - bw - 2(a^2 - bd),$$

$$G(z, w) = zw + dz + aw + (ad - bc),$$

and

$$H(z, w) = w^2 - cz - dw - 2(d^2 - ac),$$

then $\mathcal{O}_X$ can be thought of as the quotient ring $\mathcal{O}_Y[z, w]/(F, G, H)$. If $X =$ Spec $\mathcal{O}_X$ and $Y =$ Spec $\mathcal{O}_Y$, then $X$ is embedded as a codimension 2 subvariety of $\mathbf{A}^2_Y$; since $\mathcal{O}_X$ is free over $\mathcal{O}_Y$, it is Cohen-Macauley over $\mathcal{O}_Y$ and therefore the embedding of $X$ into $\mathbf{A}^2_Y$ should be determinantal [4]. In our case this representation is easy to see; $F$, $G$ and $H$ are the 2 x 2 minors of

$$\begin{pmatrix} z + a & w - 2d & c \\ b & z - 2a & w + d \end{pmatrix}.$$

**3. The global analysis.** Let $f: X \to Y$ be a flat, finite map of degree 3 between irreducible $k$-schemes $X$ and $Y$. I would like to use the results of the previous section to understand and construct such maps from a more global viewpoint. For this purpose I need to eliminate, in the local case, the dependency of the description of the relevant maps $\phi$, $\phi_1$ and $\phi_2$ on the choice of basis for the Tschirnhausen module $E$.

*Definition* 3.1. Let $E$ be a locally free rank 2 $\mathcal{O}_Y$-module. A homomorphism $\phi_2: S^2E \to E$ is a *triple cover homomorphism* if it is locally of the form

$$\phi_2(z^2) = az + bw$$

$$\phi_2(zw) = -dz - aw \tag{3.2}$$

$$\phi_2(w^2) = cz + dw$$

for some (local) basis $\{z, w\}$ of $E$ over $\mathcal{O}_Y$. The set of triple cover homomorphisms will be denoted by $\mathrm{TCHom}(S^2E, E)$.

It is not difficult to check that the above local form for a map $\phi_2 : S^2E \to E$ is independent of the choice of basis and that $\mathrm{TCHom}(S^2E, E)$ is a linear submodule of $\mathrm{Hom}(S^2E, E)$.

PROPOSITION 3.3. *There is a natural isomorphism between* $\mathrm{TCHom}(S^2E, E)$ *and* $\mathrm{Hom}(S^3E, \Lambda^2E)$.

*Proof.* Assume $\phi_2 \in \mathrm{TCHom}(S^2E, E)$ is in the above form with respect to a basis $\{z, w\}$ of $E$. The induced map

$$S^2E \otimes E \xrightarrow{\phi_2 \otimes id} E \otimes E \longrightarrow \Lambda^2E$$

sends $z^2 \otimes z$ to $-b(z \wedge w)$, $z^2 \otimes w$ and $zw \otimes z$ to $a(z \wedge w)$, $zw \otimes w$ and $w^2 \otimes z$ to $-d(z \wedge w)$, and $w^2 \otimes w$ to $c(z \wedge w)$. Hence, it factors through the canonical map from $S^2E \otimes E$ to $S^3E$ (which identifies $z^2 \otimes w$ and $zw \otimes z$ as $z^2w$ and $zw \otimes w$ and $w^2 \otimes z$ as $zw^2$), and induces a map $\Phi : S^3E \to \Lambda^2E$ which is the image of $\phi_2$ under the isomorphism.

The inverse isomorphism is constructed as follows. We seek a natural element of

$$\mathrm{Hom}(\mathrm{Hom}(S^3E, \Lambda^2E), \mathrm{Hom}(S^2E, E))$$

which is an isomorphism onto $\mathrm{TCHom}(S^2E, E)$. Note that

$$\begin{aligned}\mathrm{Hom}(\mathrm{Hom}(S^3E, \Lambda^2E), \mathrm{Hom}(S^2E, E)) &\cong \mathrm{Hom}(S^3E^* \otimes \Lambda^2E, S^2E^* \otimes E)\\ &\cong S^3E \otimes \Lambda^2E^* \otimes S^2E^* \otimes E\\ &\cong \mathrm{Hom}(\Lambda^2E \otimes S^2E, E \otimes S^3E).\end{aligned}$$

The desired element here is the map sending $e_1 \wedge e_2 \otimes e_3e_4$ in $\Lambda^2E \otimes S^2E$ to $e_1 \otimes e_2e_3e_4 - e_2 \otimes e_1e_3e_4$, where $e_i \in E$. In this form it is apparent that the isomorphism does not depend upon a choice of basis for $E$ and is therefore natural. To check that this does map $\mathrm{Hom}(S^3E, E)$ isomorphically onto $\mathrm{TCHom}(S^2E, E)$ and is the inverse of the map above, choose a basis $\{z, w\}$ for $E$. Write $\phi_2$ as in (3.2). Then the induced map $\Phi : S^3E \to \Lambda^2E$ is

$$\begin{aligned}\Phi(z^3) &= -b(z \wedge w)\\ \Phi(z^2w) &= a(z \wedge w)\\ \Phi(zw^2) &= -d(z \wedge w)\\ \Phi(w^3) &= c(z \wedge w).\end{aligned} \tag{3.4}$$

Conversely, the natural transformation given above sends

$$(z \wedge w) \otimes z^2 \quad \text{to} \quad z \otimes z^2w - w \otimes z^3,$$

$$(z \wedge w) \otimes zw \quad \text{to} \quad z \otimes zw^2 - w \otimes z^2w,$$

and

$$(z \wedge w) \otimes w^2 \quad \text{to} \quad z \otimes w^3 - w \otimes zw^2$$

as an element in $\mathrm{Hom}(\Lambda^2E \otimes S^2E, E \otimes S^3E)$, and is therefore the element

$$\begin{aligned}&(z \wedge w)^* \otimes (z^2)^* \otimes (z \otimes z^2w - w \otimes z^3)\\ &\quad + (z \wedge w)^* \otimes (zw)^* \otimes (z \otimes zw^2 - w \otimes z^2w)\\ &\quad + (z \wedge w)^* \otimes (w^2)^* \otimes (z \otimes w^3 - w \otimes zw^2)\end{aligned}$$

in $\Lambda^2E^* \otimes S^2E^* \otimes E \otimes S^3E$, using the obvious notation for the dual bases. As an element in $\mathrm{Hom}(S^3E^* \otimes \Lambda^2E, S^2E^* \otimes E)$, it sends

$$(z^3)^* \otimes (z \wedge w) \quad \text{to} \quad -(z^2)^* \otimes w,$$

$$(z^2w)^* \otimes (z \wedge w) \quad \text{to} \quad (z^2)^* \otimes z - (zw)^* \otimes w,$$

$$(zw^2)^* \otimes (z \wedge w) \quad \text{to} \quad (zw)^* \otimes z - (w^2)^* \otimes w,$$

and

$$(w^3)^* \otimes (z \wedge w) \quad \text{to} \quad (w^2)^* \otimes z.$$

If $\Phi$ is in the form of (3.4), then it is the element $(-b(z^3)^* + a(z^2w)^* -$

$d(zw^2)^* + c(w^3)^*) \otimes z \wedge w$ in $S^3E^* \otimes \Lambda^2 E$, so the image of $\Phi$ under the natural transformation is $b(z^2)^* \otimes w + a((z^2)^* \otimes z - (zw)^* \otimes w) - d((zw)^* \otimes z - (w^2)^* \otimes w) + c(w^2)^* \otimes z$ in $S^2E^* \otimes E$. As an element in $\mathrm{Hom}(S^2E, E)$, it sends $z^2$ to $az + bw$, $zw$ to $-dz - aw$, and $w^2$ to $cz + dw$; therefore, it is the desired map $\phi_2$. Q.E.D.

PROPOSITION 3.5. *There is a natural transformation from* $\mathrm{TCHom}(S^2E, E)$ *to* $\mathrm{Hom}(S^2E, \mathcal{O}_Y)$ *which sends a map* $\phi_2$ *in the form of* (3.2) *to a map* $\phi_1$ *which has the form*

$$\phi_1(z^2) = 2(a^2 - bd)$$

$$\phi_1(zw) = -(ad - bc)$$

$$\phi_1(w^2) = 2(d^2 - ac),$$

*i.e.* $\phi_1$ *is the first coordinate of the multiplication map* $\phi$ *for* $\mathcal{O}_Y \oplus E$.

*Proof.* The map $\phi_2$ induces a map $\Lambda^2\phi_2 : \Lambda^2 S^2 E \to \Lambda^2 E$, or equivalently a map $\alpha : \Lambda^2 S^2 E \otimes \Lambda^2 E^* \to \mathcal{O}_Y$. The desired map $\phi_1$ is then the composition of $\alpha$ with a natural transformation in $\mathrm{Hom}(S^2E, \Lambda^2 S^2 E \otimes \Lambda^2 E^*) \cong \mathrm{Hom}(\Lambda^2 E \otimes S^2E, \Lambda^2 S^2 E)$. This transformation sends $(e_1 \wedge e_2) \otimes e_3 e_4$ to $-(e_1e_3 \wedge e_2e_4 + e_1e_4 \wedge e_2e_3)$. If $\phi_2$ is in the form (3.2), then $\Lambda^2\phi_2$ sends

$$z^2 \wedge zw \quad \text{to} \quad (-a^2 + bd) z \wedge w,$$

$$z^2 \wedge w^2 \quad \text{to} \quad (ad - bc) z \wedge w,$$

and

$$zw \wedge w^2 \quad \text{to} \quad (-d^2 + ac) z \wedge w;$$

therefore, $\alpha$ is the map sending

$$(z^2 \wedge zw) \otimes (z \wedge w)^* \quad \text{to} \quad bd - a^2,$$

$$(z^2 \wedge w^2) \otimes (z \wedge w)^* \quad \text{to} \quad ad - bc,$$

and

$$(zw \wedge w^2) \otimes (z \wedge w)^* \quad \text{to} \quad ac - d^2.$$

The natural transformation above sends

$$z^2 \quad \text{to} \quad -(z^2 \wedge zw + z^2 \wedge zw) \otimes (z \wedge w)^* = -2(z^2 \wedge zw) \otimes (z \wedge w)^*,$$

$$zw \quad \text{to} \quad -(z^2 \wedge w^2 + zw \wedge zw) \otimes (z \wedge w)^* = -(z^2 \wedge w^2) \otimes (z \wedge w)^*,$$

and

$$w^2 \quad \text{to} \quad -(zw \wedge w^2 + zw \wedge w^2) \otimes (z \wedge w)^* = -2(zw \wedge w^2) \otimes (z \wedge w)^*.$$

Hence the composite map with $\alpha$ sends

$$z^2 \quad \text{to} \quad 2(a^2 - bd),$$

$$zw \quad \text{to} \quad -(ad - bc),$$

and

$$w^2 \quad \text{to} \quad 2(d^2 - ac),$$

and is exactly the desired map $\phi_1 : S^2E \to \mathcal{O}_Y$. Q.E.D.

Because the relevant maps are now described without coordinates, using natural transformations, the local analysis of Section 2 'sheafifies' to give the following theorem.

THEOREM 3.6. *Let $f : X \to Y$ be a flat finite map of degree 3 between irreducible k-schemes. Then*

(a) *$f_*\mathcal{O}_X = \mathcal{O}_Y \oplus E$ where $E$ is the locally free rank 2 $\mathcal{O}_Y$-submodule of $\mathcal{O}_X$ consisting (locally) of elements whose minimal cubic polynomial has no square term. $E$ will be called the* Tschirnhausen module *for $f : X \to Y$.*

(b) *The multiplication in $\mathcal{O}_X$ is determined by a map $\phi : S^2E \to \mathcal{O}_Y \oplus E$ whose second coordinate $\phi_2$ is a triple cover homomorphism and whose first coordinate $\phi_1$ is the image of $\phi_2$ of the natural transformation from* $\mathrm{TCHom}(S^2E, E)$ *to* $\mathrm{Hom}(S^2E, \mathcal{O}_Y)$ *given in Proposition* 3.5.

(c) *The triple cover homomorphism $\phi_2$ determines a unique $\mathcal{O}_Y$-linear map $\Phi : S^3E \to \Lambda^2E$.*

(d) *Conversely, given a locally free rank 2 $\mathcal{O}_Y$-module $E$, any $\mathcal{O}_Y$-linear map $\Phi : S^3E \to \Lambda^2E$ determines a unique triple cover homomorphism $\phi_2$ via the natural isomorphism from* $\mathrm{Hom}(S^3E, \Lambda^2E)$ *to*

TCHom($S^2E$, $E$) *given in Proposition* 3.3. *If* $\phi_1$ *is the image of* $\phi_2$ *in* Hom($S^2E$, $\mathcal{O}_Y$), *then* $\phi = \phi_1 \oplus \phi_2 : S^2E \to \mathcal{O}_Y \oplus E$ *defines a commutative and associative multiplication on* $\mathcal{O}_Y \oplus E$ *for which* $E$ *is the Tschirnhausen module. If* $X = \mathrm{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus E)$, *then the canonical map* $f: X \to Y$ *is a flat finite map of degree* 3.

*Definition* 3.7. Fix the variety $Y$. A flat finite map $f: X \to Y$ of degree 3 will be called a *triple cover* of $Y$. A pair $(E, \Phi)$, where $E$ is a locally free rank 2 $\mathcal{O}_Y$-module and $\Phi : S^3E \to \Lambda^2E$ is an $\mathcal{O}_Y$-linear map, will be called *triple cover data* over $Y$. The map $\Phi$ will be called the *building map* and if $(E, \Phi)$ induces the triple cover $f: X \to Y$ as in (3.6(d)), I will say that $\Phi$ *builds* $f: X \to Y$, or *builds* $X$ on $Y$. Two pairs $(E_1, \Phi_1)$ and $(E_2, \Phi_2)$ are *isomorphic* as triple cover data if there exists an isomorphism $\alpha : E_1 \to E_2$ such that the diagram

$$\begin{array}{ccc} S^3E_1 & \xrightarrow{S^3\alpha} & S^3E_2 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_2} \\ \Lambda^2E_1 & \xrightarrow{\Lambda^2\alpha} & \Lambda^2E_2 \end{array}$$

commutes; isomorphic triple cover data corresponds to isomorphic triple covers, i.e. an isomorphism $g : X_1 \to X_2$ making the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & Y & \end{array}$$

commute (where $\Phi_1$ builds $X_1$ and $\Phi_2$ builds $X_2$).

It is clear from Theorem 3.6 that given a triple cover $f: X \to Y$ between irreducible $k$-schemes, one gets unique triple cover data $(E, \Phi)$ over $Y$ up to isomorphism. Conversely, given triple cover data $(E, \Phi)$, there is a triple cover $f: X \to Y$ which $\Phi$ builds.

*Remark.* I will sometimes abuse notation and regard the building map $\Phi$ as a section of $\Lambda^2E \otimes S^3E^*$.

Proposition 3.8. *If* $f: X \to Y$ *is a triple cover with Tschirnhausen module* $E$, *then* $X$ *naturally embeds into the geometric rank* 2 *vector bundle* $\mathbf{V}(E)$ *as a codimension two subvariety.*

*Proof.* If $S(E) = \oplus_{m \geq 0} S^m E$ is the symmetric algebra of $E$, then the multiplication in $f_* \mathcal{O}_X$ induces a surjection $S(E) \to \mathcal{O}_Y \oplus E = f_* \mathcal{O}_X$ (which is the identity on $\mathcal{O}_Y \oplus E \subset S(E)$). This surjection corresponds to the embedding $X = Spec_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus E) \hookrightarrow Spec_{\mathcal{O}_Y} S(E) = \mathbf{V}(E)$. Q.E.D.

It will become necessary to know the kernel of the map $\phi_1 : S^2E \to \mathcal{O}_Y$ in Section 10.

PROPOSITION 3.9. *Given the building map* $\Phi$, *there is a naturally induced* $\mathcal{O}_Y$*-linear map* $\psi : E \otimes \Lambda^2 E \to S^2 E$ *such that the sequence* $0 \to E \otimes \Lambda^2 E \xrightarrow{\psi} S^2 E \xrightarrow{\phi_1} \mathcal{O}_Y$ *is exact, unless* $\phi_1$ *is the zero map.*

*Proof.* We seek a natural element of

$$\mathrm{Hom}(\mathrm{Hom}(S^3E, \Lambda^2E), \mathrm{Hom}(E \otimes \Lambda^2E, S^2E))$$

which carries the building map $\Phi$ to the desired map $\psi$. Note that

$$\begin{aligned} \mathrm{Hom}(\mathrm{Hom}(S^3E, \Lambda^2E), \mathrm{Hom}(E \otimes \Lambda^2E, S^2E)) & \\ &\cong \mathrm{Hom}(S^3E^* \otimes \Lambda^2E, E^* \otimes \Lambda^2E^* \otimes S^2E) \\ &\cong S^3E \otimes \Lambda^2E^* \otimes E^* \otimes \Lambda^2E^* \otimes S^2E \\ &\cong \mathrm{Hom}(\Lambda^2E \otimes E \otimes \Lambda^2E, S^3E \otimes S^2E) \end{aligned}$$

via natural isomorphisms. Consider the element of this final Hom group which sends $(e_1 \wedge e_2) \otimes e_3 \otimes (e_4 \wedge e_5)$ to

$$e_1e_3e_4 \otimes e_2e_5 - e_1e_3e_5 \otimes e_2e_4 - e_2e_3e_4 \otimes e_1e_5 + e_2e_3e_5 \otimes e_1e_4.$$

I'll leave it to the reader to check that this element, considered as a transformation from $\mathrm{Hom}(S^3E, \Lambda^2E)$ to $\mathrm{Hom}(E \otimes \Lambda^2E, S^2E)$, sends a building map $\Phi$ in the local form of (3.4) to a map $\psi$ which is locally in the form

$$\psi(z \otimes (z \wedge w)) = -dz^2 - 2azw - bw^2$$

$$\psi(w \otimes (z \wedge w)) = cz^2 + 2dzw + aw^2.$$

This map $\psi$ is injective because the local functions $A = a^2 - bd$, $B = ad - bc$ and $C = d^2 - ac$ are not identically zero and they are the 2 x 2 minors of the local matrix for $\psi$. The image of $\psi$ is in the kernel of $\phi_1$ because of the identities $dA - aB + bC = cA - db + aC \equiv 0$. This proves the proposition. Q.E.D.

**4. The ramification and branch locus of a triple cover.** To determine the ramification for the triple cover $fX \to Y$ defined by the map $\phi : S^2E \to \mathcal{O}_Y \oplus E$, I will work locally at first. Let $\mathcal{O}_Y$ be a local integral $k$-algebra and let $E$ be a free $\mathcal{O}_Y$-module of rank 2, with basis $\{z, w\}$. Let $\phi : S^3E \to \Lambda^2 E$ build $X$, where $\Phi$ is given by (3.4). I will consider $X$ as embedded in $\mathbf{A}^2 \times Y$ as in 2.8.2. Let

(4.1)
$$A = a^2 - bd,$$
$$B = ad - bc,$$

and

$$C = d^2 - ac.$$

Then $X \subseteq \mathbf{A}^2 \times Y$ is defined by the three equations

(4.2)
$$F(z, w) = z^2 - az - bw - 2A,$$
$$G(z, w) = zw + dz + aw + B,$$

and

$$H(z, w) = w^2 - cz - dw - 2C.$$

Let

$$P(z, w) = az + bw + A,$$

(4.3)
$$Q(z, w) = dz + aw + \frac{1}{2}B,$$

and

$$R(z, w) = cz + dw + C.$$

LEMMA 4.4. *The ramification locus in $X$ is defined by the ideal* $(P, Q, R)$ in $\mathcal{O}_X$.

*Proof.* The ramification locus is the locus where the jacobian matrix for $F$, $G$ and $H$ with respect to the variables $z$ and $w$ does not have maximal rank. This matrix is

$$\begin{pmatrix} 2z - a & -b \\ w + d & z + a \\ -c & 2w - d \end{pmatrix},$$

and its 2 x 2 minors are

$$\begin{aligned}(2z - a)(z + a) + b(w + d) &= 2z^2 + az + bw - A \\ &= 2F + 3P, \\ (2z - a)(2w - d) - bc &= 4zw - 2dz - 2aw + B \\ &= 4G - 6Q,\end{aligned}$$

and

$$\begin{aligned}(2w - d)(w + d) + c(z + a) &= 2w^2 + cz + dw - C \\ &= 2H + 3R.\end{aligned}$$

Hence the ideal of the ramification locus is $(2F + 3P, 4G - 6Q, 2H + 3R) = (P, Q, R)$ in $\mathcal{O}_X$. Q.E.D.

Let

$$\begin{aligned} D &= \mathrm{B}^2 - 4AC \\ &= b^2c^2 - 3a^2d^2 + 4a^3c + 4bd^3 - 6abcd. \end{aligned}$$

LEMMA 4.5. *The branch locus in $Y$ of the triple cover is defined by* $D = 0$.

*Proof.* Let us work in the ambient space $\mathbf{A}^2 \times Y$. By the previous lemma, any point of ramification in $X$ must satisfy the linear equations $P = Q = R = 0$. These three equations in the two unknowns $z$ and $w$ have a solution in $\mathbf{A}^2$ if and only if the determinant

$$\begin{vmatrix} a & b & A \\ d & a & \frac{1}{2}B \\ c & d & C \end{vmatrix} = -\frac{1}{2}D$$

equals zero. Hence the branch locus is contained in the locus $D = 0$. An alternate proof of this is that because of the identities

$$cA - dB + aC = dA - aB + bC = 0,$$

one can write $D = -\frac{1}{2}(CP - BQ + AR)$, and so $D$ is in the ramification ideal $(P, Q, R)$.

To show that if $D = 0$, then there is ramification of the triple cover, I must show that there is a solution to $P = Q = R = 0$ which satisfy $F$, $G$ and $H$ and so lies in $X$. There are several cases to consider. The rank of the above determinant is

three if $D \neq 0$,

two if $D = 0$ but one of $A$, $B$ or $C \neq 0$,

one if $A = B = C = 0$ but one of $a$, $b$, $c$ or $d \neq 0$,

and

zero if $a = b = c = d = 0$.

*Case* 3. $D \neq 0$ at the closed point $y \in Y$. Then there is no ramification over $y$ and $f: X \to Y$ is étale at the three distinct points lying over $y$.

*Case* 2. $D = 0$ but one of $A, B, C \neq 0$ at $y \in Y$. Here, then, there is a unique solution $(z_0, w_0)$ to the system $P = Q = R = 0$. There are several subcases to consider.

*Case* 2A. $D = 0, A \neq 0$ at $y$. In this case $P = Q = 0$ define $(z_0, w_0)$ and therefore

$$z_0 = \frac{1}{A}\left(-aA + \frac{b}{2}B\right), \quad \text{and} \quad w_0 = \frac{1}{A}\left(dA - \frac{a}{2}B\right).$$

Note that since $A \neq 0$, the matrix $\begin{pmatrix} -a & b/2 \\ d & -a/2 \end{pmatrix}$ is nonsingular, so that if $z_0 = w_0 = 0$, necessarily $A = B = 0$, a contradiction. Hence $(z_0, w_0) \neq (0, 0)$.

One can easily verify that $(z_0, w_0)$ satisfy $F$, $G$ and $H$; in fact, we have the identities

$$A^2F = \left(Az - \left(-aA + \frac{b}{2}B\right)\right)(aP - bQ) - A^2P + \frac{b^2}{4}D,$$

$A^2G = (aP - bQ)(aQ - dP) + A^2Q - (ab/4)D + (-aA + (b/2)B)(aQ - dP) + (dA - (a/2)B)(aP - bQ)$, and

$$A^2H = \left(Aw + \left(dA - \frac{a}{2}B\right)\right)(aQ - dP) - A^2R + \frac{a^2}{4}D,$$

so that $F$, $G$ and $H$ vanish if $P$, $Q$, $R$ and $D$ do, and $A \neq 0$.

A calculation shows that $(z_1, w_1) = (-2z_0, -2w_0)$ also satisfy $F$, $G$ and $H$ in this case:

$$\begin{aligned} F(z_1, w_1) &= 4z_0^2 + 2az_0 + 2bw_0 - 2A \\ &= 4F(z_0, w_0) + 6P(z_0, w_0) = 0, \\ G(z_1, w_1) &= 4z_0w_0 - 2dz_0 - 2aw_0 + B \\ &= 4G(z_0, w_0) - 6Q(z_0, w_0) = 0, \end{aligned}$$

and

$$\begin{aligned} H(z_1, w_1) &= 4w_0^2 + 2cz_0 + 2dw_0 - 2C \\ &= 4H(z_0, w_0) + 6R(z_0, w_0) = 0. \end{aligned}$$

Since $(z_0, w_0) \neq (0, 0)$, $(z_1, w_1) \neq (z_0, w_0)$ and over $y$ there is ramification of type $(2, 1)$: $f: X \to Y$ is simply ramified at $(z_0, w_0)$ and is étale at $(z_1, w_1)$.

*Case* 2C. $D = 0$, $C \neq 0$. In this case $Q = R = 0$ define $(z_0, w_0)$, and

$$z_0 = \frac{1}{C}\left(-\frac{d}{2}B + aC\right), \qquad w_0 = \frac{1}{C}\left(\frac{c}{2}B - dC\right),$$

with $(z_0, w_0) \neq (0, 0)$. If $(z_1, w_1) = (-2z_1, -2w_1)$, then both $(z_0, w_0)$ and $(z_1, w_1)$ lie on $X$ and $f: X \to Y$ is simply ramified at $(z_0, w_0)$ and étale at $(z_1, w_1)$.

*Case* 2B. $D = 0, B \neq 0$. Since $D = B^2 - 4AC$, $AC \neq 0$ too so actually both of the above subcases hold. However, to be complete, $P = R = 0$ define $(z_0, w_0)$, and

$$z_0 = \frac{1}{B}(-dA + bC), \qquad w_0 = \frac{1}{B}(cA - aC),$$

and $(z_0, w_0) \neq (0, 0)$.

*Cases* 1, 0. $A = B = C = 0$ at $y$. In this case there is no longer a unique solution to $P = Q = R = 0$ and one can't expect all solutions to lie on $X$.

*Claim.* If $A = B = C = 0$ at $y$, the only point of $X$ lying over $y$ is $(z, w) = (0, 0)$, and $f: X \to Y$ is totally ramified there.

*Proof of the claim.* Under the hypotheses, $z$ and $w$ must satisfy

$$z^3 = (z + a)F + bG \quad \text{and} \quad w^3 = (w + d)H + cG,$$

so that if $F(z, w) = G(z, w) = H(z, w) = 0$, necessarily $z = w = 0$. The point $(0, 0)$ certainly satisfies $P = Q = R = 0$, so $f: X \to Y$ must be totally ramified here.

This completes the proof of Lemma 4.5. Q.E.D.

COROLLARY 4.6. *The locus in Y over which there is total ramification is defined by the ideal $(A, B, C)$ in $\mathcal{O}_Y$ and is generally codimension two in Y.*

For a more global viewpoint, there is the following:

PROPOSITION 4.7. *Let $f: X \to Y$ be a triple cover with Tschirnhausen module E. Then the branch locus $\{D = 0\}$ in Y is a divisor whose associated line bundle is $(\Lambda^2 E)^{-2}$.*

*Proof.* Consider the multiplication map $\phi: S^2E \to \mathcal{O}_Y \oplus E$. This induces $\Lambda^3\phi: \Lambda^3 S^3 E \to \Lambda^3(\mathcal{O}_Y \oplus E)$ which (locally) sends

$$z^2 \wedge zw \wedge w^2 \quad \text{to} \quad D(1 \wedge z \wedge w).$$

Hence $\{D = 0\}$ is the zero locus of this map between two line bundles. In particular we can consider $D$ as a section of $\Lambda^3 S^2 E^* \otimes \Lambda_3(\mathcal{O}_Y \oplus E)$. Since $E$ has rank 2, $\Lambda^3 S^2 E \cong (\Lambda^2 E)^3$ and $\Lambda^3(\mathcal{O}_Y \oplus E) \cong \Lambda^2 E$; the result follows. Q.E.D.

It is clear from the equation $D$ for the branch locus that even when $X$ and $Y$ are smooth, $\{D = 0\}$ may be singular. This contrasts with the case of double covers where, if $Y$ is smooth, then $X$ is smooth if and only if the branch locus is smooth. More precisely, we have:

LEMMA 4.8. *Assume $f: X \to Y$ is a triple cover and $X$ and $Y$ are both smooth. Then the branch locus $\{D = 0\}$ is singular at a point $p \in Y$ if and only if there is total ramification over $p$, i.e., if $A$, $B$ and $C$ vanish at $p$.*

*Proof.* If there is only simple ramification over $p$, then locally on $X$ the covering map is either étale or is analytically a double cover over $Y$. Therefore, since $X$ and $Y$ are smooth, the branch locus must be smooth. Conversely, if there is total ramification over $p$, then the local functions $A$, $B$ and $C$ must all vanish at $p$ by (4.6), so $D = B^2 - 4AC$ is at least double at $p$. Q.E.D.

Slightly more can be said about the precise nature of the singularities of the branch locus when both $X$ and $Y$ are smooth, especially in low dimensions. I will defer this until the next section, in which explicit criteria are given for the nonsingularity of $X$, given that $Y$ is smooth.

## 5. The singularities of a triple cover.

To analyze the singularities of the variety $X$, where $f: X \to Y$ is a triple cover, I will again work locally on $Y$ and in addition assume that $Y$ is *smooth* at a point $p$. Choose a system of local parameters $\underline{y} = \{y_1, \ldots, y_n\}$ at $p$.

LEMMA 5.1. *Assume that $X$ is defined by an equation of the form*

$$f(z) = z^3 + g(\underline{y})z + h(\underline{y})$$

*as a subset of $\mathbf{A}^1 \times Y$, where $g$ and $h$ are regular functions on $Y$ at $p$. Let $\underline{m}$ be the maximal ideal at $p$, and let $d(\underline{y}) = 4g(\underline{y})^3 + 27h(\underline{y})^2$ be the discriminant function for $f$. Then $X$ is singular over $p$ if and only if either*

(a) *$g \in \underline{m}$ and $h \in \underline{m}^2$, or*
(b) *$g \notin \underline{m}$ but $d \in \underline{m}^2$. (In this case necessarily $h \notin \underline{m}$.)*

*Proof.* Assume (a). Then the equation $f(z)$ defining $X$ is in the square of the maximal ideal $(z, \underline{m})$ in $\mathcal{O}_{\mathbf{A}^1} \otimes \mathcal{O}_Y$, so $X$ is singular at the point $z = 0$ over $p$.

Assume (b). Set $\bar{z} = z + 3h(\underline{y})/2g(\underline{y})$. Then changing coordinates in $\mathbf{A}^1$ to $\bar{z}$ transforms $f(z)$ to

$$\bar{f}(\bar{z}) = \bar{z}^3 - \frac{9h}{2g}\bar{z}^2 + \frac{d}{4g^2}\bar{z} - \frac{hd}{8g^3},$$

and so $\bar{f}$ is in the square of the maximal ideal $(\bar{z}, \underline{m})$ of $\mathcal{O}_{\mathbf{A}^1} \otimes \mathcal{O}_Y$. Therefore $X$ is singular at the point $z = -3h/2g$ over $p$.

For the converse, assume $X$ is singular at a point $q$ over $p$; then necessarily $d \in \underline{m}$. Let $\underline{m}_q$ be the maximal ideal of $\mathcal{O}_X$ at $q$.

Assume $g \in \underline{m}$. Since $d \in \underline{m}$, this forces $h \in \underline{m}$, and since $f \in \underline{m}_q^2$, $z$ must be in $\underline{m}_q$. Therefore $h = f - z^3 - gz$ is in $\underline{m}^2$ and we have (a).

Assume $g \notin \underline{m}$. Make the change of coordinates to $\bar{z} = z + 3h/2g$, transforming $f(z)$ to $\bar{f}(\bar{z})$ as above. Since $d \in \underline{m}$, $\bar{f}(\bar{z})$ has a double root at $\bar{z} = 0$ and a single root at $\bar{z} = 9h/2g$ over $p$. Therefore $q$ is the point $\bar{z} = 0$ over $p$, and $\bar{z} \in \underline{m}_q$. Since $\bar{f} \in \underline{m}_q^2$,

$$\frac{hd}{8g^3} = \bar{z}^3 - \frac{9h}{2g}\bar{z}^2 + \frac{d}{4g^2}\bar{z} - \bar{f}(\bar{z})$$

is in $\underline{m}^2$; since $h$ and $g$ are units at $p$, $d \in \underline{m}^2$ and we have (b). Q.E.D.

The above lemma is preliminary to the analysis in the general case.

PROPOSITION 5.2. *Assume $f: X \to Y$ is a triple cover, locally defined at $p \in Y$ as a subset of $\mathbf{A}^2 \in Y$ by the three equations $F$, $G$ and $H$ as in* (4.2). *Let $\underline{m}$ be the maximal ideal of $p$. Then $X$ is singular over $p$ if and only if one of the following conditions hold*:

(i) $a, b, c, d \in \underline{m}$
(ii) $a, c \in \underline{m}$, $b \in \underline{m}^2$
(iii) $b, d \in \underline{m}$, $c \in \underline{m}^2$
(iv) $b \notin \underline{m}$, $A \in \underline{m}$, $bB - 2aA \in \underline{m}^2$
(v) $c \notin \underline{m}$, $C \in \underline{m}$, $cB - 2dC \in \underline{m}^2$
(vi) $b \notin \underline{m}$, $A \notin \underline{m}$, $D \in \underline{m}^2$
(vii) $c \notin \underline{m}$, $C \notin \underline{m}$, $D \in \underline{m}^2$.

*Proof.* Assume $b \notin \underline{m}$. Then, using the equation $F$, one may solve for $w$ in terms of $z$ and $X$ will be defined by the equation

$$z^3 - 3Az + (bB - 2aA) = 0$$

(see Remark 2.8.1) as a subset of $\mathbf{A}^1 \times Y$. Therefore, by the previous lemma, $X$ is singular over $p$ if and only if either

$$-3A \in \underline{m} \quad \text{and} \quad bB - 2aA \in \underline{m}^2, \qquad \text{or} \quad -3A \notin \underline{m} \quad \text{and} \quad D \in \underline{m}^2,$$

since $4(-3A)^2 + 27(bB - 2aA)^3 = 27b^2D$. In the first case we have (iv) and in the second case, (vi).

If $c \notin \underline{m}$, by symmetry $X$ will be singular over $p$ if and only if (v) or (vii) hold. Therefore let us assume that $b$ and $c$ are both in $\underline{m}$. $X$ will be singular over $p$ if and only if the 3 x $(n + 2)$ jacobian matrix of $F$, $G$ and $H$ with respect to the coordinates $z, w, y_1, \ldots, y_m$ has rank one at some point $q$ over $p$. This matrix is

$$\begin{pmatrix} 2z - a & -b & (-2A_i - a_iz - b_iw) \\ w + d & z + a & (B_i + d_iz + a_iw) \\ -c & 2w - d & (-2C_i - c_iz - d_iw) \end{pmatrix}$$

where $a_i, b_i, \ldots$, etc., are the derivatives with respect to $y_i$.

Since $b$ and $c$ both vanish at $p$, the equations $F$, $G$ and $H$ can be easily solved for $z$ and $w$ over $p$; the three solutions are $(z, w) = (-a, -d)$, $(-a, 2d)$, and $(2a, -d)$. Since $D$ must vanish at $p$ for $X$ to be singular over $p$, and $D = -3a^2d^2$ modulo $\underline{m}$, we may assume either $a \in \underline{m}$ or $d \in \underline{m}$.

If $a \in \underline{m}$, then $z = 0$ over $p$ and $w = -d$ at the ramification point $q$ over $p$. The jacobian matrix at $q = (0, -d)$ over $p$ is now

$$\begin{pmatrix} 0 & 0 & (3db_i) \\ 0 & 0 & 0 \\ 0 & -3d & (3dd_i) \end{pmatrix}$$

modulo $\underline{m}$, so the vanishing of the 2 x 2 minors $9d^2b_i$ and $-9d^2(b_id_j - b_jd_i)$ is equivalent to the singularity of $X$ at $q$. Assume $d \in \underline{m}$; this gives condition (i). If $d \notin \underline{m}$, then $b_i \in \underline{m}$ for each $i$, so that $b \in \underline{m}^2$; this is (ii).

If $a \notin \underline{m}$, then $d \in \underline{m}$ and by symmetry we get condition (iii). Q.E.D.

The seven conditions of Proposition 5.2 may seem a bit daunting at first. However, there is a simple and surprising corollary.

COROLLARY 5.3. *Assume the dimension of Y is at least* 4. *Then the general triple cover of Y is singular.*

*Proof.* Let $\Phi$ be a building map for a triple cover $f:X \to Y$ and consider $\Phi$ as a section of the rank 4 bundle $S^3E^* \otimes \Lambda^2 E$. For general $E$, such a section will have a zero if dim $Y \geq 4$. Using the local description (3.4) for $\Phi$, one sees that a zero of $\Phi$ is precisely a point $p \in Y$ where $a$, $b$, $c$ and $d$ all vanish; by condition (i) of the proposition, $X$ will be singular over such a $p$. Q.E.D.

Note that there may well be special Tschirnhausen bundles $E$ for which the general building map $\Phi$ has no zeros, so no singularity of the triple cover is forced in this case. If this is so, then necessarily the top chern class $c_4$ of $S^3E^* \otimes \Lambda^2 E$ must vanish; a chern class computation shows that $c_4(S^3E^* \otimes \Lambda^2 E) = c_2(E)(9c_2(E) - 2c_1(E)^2)$.

COROLLARY 5.4. *Assume $f:X \to Y$ is a triple cover with Tschirnhausen module E. If X and Y are both smooth, then*

$$c_2(E)(9c_2(E) - 2c_1(E)^2) = 0$$

*in the Chow ring of Y.*

*Example* 5.5. Let $Y = \mathbf{P}^n$, and let $E = \mathcal{O}_{\mathbf{P}^n}(-1) \oplus \mathcal{O}_{\mathbf{P}^n}(-2)$. Identifying $A^2(Y)$ with $\mathbf{Z}$, we have $c_2(E) = 2$ and $c_1(E)^2 = 9$, so a triple cover with Tschirnhausen module $E$ could be smooth. In fact, as we will see later, such covers are the cubic hypersurfaces in $\mathbf{P}^{n+1}$ and the covering map $f$ is a projection from a point not on the cubic; certainly the general such cover is smooth.

We are now in a position to analyze the singularities of the branch locus $\{D = 0\}$ more closely. Let $p \in Y$ be a point over which there is total ramification of the cover $f:X \to Y$ and assume that $X$ is smooth at the point $q$ over $p$.

LEMMA 5.6. *With the above assumptions, the local functions b and c cannot both vanish at p.*

*Proof.* Assume $b$ and $c$ are both zero at $p$. Then, at $p$, $A = a^2$, $B = ad$, and $C = d^2$ all vanish by Corollary 4.6, so $a$ and $d$ must vanish at $p$. In this case $X$ is singular over $p$, by Proposition 5.2(i). Q.E.D.

Let us assume that $b$ is not zero at $p$; then, locally near $p$, we can solve for $w$ in terms of $z$ and $X$ will be defined by an equation of the form $z^3 + g(\underline{y})z + h(\underline{y}) = 0$ over a neighborhood of $p$, as in Lemma 5.1.

LEMMA 5.7. *If $\underline{m}$ is the maximal ideal at $p$, then the local functions $g$ and $h$ are both in $\underline{m}$ and $h \notin \underline{m}^2$.*

*Proof.* Since there is total ramification over $p$, the discriminant $d = 4y^3 + 27h^2 \in \underline{m}^2$; by Lemma 5.1(b), $g \in \underline{m}$ since $X$ is smooth. Therefore, $27h^2 = d - 4g^3 \in \underline{m}$, forcing $h \in \underline{m}$ also. However, $h \notin \underline{m}^2$ by Lemma 5.1(a). Q.E.D.

COROLLARY 5.8. *Assume that $f: X \to Y$ is a triple cover, $X$ and $Y$ are both smooth, and $p \in Y$ is a point over which there is total ramification.*

(i) *If the dimension of $X$ and $Y$ is one, then the branch locus $\{D = 0\}$ has an ordinary double point at $p$.*
(ii) *If the dimension of $X$ and $Y$ is two, then the branch locus $\{D = 0\}$ has a double point at $p$, with one tangent. Generally, $\{D = 0\}$ has an ordinary cusp at $p$.*

*Proof.* If $X$ is defined by an equation of the form $z^3 + gz + h = 0$, then the branch locus is defined by $d = 4g^3 + 27h^2 = 0$ at $p$. By Lemma 5.7, we may take $h$ to be a local parameter for $Y$ at $p$. If the dimension is one, then $h$ must divide $g$ so that $d = h^2(27 + 4g')$ defines an ordinary double point at $p$. If the dimension is two, then $d = 4g^3 + 27h^2$ is double at $p$ with the one tangent $h = 0$. Generically, $g$ will be another parameter at $p$, so the branch locus will have an ordinary cusp at $p$. Q.E.D.

In the case of surfaces, even though the branch locus $D$ is singular at a point over which there is total ramification, generically the ramification divisor $R$ and the residual divisor $R_0 = f^*D - 2R$ is smooth.

LEMMA 5.9. *Assume that $f: X \to Y$ is a triple cover, $X$ and $Y$ are both smooth surfaces, and $p \in Y$ is an isolated singular point of the branch locus which is an ordinary cusp. Then both the ramification divisor $R$ and the residual divisor $R_0$ in $X$ are smooth over $p$ and they are tangent there.*

*Proof.* We may assume as above that $X$ is defined by $z^3 + gz + h = 0$. If $d = 4g^3 + 27h^2 = 0$ defines an ordinary cusp at $p$, then $g$ and $h$ must form a system of parameters for $Y$ at $p$. The branch locus can then be parametrized near $p$ by $g = -3t^2$, $h = 2t^3$ for some parameter $t$. The inverse image of this branch locus is now described by the equation $z^3 - 3t^2z + 2t^3 = 0$ over $p$ which factors as $(z-t)^2(z+2t) = 0$. The ramification divisor is clearly the locus $z = t$ and the residual divisor is the locus $z = -2t$ (confirming the formulas determined in the proof of Lemma 4.5); both are smooth over $p$ and they meet transversally. Q.E.D.

**6. The split case.** In this section I will analyze the triple cover data in the case where the Tschirnhausen module $E$ is split. Assume that $E \cong L^{-1} \oplus M^{-1}$ where $L$ and $M$ are line bundles on $Y$. Then $S^3E \cong L^{-3} \oplus L^{-2}M^{-1} \oplus L^{-1}M^{-2} \oplus M^{-3}$ and $\Lambda^2 E \cong L^{-1} \otimes M^{-1}$, so that the building map $\Phi$ is a section of $S^3E^* \otimes \Lambda^2 E \cong (L^2 \otimes M^{-1}) \oplus L \oplus M \oplus (L^{-1} \otimes M^2)$. If we choose a local basis $z$ and $w$ for $E$ such that $z$ generates $L^{-1}$ and $w$ generates $M^{-1}$, then using the notation of (3.4) it follows that

$$a \in H^0(L)$$

$$b \in H^0(L^2 \otimes M^{-1}) \tag{6.1}$$

$$c \in H^0(L^{-1} \otimes M^2),$$

and

$$d \in H^0(M).$$

Moreover, if $X$ is to be irreducible neither $b$ nor $c$ can be identically zero; therefore,

$$L^2 \geq M \quad \text{and} \quad M^2 \geq L. \tag{6.2}$$

The branch locus $\{D = 0\}$ is in this case a divisor whose line bundle is $L^2 \otimes M^2$.

**7. The Galois case and normalization.** Let $f: X \to Y$ be a triple cover which is the quotient of a $\mu_3$ action on $X$. Such a cover will be called a *Galois triple cover*. Let $\zeta$ be a primitive cube root of unity, generating $\mu_3$.

PROPOSITION 7.1. *If $f: X \to Y$ is a Galois triple cover, then*:

(a) *$f_* \mathcal{O}_X$ splits into eigenspaces as $\mathcal{O}_Y \oplus L^{-1} \oplus M^{-1}$ where $\mathcal{O}_Y$, $L^{-1}$ and $M^{-1}$ are the eigenspaces for* 1, *$\zeta$ and $\zeta^2$, respectively.*

(b) *The Tschirnhausen module $E$ for $f$ is the sum of eigenspaces $L^{-1} \oplus M^{-1}$.*

(c) *The building map* $\Phi$ *for* $f$ *sends*

$$z^3 \quad to \quad -b(z \wedge w)$$

$$w^3 \quad to \quad c(z \wedge w),$$

*and*

$$z^2 w \quad and \quad zw^2 \text{ to } 0,$$

*where* $z$ *generates* $L^{-1}$ *and* $w$ *generates* $M^{-1}$ *(locally) and* $b$ *and* $c \in \mathcal{O}_Y$.

(d) *The multiplication map* $\phi\colon S^2E \to \mathcal{O}_Y \oplus E$ *has the form*

$$\phi(z^2) = bw$$

$$\phi(zw) = bc$$

$$\phi(w^2) = cz.$$

*Proof.* It is clear that $f_* \mathcal{O}_X$ must split into eigenspaces for the action of $\mu_3$. Since $f: X \to Y$ is the quotient space, the eigenspace for 1 is exactly $\mathcal{O}_Y$. Both other eigenvalues $\zeta$ and $\zeta^2$ must occur since if $x$ is in $\mathcal{O}_X$ and is in the eigenspace for $\zeta$, then $x^2$ is in the eigenspace for $\zeta^2$ and vice-versa. This proves (a).

To show (b), it suffices to prove that $E$ is preserved by the $\mu_3$ action; it must then be a sum of eigenspaces. Let $z$ be in $E$, i.e., assume that the minimal cubic polynomial for $z$ has the form $z^3 + rz + s$. Then the minimal cubic polynomial for $\bar{z} = \zeta z$ is $\bar{z}^3 + (\zeta^2 r)\bar{z} + s$, so $\bar{z}$ is in $E$ also.

For the rest, work locally on $Y$; choose $z \in L^{-1}$ and $w \in M^{-1}$ generating $E$. Then $z^2 \in M^{-1}$, $zw \in \mathcal{O}_Y$ and $w^2 \in L^{-1}$ since they are also eigenvectors. However, by Theorem 2.7.1(b),

$$z^2 = az + bw + 2A,$$

$$zw = -dz - aw - B,$$

and

$$w^2 = cz + dw + 2C,$$

with $a$, $b$, $c$ and $d \in \mathcal{O}_Y$, so that necessarily $a = 2A = -d = -a = d = 2C = 0$ or equivalently, $a = d = 0$. This proves (c) and (d). Q.E.D.

Note that the form of 7.1(c) for the building map $\Phi$ is exactly what is required to have $\Phi$ be compatible with the induced actions of $\mu_3$ on $S^3E$ and $\Lambda^2E$.

*Definition* 7.2. *Galois triple cover data* over $Y$ consists of a pair of line bundles $L$ and $M$ on $Y$ and two sections

$$b \in H^0(L^2 \otimes M^{-1}), \qquad c \in H^0(L^{-1} \otimes M^2).$$

By the above proposition and the analysis of Section 6, the giving of Galois triple cover data over $Y$ is equivalent to giving triple cover data over $Y$ which build a Galois triple cover.

The explicit local description of $X$ in this case is quite simple:

COROLLARY 7.3. *If $f: X \to Y$ is a Galois triple cover and $Y$ is affine, then* $X \cong \operatorname{Spec} \mathcal{O}_Y[z, w]/(z^2 - bw, zw - bc, w^2 - cz)$.

The analysis of singularities in the Galois case can be carried out more explicitly (and more simply) than in the general case.

PROPOSITION 7.4. *Let $\mathcal{O}_Y$ be an integral k-algebra and let $\mathcal{O}_X = \mathcal{O}_Y[z, w]/(z^2 - bw, zw - bc, w^2 - cz)$ for $b, c \in \mathcal{O}_Y$. Let $K_Y$ be the fraction field of $\mathcal{O}_Y$. Then*

(a) *$\mathcal{O}_X$ is an integral domain $\Leftrightarrow$ $b^2c$ is not a cube in $K_Y$ $\Leftrightarrow$ $bc^2$ is not a cube in $K_Y$.*

(b) *The discriminant $D = b^2c^2$ defines the branch locus on $Y$.*

(c) *Assume $b^2c$ is not a cube in $K_Y$ and that $\mathcal{O}_Y$ is a U.F.D. Let $\bar{D} = bc$. Then $X$ is normal $\Leftrightarrow$ $\bar{D}$ has no square factors.*

(d) *Assume $Y$ is smooth. Then $X$ is singular $\Leftrightarrow$ $\{\bar{D} = 0\}$ is singular (scheme-theoretically).*

*Proof.* Assume $\mathcal{O}_X$ is an integral domain and let $K_X$ be its fraction field. The minimal polynomial of $z$ over $K_Y$ is $z^3 - b^2c$; this must be irreducible, so $b^2c$ can't be a cube in $K_Y$.

If $b^2c$ is not a cube in $K_Y$, then $c \neq 0$ and $b^2c = (bc^2)^2c^{-3}$; therefore, $bc^2$ can't be a cube either.

To finish the proof of (a), assume $bc^2$ is not a cube. Then $c \neq 0$, and the minimal cubic polynomial $w^3 - bc^2$ is irreducible over $K_Y$. Therefore, $K_Y(w)$ is a field and since $c \neq 0$, $z = c^{-1}w^2$ and $w$ are both in $K_Y(w)$, so $\mathcal{O}_X \subseteq K_Y(w)$. Therefore, $\mathcal{O}_X$ is an integral domain.

Statement (b) is merely the definition of $D$ given in Section 4. To prove (c), assume that $bc$ has a square factor $x^2$ where $x$ is irreducible and is not a unit in $\mathcal{O}_Y$. Since $\mathcal{O}_Y$ is a U.F.D., there are three cases to consider.

*Case* 1. $x^2$ divides $b$. Then $(x^{-1}z)^2 = (x^{-2}b)w$ in $K_X$, so $x^{-1}z$ is in the normalization $\tilde{\mathcal{O}}_X$ of $\mathcal{O}_X$ but is not in $\mathcal{O}_X$.

*Case* 2. $x^2$ divides $c$. Then, as in Case 1, $(x^{-1}w)^2 = (x^{-2}c)z$, so $x^{-1}w$ is in $\tilde{\mathcal{O}}_X - \mathcal{O}_X$.

*Case* 3. $x$ divides both $b$ and $c$. Then $(x^{-1}z)^3 = (x^{-1}b)^2(x^{-1}c)$ and $(x^{-1}w)^3 = (x^{-1}b)(x^{-1}c)^2$, so both $x^{-1}z$ and $x^{-1}w$ are in $\tilde{\mathcal{O}}_X - \mathcal{O}_X$.

For the converse, assume that $p$ is in $\tilde{\mathcal{O}}_X - \mathcal{O}_X$. The action of $\mu_3$ extends to $\tilde{\mathcal{O}}_X$, which then splits into eigenspaces as $\mathcal{O}_Y \oplus \widetilde{L^{-1}} \oplus \widetilde{M^{-1}}$. Writing $p = r + s\tilde{z} + t\tilde{w}$ (where $\tilde{z}$ and $\tilde{w}$ generate $\widetilde{L^{-1}}$ and $\widetilde{M^{-1}}$, respectively), it is clear that one of $\tilde{z}$ or $\tilde{w}$ must be in $\tilde{\mathcal{O}}_X - \mathcal{O}_X$. Hence we may assume that $p = \tilde{z}$. Since $\tilde{z} \in K_X$, it must be a $K_Y$-linear combination of $w^{-1}$, $z$ and $z^{-1}w$. Write $\tilde{z} = \tilde{r}w^{-1} + \tilde{s}z + \tilde{t}z^{-1}w$, with $\tilde{r}$, $\tilde{s}$ and $\tilde{t}$ in $K_Y$. Then $\tilde{z} = (\tilde{r}z + \tilde{s}z^2w + \tilde{t}w^2)/zw = z \cdot (\tilde{r} + \tilde{s}bc + \tilde{t}c)/bc$ so in fact $\tilde{z} \in K_Y \cdot z$. Write $\tilde{z} = (x^{-1}y)z$ with $x$, $y$ in $\mathcal{O}_Y$. If $x^{-1}z$ is in $\mathcal{O}_X$ then so is $\tilde{z}$, so we may assume $\tilde{z} = x^{-1}z$. The minimal cubic polynomial for $\tilde{z}$ is of the form $\tilde{z}^3 - f$ for some $f \in \mathcal{O}_Y$ since $\tilde{z}$ is in the eigenspace for $\zeta$. Therefore, $b^2c = z^3 = x^3(x^{-1}z)^3 = x^3 f$, so $x^3$ divides $b^2c$, implying $x_1^2$ divides $bc$ where $x_1$ is a prime factor of $x$.

The proof of (d) is a jacobian calculation. Let $\{y_1, \ldots, y_n\}$ be local parameters at a point $p \in Y$. Then $X$ is singular at a point $q$ over $p$ if and only if the jacobian matrix

$$\begin{pmatrix} 2z & -b & (-b_i w) \\ w & z & (-b_i c - bc_i) \\ -c & 2w & (-c_i z) \end{pmatrix}$$

has rank 1 at $q$ where $b_i$ and $c_i$ are the derivatives of $b$ and $c$ with respect to $y_i$. Since $D$ must vanish at a singular point, either $b$ or $c$ must be zero; we may assume by symmetry that $b = 0$ at $p$. This forces $z = w = 0$ over $p$, so the top row of the above matrix is zero over $p$. Therefore, the matrix has rank 1 if and only if either $c = 0$ or $(bc)_i = 0$ for every $i$ at $p$. In our situation, if $c = 0$, then $(bc)_i = 0$ for every $i$ automatically and the result follows.

An alternate proof is provided by using the conditions of Proposition 5.2. In this case $a = d = A = C = 0$, and $B = -bc = -\bar{D}$, so that the seven conditions of 5.2 reduce to the following three:

(i) $b, c \in \underline{m}$
(ii) $b \notin \underline{m}$, $c \in \underline{m}^2$
(iii) $c \notin \underline{m}$, $b \in \underline{m}^2$.

It is easy to see that these are equivalent to the single condition $bc \in \underline{m}^2$ which is (d). Q.E.D.

The proof of (7.4c) actually indicates the normalization process, as follows.

PROPOSITION 7.5. *Let $Y$ be a factorial variety over $k$ and let line bundles $L$, $M$ and sections $b \in H^0(L^2 \otimes M^{-1})$, $c \in H^0(L^{-1} \otimes M^2)$ define Galois triple cover data over $Y$ which builds the irreducible cover $f: X \to Y$. Let $\bar{D}_b$ and $\bar{D}_c$ be the divisors of zeros of $b$ and $c$, respectively, so that $\bar{D}_b + \bar{D}_c$ is the divisor of $bc = \bar{D}$. Define $L_b$ and $M_c$ to be the largest effective divisors on $Y$ such that $2L_b \leq \bar{D}_b$ and $2M_c \leq \bar{D}_c$; define $N$ to be the largest effective divisor on $Y$ such that $2N \leq \bar{D}_b + \bar{D}_c - 2L_b - 2M_c$. Writing $L_b$, $M_c$ and $N$ for the line bundles associated to these divisors also, let $f$, $g$ and $h$ be the sections of $L_b$, $M_c$ and $N$ defining these divisors. Then the Galois triple cover data given by the line bundles*

$$\tilde{L} = L \otimes L_b^{-1} \otimes N^{-1}, \qquad \tilde{M} = M \otimes M_c^{-1} \otimes N^{-1},$$

*and the sections*

$$\tilde{b} = f^{-2}gh^{-1}b \in H^0(\tilde{L}^2 \otimes \tilde{M}^{-1}), \qquad \tilde{c} = fg^{-2}h^{-1}c \in H^0(\tilde{L}^{-1} \otimes \tilde{M}^2)$$

*builds the normalization $\tilde{X}$ of $X$.*

*Proof.* Locally, let $z$ and $w$ generate $L^{-1}$ and $M^{-1}$, respectively. Then $\tilde{z} = f^{-1}h^{-1}z$ and $\tilde{w} = g^{-1}h^{-1}w$ generates $\tilde{L}^{-1}$ and $\tilde{M}^{-1}$. Moreover,

$$\tilde{z}^2 = f^{-2}h^{-2}z^2 = (f^{-2}gh^{-1}b)(g^{-1}h^{-1}w) = \tilde{b}\tilde{w},$$

$$\tilde{z}\tilde{w} = f^{-1}g^{-1}h^2zw = f^{-1}g^{-1}h^{-2}bc = \tilde{b}\tilde{c},$$

and

$$\tilde{w}^2 = g^{-2}h^{-2}w^2 = (fg^{-2}h^{-1}c)(f^{-1}h^{-1}z) = \tilde{c}\tilde{z},$$

so that the above Galois triple cover data builds a triple cover $\tilde{X}$ to which the normalization of $X$ maps. However, by the definition of $L_b$, $M_c$ and $N$, the function $\tilde{b}\tilde{c}$ has no square factors locally, so this triple cover is normal by (7.4c). Hence $\tilde{X}$ is the normalization of $X$. Q.E.D.

*Remark* 7.6. In the special case when $M = L^2$, one gets the more familiar construction of a Galois triple cover. In this case $b \in H^0(\mathcal{O}_Y)$, so if $Y$ is complete, $b$ must be constant. We can normalize $b$ to be 1 and solve for $w$; then locally $\mathcal{O}_X$ is generated by 1, $z$ and $z^2 = w$ with the relation $z^3 = c \in H^0(L^3)$. Therefore, $f: X \to Y$ is completely specified by a line bundle $L$ on $Y$ and a section $c$ of $L^3$.

If the triple cover $f: X \to Y$ is Galois and *étale*, and $Y$ is complete, then it must be of the above form. For then both $b$ and $c$ are nowhere zero so both $L^2 \otimes M^{-1}$ and $L^{-1} \otimes M^2$ are trivial; this forces $M = L^2$ and $L^3 = \mathcal{O}_Y$.

**8. Trisections of ruled varieties.** Let $Y$ be an irreducible $k$-variety and let $F$ be a locally free rank 2 $\mathcal{O}_Y$-module. Let $\mathbf{P} = \mathbf{P}(F)$ be the associated projective line bundle with $\pi: \mathbf{P} \to Y$ as structure map. Let $S$ be a divisor in $\mathbf{P}$ whose line bundle is $\mathcal{O}_{\mathbf{P}}(1)$.

Assume that $X \subseteq \mathbf{P}$ is a trisection of $\pi$ and that $f = \pi|_X: X \to Y$ is flat so that $f$ is a triple covering. As a divisor in $\mathbf{P}$, $X$ is linearly equivalent to $3S + \pi^*T$ for some divisor $T$ on $Y$.

PROPOSITION 8.1. *In the above situation, the Tschirnhausen module for $E$ is $(F \otimes \Lambda^2 F \otimes \mathcal{O}_Y(T))^*$.*

*Proof.* The exact sequence which defines $\mathcal{O}_X$ is

$$0 \to \mathcal{O}_{\mathbf{P}}(-3) \otimes \pi^*\mathcal{O}_Y(-T) \to \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_X \to 0.$$

Since $\pi_*(\mathcal{O}_{\mathbf{P}}(-3) \otimes \pi^*\mathcal{O}_Y(-T)) = 0$ and $\pi_*\mathcal{O}_{\mathbf{P}} = \mathcal{O}_Y$, one gets

$$0 \to \mathcal{O}_Y \to \pi_*\mathcal{O}_X \to (R^1\pi_*\mathcal{O}_{\mathbf{P}}(-3)) \otimes \mathcal{O}_Y(-T) \to 0$$

after applying $\pi_*$ and using the projection formula. Therefore, $E \cong (R^1\pi_*\mathcal{O}_{\mathbf{P}}(-3)) \otimes \mathcal{O}_Y(-T)$ which is $(F \otimes \Lambda^2 F)^* \otimes \mathcal{O}_Y(-T)$ by duality. Q.E.D.

Note that if $S$ is an effective irreducible divisor (which can always be achieved by tensoring $F$ with a line bundle), then the restriction of $\pi$ to $S$ will be an isomorphism. In particular, $\pi_*(S \cdot \pi^* T) = T$ so that

$$\mathcal{O}_Y(T) = \mathcal{O}_Y(\pi_*(X \cdot S)) \otimes (\Lambda^2 F)^{-3} \tag{8.2}$$

since $\pi_*(S \cdot S) \cong \Lambda^2 F$. Therefore, the Tschirnhausen module can also be realized as

$$E \cong (F \otimes (\Lambda^2 F)^{-2} \otimes \mathcal{O}_Y(X))^* \tag{8.3}$$

where $\mathcal{O}_Y(X)$ is the line bundle of $\pi_*(X \cdot S)$.

COROLLARY 8.4. *Assume that $X$ is a trisection of an affine line bundle $\mathbf{V}(L^{-1})$ for some $L \in \operatorname{Pic} Y$ and that the structure map $f: X \to Y$ is a triple cover. Then the Tschirnhausen module for $X$ is $L^{-1} \oplus L^{-2}$.*

*Proof.* By embedding $\mathbf{V}(L^{-1})$ into $\mathbf{P} = \mathbf{P}(\mathcal{O}_Y \oplus L^{-1})$, we can consider $X$ as a trisection of $\mathbf{P}$ which does not meet the section $S$ at infinity. Hence, $\mathcal{O}_Y(X) \cong \mathcal{O}_Y$ and the result follows from (8.3). Q.E.D.

In the above situation, if $Y$ is complete, then the section $b \in H^0(L^2 \otimes L^{-2}) = H^0(\mathcal{O}_Y)$ must be a constant; therefore, one can globally solve for $w$ in terms of $z$. Hence $X$ is given, as a subvariety of $\mathbf{V}(L^{-1})$, by a single equation of the form $z^3 + rz + s = 0$ where $z$ is the global coordinate in the fibers of $\mathbf{V}(L^{-1})$.

COROLLARY 8.5. *Let $X \subseteq \mathbf{P}^{N+1}$ be a hypersurface of degree $d \geq 3$. Let $p \in \mathbf{P}^N$ be a point of multiplicity $d - 3$ on $X$. (If $d = 3$, $p$ is not on $X$.) Then projection from $p$ gives a triple cover $f: \tilde{X} \to \mathbf{P}^N$ (where $\tilde{X}$ is the proper transform of $X$ in the blow-up of $\mathbf{P}^{N+1}$ at $p$) and the Tschirnhausen module for $f$ is*

$$E \cong \mathcal{O}_{\mathbf{P}^N}(2 - d) \oplus \mathcal{O}_{\mathbf{P}^N}(1 - d).$$

*Proof.* Let $\widetilde{\mathbf{P}^{N+1}}$ be the blow-up of $\mathbf{P}^{N+1}$ at $p$; then $\widetilde{\mathbf{P}^{N+1}}$ is naturally the $\mathbf{P}^1$-bundle $\mathbf{P} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^N} \oplus \mathcal{O}_{\mathbf{P}^N}(-1))$ over $\mathbf{P}^N$ in which $\tilde{X}$ sits as a trisection. The intersection of $\tilde{X}$ with the canonical section $S$ in $\mathcal{O}_{\mathbf{P}}(1)$ is a hypersurface (in $S \cong \mathbf{P}^N$) of degree $d - 3$. Therefore, by (8.3), $E \cong [[\mathcal{O}_{\mathbf{P}^N} \oplus \mathcal{O}_{\mathbf{P}^N}(-1)] \otimes \mathcal{O}_{\mathbf{P}^N}(2) \otimes \mathcal{O}_{\mathbf{P}^N}(d - 3)]^*$, which gives the result. Q.E.D.

**9. Triple covers of curves and trigonal curves.** In this section I will restrict myself to triple covers $f: X \to Y$ between smooth curves. If $E$ is the Tschirnhausen module for $f$, then the degree of the ramification divisor on $X$ is the degree of the branch locus on $Y$, and therefore it is the degree of $(\Lambda^2 E)^{-2}$. Hence, by the Hurwitz formula,

$$2g(X) - 2 = 3(2g(Y) - 2) + \text{degree}(\Lambda^2 E),$$

or

$$g(X) = 3g(Y) - 2 + \text{degree}(\Lambda^2 E)^{-1}, \tag{9.1}$$

where $g(X)$ and $g(Y)$ are the genera of $X$ and $Y$, respectively.

Let us specialize to the case of *trigonal curves*, i.e., triple covers of $\mathbf{P}^1$. In this case the Tschirnhausen module $E$ must split as $E \cong \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n)$ (i.e., $E$ is of type $(m, n)$), so the analysis of Section 6 applies with $L = \mathcal{O}_{\mathbf{P}^1}(m)$ and $M = \mathcal{O}_{\mathbf{P}^1}(n)$. By (6.1), the local functions $a$, $b$, $c$ and $d$ are forms of degrees $n$, $2n - m$, $2m - n$ and $m$, respectively; moreover, $n \le 2m$ and $m \le 2n$ by (6.2) and the branch locus is of degree $2(m + n)$. This forces $m \ge 0$ and $n \ge 0$ with $m = 0 \Leftrightarrow n = 0$. However, if $m = n = 0$, then the local functions $a$, $b$, $c$ and $d$ must be constants and the cover $X$ is pulled back from a triple cover of Spec $k$, so cannot be irreducible. Therefore,

$$m > 0, \qquad n > 0, \qquad m \le 2n, \qquad \text{and} \quad n \le 2m. \tag{9.2}$$

Since $\Lambda^2 E = \mathcal{O}_{\mathbf{P}^1}(-m - n)$, the genus of the cover $X$ is

$$g(X) = m + n - 2. \tag{9.3}$$

It is a result of Petri that all trigonal curves lie on scrolls [3]. Assume that $X$ is trigonal and that $X$ is a trisection in $\mathbf{F}_k = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-k))$, with $k \ge 0$. Write $X \sim 3S + \pi^* T$ as in Section 8, where $T$ is a divisor on $\mathbf{P}^1$ of degree $t$ and assume that the Tschirnhausen module for $X$ is $\mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n)$, with $m \le n$. Then, by (8.1),

$$\begin{aligned}\mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(n) &\cong (\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-k)) \otimes \mathcal{O}_{\mathbf{P}^1}(-k) \otimes \mathcal{O}_{\mathbf{P}^1}(t) \\ &\cong \mathcal{O}_{\mathbf{P}^1}(t - 2k) \oplus \mathcal{O}_{\mathbf{P}^1}(t - k),\end{aligned}$$

so that

$$m = t - 2k, \qquad n = t - k, \qquad \text{and} \quad k = n - m, \qquad t = 2m - n.$$

The conditions (9.2) on $m$ and $n$ are equivalent to

(9.4) $\quad$ either $\; k = 0 \;$ and $\; t \geq 1, \quad$ or $\; k \geq 1, \quad t \geq 3k.$

These conditions are, of course, exactly the conditions for which an irreducible curve in the class of $3S + \pi^*T$ exists in $\mathbf{F}_k$.

The number of moduli for trigonal curves with $E \cong \mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n)$ can be computed as $\dim H^0(S^3E^* \otimes \Lambda^2 E) - \dim \mathrm{Aut}_{\mathbf{P}^1}(E) - \dim \mathrm{Aut}_k(\mathbf{P}^1)$, considering the building map $\Phi$ as a section of $S^3E^* \otimes \Lambda^2 E$. Since $S^3E^* \otimes \Lambda^2 E \cong \mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(2m - n) \oplus \mathcal{O}_{\mathbf{P}^1}(2n - m) \oplus \mathcal{O}_{\mathbf{P}^1}(n)$, $\dim \mathrm{Aut}_{\mathbf{P}^1}(E) = 4$ (if $m = n$) or $n - m + 3$ (if $m < n$), and $\dim \mathrm{Aut}_k(\mathbf{P}^1) = 3$, we have

(9.5) the number of moduli of trigonal curves with $E$ of type $(m, n)$

$$= \begin{cases} 2(m + n) - 3 & \text{if} \quad m = n \\ 3m + n - 2 & \text{if} \quad m < n. \end{cases}$$

For fixed genus $g$, the bundles $E$ with fixed $m + n$ which are most general are those with minimum $n - m$. Hence, for even genus $g = 2p$, the most general trigonal curve of genus $g$ has $m = n = p + 1$; for odd genus $g = 2p + 1$, the general curve has $m = p + 1$, $n = p + 2$. Using (9.5), the number of moduli for the general trigonal curve of genus $g$ is equal to $2g + 1$ in either case.

Of course, this is the moduli of the covering map $f: X \to \mathbf{P}^1$, not just the curve $X$. Even for $X = \mathbf{P}^1$ there is one modulus for this problem. For low values of $g$, the moduli can be seen quite geometrically.

If $g = 0$, the map $f: \mathbf{P}^1 \to \mathbf{P}^1$ will have 4 branch points; the cross-ratio of these 4 points is a function of the modulus.

For $g = 1$, every elliptic curve $X$ should be trigonal in dimension 2 ways since the moduli space for the curves themselves has dimension 1. To see this more geometrically, let $N$ be the line bundle associated to the triple cover $f: X \to \mathbf{P}^1$. The degree of $N$ is three and since all degree 3 line bundles on $X$ are equivalent under the action of internal translation of $C$, there is no moduli for $N$. The map $f$ is obtained by using the sections of $N$ to map $X$ to $\mathbf{P}^2$ and then projecting $X$ from some point $p \in \mathbf{P}^2$ to a line. It is the point $p$ which has two moduli.

If $2 \leq g \leq 4$, then $2g + 1 \geq 3g - 3$, so the general curve of genus $g$ should be trigonal in dimension $(2g + 1) - (3g - 3) = 4 - g$ ways.

If $g = 2$, any divisor on $X$ of degree 3 maps $X$ to $\mathbf{P}^1$ and the 2 moduli come from the jacobian variety of $X$ which parametrizes line bundles of degree 3.

If $g = 3$, the general curve $X$ should be trigonal in dimension 1 ways. If $X$ is nonhyperelliptic, these ways are parametrized by the points of $X$; given a point $p \in X$, embed $X$ into $\mathbf{P}^2$ canonically as a quartic and project from $p$ to a line.

If $g = 4$, one expects only a finite number of trigonal structures on the general curve $X$. As is well known [1], there are two: the canonical model of $X$ sits in $\mathbf{P}^3$ as a curve of type (3, 3) on a quadric $Q$, and the two rulings of $Q$ give two $g_3^1$'s on $X$.

If $g \geq 5$, the general curve of genus $g$ is not trigonal. However, the following classical result should be clear.

PROPOSITION 9.6. *The moduli space of trigonal curves of genus g is connected and unirational.*

*Proof.* An open set of this locus is covered by an open set of $\mathbf{P}(H^0(S^3E^* \otimes \Lambda^2 E))$, where $E$ is of type $(m, n)$ and $m$ and $n$ are chosen to minimize $|n - m|$ subject to $m + n - 2 = g$. Q.E.D.

Note that in the case $g = 1$, the cover $f: X \to \mathbf{P}^1$ can be the quotient of a $\mu_3$ action if and only if $j(X) = 0$, where $j$ is the elliptic modulus. In such a case, if $f$ is represented as the projection of the cubic curve $X$ from a point $p$ not on $X$, then the three ramification points of $f$ must be flexes of $X$ and the three flexed lines are concurrent at $p$. Hence, we have recovered the following well-known proposition.

PROPOSITION 9.7. *Three flexed lines to a smooth cubic $X \subseteq \mathbf{P}^2$ are concurrent if and only if $X$ is the Fermat cubic.*

**10. Triple covers of surfaces.** Let $f: X \to Y$ be a triple cover. In this section I will assume that $X$ and $Y$ are smooth complete surfaces. Moreover, I will restrict the discussion to *general* triple covers, by which I mean that $f$ has no total ramification in codimension one and that the only singular points of the branch locus $D$ are ordinary cusps. The goal is to compute the standard invariants of $X$ in terms of those of $Y$ and the Tschirnhausen module $E$. Let $c_i$ denote the $i^{\text{th}}$ chern class of $E$.

LEMMA 10.1. *The number of cusps of the branch locus D is* 3 *degree* $c_2(E)$.

*Proof.* Let $I$ be the ideal sheaf of the points of $Y$ over which there is total ramification. By Corollary 4.6, $I$ is locally generated by $A$, $B$ and $C$ and is exactly the image of $\phi_1 : S^2E \to \mathcal{O}_Y$. Therefore, by Proposition 3.9 we have an exact sequence

$$0 \to E \otimes \Lambda^2 E \to S^2 E \to I \to 0,$$

so the chern polynomial $c_t(I)$ is the quotient $c_t(S^2E)/c_t(E \otimes \Lambda^2 E)$ in the Chow ring of $Y$. A computation shows that

$$c_t(S^2E) = 1 + 3c_1 t + (2c_1^2 + 4c_2)t^2$$

and

$$c_t(E \otimes \Lambda^2 E) = 1 + 3c_1 t + (2c_1^2 + c_2)t^2.$$

Long division gives $c_t(I) = 1 + 3c_2 t^2$; since the number of cusps of $D$ is the degree of $c_2(I)$, the result follows. Q.E.D.

Let $R$ be the ramification divisor and $R_0 = f^*D - 2R$ be the residual divisor on $X$. By Lemma 5.9, they are both smooth and therefore they are both isomorphic to the normalization of $D$. In the following, I will abuse notation and consider elements of the codimension two-piece Chow ring of $Y$ as integers via the degree map.

LEMMA 10.2. *The genus of R is* $2c_1^2 - c_1 K_Y + 1 - 3c_2$.

*Proof.* The arithmetic genus of the branch locus $D$ is

$$p_a(D) = \frac{1}{2}(D^2 + DK_Y) + 1$$

by Riemann-Roch; since $D = -2c_1$ by Proposition 4.7, and since the genus of $R$ differs from the arithmetic genus of $D$ by the number of cusps, the result follows from the previous lemma. Q.E.D.

I will use the standard notation for invariants of curves and surfaces in what follows.

PROPOSITION 10.3. *The following formulas compute the standard invariants of X.*

(i) $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_Y) + h^i(E)$ *for* $i \geq 0$.
(ii) $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \chi(E) = 3\chi(\mathcal{O}_Y) + (1/2)c_1^2 - (1/2)c_1K_Y - c_2$.
(iii) $K_X^2 = 3K_Y^2 - 4c_1K_Y + 2c_1^2 - 3c_2$.
(iv) $e(X) = 3e(Y) - 2c_1K_Y + 4c_1^2 - 9c_2$.

*Proof.* The first statement follows from the splitting $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus E$ and the finiteness of $f$. Statement (ii) is Riemann-Roch for the bundle $E$ which calculates $\chi(E)$. By decomposing $X$ into strata, one sees that

$$e(X) = e(X - f^{-1}(D)) + e(f^{-1}(D) - R) + e(R).$$

Since $f: X - f^{-1}(D) \to Y - D$ is an unbranched cover,

$$e(X - f^{-1}(D)) = 3e(Y - D) = 3(e(Y) - e(D)).$$

The stratum $f^{-1}(D) - R$ is the residual divisor $R_0$ minus the $3c_2$ points of total ramification; hence, $e(f^{-1}(D) - R) = e(R_0) - 3c_2 = e(D) - 3c_2$. In addition, $e(R) = e(D) = 2 - 2(2c_1^2 - c_1K_Y + 1 - 3c_2)$; putting this together yields

$$\begin{aligned} e(X) &= 3e(Y) - e(D) - 3c_2 \\ &= 3e(Y) - 2 + 4c_1^2 - 2c_1K_Y + 2 - 6c_2 - 3c_2 \\ &= 3e(Y) - 2c_1K_Y + 4c_1^2 - 9c_2. \end{aligned}$$

The final statement (iii) follows from (ii) and (iv) using Noether's formula. Q.E.D.

If $Y = \mathbf{P}^2$ and the Tschirnhausen bundle is split, we have the following corollary.

COROLLARY 10.4. *Assume that* $f: X \to \mathbf{P}^2$ *is a triple cover with Tschirnhausen module* $\mathcal{O}_{\mathbf{P}^2}(-m) \oplus \mathcal{O}_{\mathbf{P}^2}(-n)$. *Then*

(i) $q(X) = 0$
(ii) $p_g(X) = 1/2(m^2 + n^2 - 3m - 3n) + 2$

(iii) $K_X^2 = 2(m + n - 3)^2 - 3(mn - 3)$
(iv) $e(X) = 9 + 4(m + n)^2 - 6(m + n) - 9mn$.

As in the case of curves, for low values of $m$ and $n$ these surfaces are quite familiar.

Table 10.5

Invariants of triple covers of $\mathbf{P}^2$, with $E$ of type $(m, n)$

| $m$ | $n$ | $p_g(X)$ | $K_X^2$ | $e(X)$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 8 | 4 |
| 1 | 2 | 0 | 3 | 9 |
| 2 | 2 | 0 | −1 | 13 |
| 2 | 3 | 1 | −1 | 25 |
| 2 | 4 | 3 | 3 | 45 |
| 3 | 3 | 2 | 0 | 36 |
| 3 | 4 | 4 | 5 | 55 |

If $(m, n) = (1, 1)$, the surface $X$ is the Steiner cubic in $\mathbf{P}^4$ and the triple cover map is projection. If $(m, n) = (1, 2)$, $X$ is a cubic hypersurface in $\mathbf{P}^3$ (as Corollary 8.5 indicates).

If $(m, n) = (2, 2)$, the surface $X$ is the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at nine points; the map to the plane is given by curves of bidegree (2, 3) through the nine base points. When $(m, n) = (2, 3)$, $X$ is a quartic surface blown up at one point and the map is projection from the point. When $(m, n) = (2, 4)$, $X$ is a surface of general type with $p_g = K_X^2 = 3$ and $f$ is the canonical map. In the (3, 3) case, $X$ is an elliptic surface over $\mathbf{P}^1$ (the elliptic structure being given by the canonical map) and the triple covering is defined by a linear system of genus 4 trisections of the elliptic structure. If $(m, n) = (3, 4)$, $X$ is a quintic surface in $\mathbf{P}^3$ with a double point $p$ and $f$ is projection from $p$.

Of course, there are other familiar bundles on $\mathbf{P}^2$ which could be used to construct triple covers. As an example, we have the following calculation.

COROLLARY 10.6. *Assume that* $f: X \to \mathbf{P}^2$ *is a triple cover with Tschirnhausen module* $\Omega^1_{\mathbf{P}^2}(-m)$, *with* $m \geq 0$. *Then*

(i) $q(X) = 0$ *if* $m \geq 1$; $q(X) = 1$ *if* $m = 0$.
(ii) $p_g(X) = m^2 - 1$ *if* $m \geq 1$; $p_g(X) = 0$ *if* $m = 0$.
(iii) $K_X^2 = m(5m - 9)$.
(iv) $e(X) = m(7m + 9)$.

When $m = 0$, $X$ is a ruled surface and when $m = 1$, $X$ is the 13-fold blow-up of the plane, mapped to $\mathbf{P}^2$ via quartics through 13 base points which impose only 12 conditions on quartics.

I will close with a brief sketch of a construction of surfaces of general type $X$ which have $c_1^2/c_2$ arbitrarily close to the upper bound 3 [2]. The construction is iterative; assume that the surface $X_N$ has been built. Let $E_N$ be a rank two bundle on $X_N$ such that $c_1(E_N) = -3K_{X_N}$, and $c_2(E_N) = 4K_{X_N}^2$. Let $\Phi_N: S^3E_N \to \Lambda^2E_N$ be a building map, and let $X_{N+1}$ be the triple cover of $S_N$ which $\Phi_N$ builds. By Proposition 10.3, $K_{X_{N+1}}^2 = 21K_{X_N}^2$, and $e(X_{N+1}) = 6K_{X_N}^2 + 3e(X_N)$. If $\alpha_N = K_{X_N}^2/e(X_N)$, then $\alpha_{N+1} = 7\alpha_N/(2\alpha_N + 1)$. As $N \to \infty$, $\alpha_{N+1} \to 3$ monotonically from below.

There is no trouble in constructing the bundles $E_N$ with the proper chern classes; see [1], page 731, for example, the weakness of this method is that I have not yet been able to show that sufficiently general building maps $\Phi_N$ exist, to ensure that $X_{N+1}$ is a smooth surface.

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