ON THE STABILITY OF PENCILS OF CUBIC CURVES

By RICK MIRANDA

1. Introduction. Fix an algebraically closed field k of characteristic 0. In this paper we study the classification of pencils of cubic curves in $\mathbf{P}_k{}^2$ up to projective automorphism. In particular, we construct a classification space for cubic pencils, using geometric invariant theory.

The automorphism group PGL(3) of $\mathbf{P}^2$ acts naturally on the space G of all cubic pencils, as described below. In the construction of the quotient variety, which is the desired classification space, the central problem is to determine the stable pencils (those whose orbit is closed and of maximal dimension); it is for these pencils that the quotient variety is an orbit space. We obtain explicit vanishing criteria on the Plücker coordinates of a pencil for both stability and semi-stability (see Propositions 4.3 and 4.4); moreover, we give the equations defining pairs of generators for stable and semi-stable pencils (Propositions 5.1 and 5.2).

Finally, a more geometric characterization of the stability of pencils with smooth members is obtained by considering the elliptic surface X_P associated to such a pencil (X_P is obtained by blowing up $\mathbf{P}^2$ at the base points of the pencil P). We prove the

THEOREM. (1) *P is a stable pencil $\Leftrightarrow$ P contains a smooth member and* **every** *fibre of X_P is reduced.*
(2) *If P contains a smooth member, then P is a semi-stable pencil $\Leftrightarrow$ X_P contains no fibre of type II*, III*, or IV** (*using Kodaira's notation for singular fibres of elliptic surfaces*).

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2. The space of pencils. Let V be the vector space of sections $\Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$, and denote by S^3V the third symmetric power of V, naturally isomorphic to $\Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3))$. The projective space $\mathbf{P}^9 \cong \mathbf{P}(S^3V^*)$ is the parameter space for cubic curves in $\mathbf{P}^2$. Let G be the Grassman variety of lines in this $\mathbf{P}^9$; a point of G then corresponds to a pencil of plane cubic curves. G is naturally embedded in the projective space $\mathbf{P}^{44} = \mathbf{P}(\Lambda^2 S^3 V^*)$ via the Plücker coordinates described briefly below.

Choose coordinates $[x, y, z]$ of $\mathbf{P}^2$, i.e., a basis for the space V. Let P be a point of G, representing the line P in $\mathbf{P}^9$. (By abuse of notation we shall use P for a point of G, the line in $\mathbf{P}^9$ it represents, or the pencil of cubics to which that line corresponds, depending on the context.) Choose any two distinct points A and B on the line P. Let A and B represent the cubics $\Sigma\, a_{ij}x^iy^jz^{3-i-j}$ and $\Sigma\, b_{ij}x^iy^jz^{3-i-j}$, respectively. Form the 2×10 matrix

$$\begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix}.$$

Let

$$m_{ijk\ell} = \begin{bmatrix} a_{ij} & a_{k\ell} \\ b_{ij} & b_{k\ell} \end{bmatrix} = a_{ij}b_{k\ell} - a_{k\ell}b_{ij}$$

be the determinant of the corresponding 2×2 minor. The 45 coordinates $(m_{ijk\ell})$ are called the Plücker coordinates of P.

THEOREM [*KL*]. *The coordinates $(m_{ijk\ell})$ of P are, up to scalar multiplication, independent of the choice of A and B and give rise to a well-defined point of $\mathbf{P}^{44}$. Moreover, the map $G \to \mathbf{P}^{44}$ given by sending a point to its Plücker coordinates is a closed embedding.*

The above theorem justifies the use of the Plücker coordinates in working with the space G. In particular, we wish to use these coordinates to study the effect of applying a projective automorphism of the plane on G. To define this action it is convenient to consider the canonical isogeny $\mathrm{SL}(V) \to \mathrm{Aut}(\mathbf{P}(V^*)) = \mathrm{Aut}(\mathbf{P}^2)$ which induces an action of $\mathrm{SL}(V)$ on $\mathbf{P}^2$; this action lifts to the natural action of $\mathrm{SL}(V)$ on V. Both groups act with the same orbits, so there is no loss of structure in this reduction. The group $\mathrm{SL}(V)$ acts canonically on V^*, S^3V^*, and on $\Lambda^2S^3V^*$ which has as coordinates the $(m_{ijk\ell})$. (See [H] for details.) This action is again linear and induces an action of SL(V) on $\mathbf{P}^{44}$ in which G is an invariant subvariety. The restricted action of SL(V) on G is the desired action.

To classify cubic pencils up to projective automorphism, we would like to construct, using geometric invariant theory, the orbit space $G/\mathrm{SL}(V)$, as a k-scheme. This quotient space unfortunately can not be given a scheme structure, essentially due to the presence of non-closed orbits. We will outline in the next section the standard method for constructing the quotient variety for the 'semi-stable' orbits and for determining which pencils are 'unstable' and must be deleted.

For the purposes of the subsequent calculation we need to write down the action explicitly for diagonal elements of SL(V). Again choose coordinates $[x, y, z]$ of $\mathbf{P}^2$, and let SL(3) act on these coordinates by the usual matrix multiplication: if $g \in \mathrm{SL}(3)$ is diagonal and is given by the matrix

$$\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix},$$

then

$$g : [x, y, z] \rightsquigarrow [ux, vy, wz].$$

If (a_{ij}) is a point of S^3V^* representing the cubic form $\Sigma\, a_{ij}x^iy^jz^{3-i-j}$, then the action of g on S^3V^* is given by

$$g : (a_{ij}) \rightsquigarrow (u^iv^jw^{3-i-j}a_{ij}).$$

Finally, g acts on the Plücker coordinates $m_{ijk\ell}$ by

$$g : (m_{ijk\ell}) \rightsquigarrow (u^{i+k}v^{j+\ell}w^{6-i-j-k-\ell}m_{ijk\ell}). \tag{1.1}$$

3. Relevant invariant theory. In this section we will collect the definitions and results of geometric invariant theory required for the sequel. The ultimate reference for all definitions and details is [GIT]; a more pedestrian account is given in [SPV].

Fix an algebraically closed field k, and a reductive algebraic group G defined over k. Let V be an n-dimensional representation of G, and let x be a vector in V. Let $G \cdot x$ denote the orbit and G_x the stabilizer of x.

Definition 3.1. (a) x is unstable if $0 \in \overline{G \cdot x}$
(b) x is semi-stable if $0 \notin \overline{G \cdot x}$
(c) x is properly stable (or simple stable) if $G \cdot x$ is closed and G_x is finite
(d) x is strictly semi-stable if x is semi-stable but not stable.

Let $\mathbf{P}(V^*)$ be the projective space of 1-dimensional subspaces of V. A point p of $\mathbf{P}(V^*)$ will be called *unstable* if any non-zero vector x of V lying over p is unstable. Analogous definitions are made for stability and semi-stability. Let V_s and V_{ss} be the open cones of stable and semi-stable vectors of V, and let $\mathbf{P}_s(V^*)$ and $\mathbf{P}_{ss}(V^*)$ be the open sets of stable and semi-stable points of $\mathbf{P}(V^*)$. The following theorem is the main result:

THEOREM 3.2. *Let V be an n-dimensional representation of G, inducing an action of G on $\mathbf{P}(V^*)$. Let $Y \subset \mathbf{P}(V^*)$ be a closed G-invariant subscheme of $\mathbf{P}(V^*)$; Y is then a projective scheme on which G acts. Let $Y_{ss} = Y \cap \mathbf{P}_{ss}(V^*)$ and $Y_s = Y \cap \mathbf{P}_s(V^*)$. Then* (a) *a universal categorical quotient (X, π) of Y_{ss} by G exists, and X is a projective k-scheme,* (b) *there is an open set X_s of X such that $(X_s, \pi \mid Y_s)$ is a universal geometric quotient of Y_s by G.*

We see that the image X_s of Y_s is an actual orbit space for those orbits in Y_s; however, in general, two orbits are identified in X if they have common closure in Y_{ss}.

The above results are testimony to the central importance of the concepts of stability, semi-stability, and instability. That it is also relatively computable makes them not only theoretically interesting notions but also honestly useful tools in modern invariant theory. In particular, there is a strong numerical criterion for stability which we now describe.

Definition 3.3. Let V be a representation of G_m. Since G_m is reductive, V splits into a sum of eigenspaces as $V = \bigoplus_{n \in \mathbf{Z}} V_n$ where the action of $t \in G_m$ on V_n is given by scalar multiplication by t^n. For any $x \in V$, write x as $\Sigma\, x_n$ where $x_n \in V_n$. The *weights of* x with respect to this representation of G_m is the set of integers n such that x_n is not zero.

Definition 3.4. Let V be a representation of an algebraic group G. Let $\lambda : G_m \to G$ be a 1-parameter subgroup of G, and let x be a point of V. The λ-weights of x are the weights of x with respect to the induced representation of G_m on V given by λ. If p is a point of $\mathbf{P}(V^*)$, the λ-weights of p are the λ-weights of any vector x in V lying over p (they are all equal).

The numerical criterion for stability can now be stated.

THEOREM 3.5. *Let G be a reductive algebraic group acting linearly on the vector space V, and let x be a vector in V. Then* (a) *x is unstable $\Leftrightarrow$ there exists a* 1*-parameter subgroup λ of G such that the λ-weights of x are all positive,* (b) *x is semi-stable $\Leftrightarrow$ no such* 1*-parameter subgroup of G exists,* (c) *x is stable $\Leftrightarrow$ x has both positive and negative weights with respect to every non-trivial* 1*-parameter subgroup of G.*

Condition (c) is often more useful in the form

(d) *x is not stable $\Leftrightarrow$ there exists a non-trivial* 1*-parameter subgroup λ of G such that the λ-weights of x are all non-negative.*

Statements (c) and (d) give numerical criteria for stable vectors, i.e., those whose orbits are closed and of maximal dimension. A slight generalization gives a criterion for any closed orbit:

(e) *$G \cdot x$ is closed $\Leftrightarrow$ for every* 1*-parameter subgroup λ of G, either the λ-weights of x are both positive and negative, or* 0 *is the only λ-weight of x.*

4. The criterion for the stability of cubic pencils. As we have seen, the analysis of stability is of central importance in the classification of cubic pencils. We shall now determine which pencils are unstable and not properly stable using the numerical criteria (3.5). The conditions for stability will be expressed in terms of the Plücker coordinates $(m_{ijk\ell})$.

By (3.5(a)), a pencil P is unstable if there exists a 1-parameter subgroup $\lambda : G_m \to SL(V)$ such that the weights of P with respect to λ are all positive. Let us compute these weights explicitly as follows:

Assume P is an unstable pencil. Let λ be the 1-parameter subgroup with respect to which the weights of P are all positive. We may choose coordinates $[x, y, z]$ of $\mathbf{P}^2$ such that the induced action of G_m on $[x, y, z]$ is diagonal. Let us say that in these coordinates the action of G_m is given by

$$\lambda(t) : [x, y, z] \qquad [t^{r_x}x, t^{r_y}y, t^{r_z}z]$$

where

$$r_x \geq r_y \geq r_z, \qquad r_x \gneq 0, \quad \text{and} \quad r_x + r_y + r_z = 0.$$

The action of $\lambda(t)$ on the Plücker coordinates $(m_{ijk\ell})$ will then be

$$\lambda(t) : (m_{ijk\ell}) \qquad (t^{r_x(i+k)+r_y(j+\ell)+r_z(6-i-j-k-\ell)} m_{ijk\ell})$$

by (1.1). The weights of the point $(m_{ijk\ell})$ with respect to λ are the exponents of t for which $m_{ijk\ell}$ is non-zero. By setting $r_z = -r_x - r_y$, the exponent can be written in terms of r_x and r_y as

$$r_x(2i + 2k + j + \ell - 6) + r_y(2j + 2\ell + i + k - 6).$$

The criterion for instability can now be stated as follows:

Assume P is an unstable pencil. Then there exists integers r_x, r_y (with $r_x \geq r_y \geq -r_x - r_y$ and $r_x > 0$) and coordinates in $\mathbf{P}^2$ such that if P is represented by the point $(m_{ijk\ell})$ of $\mathbf{P}^{44}$ in these coordinates, then $m_{ijk\ell} = 0$ whenever $r_x(2i + 2k + j + \ell - 6) + r_y(2j + 2\ell + i + k - 6) \leq 0$.

There is a slightly less cumbersome form of the numerical criterion which is useful in this case. Let $r = r_y/r_x$. The conditions $r_x \geq r_y \geq -r_x - r_y$ translate to $-1/2 \leq r \leq 1$. By dividing the inequality above by the positive integer r_x we obtain the following restatement of the numerical criterion:

PROPOSITION 4.1. *P is an unstable pencil in G if and only if there exist a rational number $r \in [-1/2, 1]$ and coordinates in $\mathbf{P}^2$ such that if P is represented by the point $(m_{ijk\ell})$ in these coordinates, then*

$$m_{ijk\ell} = 0 \quad \textit{whenever} \quad e_{ijk\ell}(r) \leq 0,$$

where $e_{ijk\ell}(r) = (2i + 2k + j + \ell - 6) + r(2j + 2\ell + i + k - 6)$.

The criterion for stable pencils in G can be expressed similarly, using (3.5(d)):

PROPOSITION 4.2. *P is not properly stable if and only if there exist a rational number $r \in [-1/2, 1]$ and coordinates in $\mathbf{P}^2$ such that if P is represented by the point $(m_{ijk\ell})$ in these coordinates, then*

$$m_{ijk\ell} = 0 \quad \textit{whenever} \quad e_{ijk\ell}(r) < 0,$$

where $e_{ijk\ell}(r)$ is as above.

We wish to remove from the above propositions the dependency on the existence of the rational number r. A priori, given a coordinate system on $\mathbf{P}^2$, one would need to check all the rationals in $[-1/2, 1]$ to determine the conditions on the $(m_{ijk\ell})$ for instability (or non-proper-stability). Note however that the conditions $e_{ijk\ell}(r) \leq 0$ (respectively $e_{ijk\ell}(r) < 0$) for all appropriate i, j, k, and ℓ subdivide the interval $[-1/2, 1]$ into a finite number of subintervals within which the truth or falsity of the inequalities are constant. Hence we need check only one rational r in each sub-interval. Moreover, a careful inspection of the conditions on the $(m_{ijk\ell})$ in each of the sub-intervals shows that these conditions are not independent and in fact the number of sub-intervals giving 'minimal' conditions on the $(m_{ijk\ell})$ for instability (or non-proper-stability) is two (respectively, three). Let us merely summarize the results of performing this calculation below.

PROPOSITION 4.3. *A pencil P is unstable if and only if there exist coordinates in* $\mathbf{P}^2$ *such that, if* $(m_{ijk\ell})$ *are the induced coordinates of P, then either*

Case 1.

$$\begin{array}{l} m_{0001}, m_{0002}, m_{0003}, m_{0010}, m_{0011}, m_{0012}, m_{0020}, m_{0021}, \\ m_{0102}, m_{0103}, m_{0110}, m_{0111}, m_{0112}, m_{0120}, m_{0121}, \\ m_{0203}, m_{0210}, m_{0211}, m_{0212}, m_{0220}, \\ m_{0310}, m_{0311}, \\ m_{1011}, m_{1012} \end{array}$$

all vanish, or

Case 2.

$$\begin{array}{l} m_{0001}, m_{0002}, m_{0003}, m_{0010}, m_{0011}, m_{0012}, m_{0020}, m_{0021}, m_{0030} \\ m_{0102}, m_{0103}, m_{0110}, m_{0111}, m_{0112}, m_{0120}, m_{0121}, \\ m_{0210}, m_{0211}, m_{0220}, \\ m_{0310}, \\ m_{1011}, m_{1012}, m_{1020} \end{array}$$

all vanish. In Case 1, *an* $r \in (-1/5, 0)$ *will exhibit P as unstable and in Case* 2 *an r in* $(1/4, 1)$ *will work, using Proposition* 4.1.

PROPOSITION 4.4. *A pencil P is not properly stable if and only if there exist coordinates of* $\mathbf{P}^2$ *such that, if* $(m_{ijk\ell})$ *are the induced coordinates of P, then either*

Case 3:

$$\begin{array}{l} m_{0001}, m_{0002}, m_{0003}, m_{0010}, m_{0011}, m_{0012}, \\ m_{0102}, m_{0103}, m_{0110}, m_{0111}, m_{0112}, \\ m_{0203}, m_{0210}, m_{0211}, m_{0212}, \\ m_{0310}, m_{0311}, m_{0312} \end{array}$$

all vanish, or

Case 4:

$$m_{0001},\ m_{0002},\ m_{0003},\ m_{0010},\ m_{0011},\ m_{0012},\ m_{0020},\ m_{0021},$$
$$m_{0102},\ m_{0103},\ m_{0110},\ m_{0111},\ m_{0112},\ m_{0120},$$
$$m_{0203},\ m_{0210},\ m_{0211},$$
$$m_{0310},$$
$$m_{1011}$$

all vanish, or

Case 5:

$$m_{0001},\ m_{0002},\ m_{0003},\ m_{0010},\ m_{0011},\ m_{0012},\ m_{0020},\ m_{0021},\ m_{0030},$$
$$m_{0102},\quad m_{0110},\ m_{0111},\quad m_{0120},$$
$$m_{0210},$$
$$m_{1011},\quad m_{1020}$$

all vanish. In Cases 3, 4, *and* 5 *the rational number r exhibiting P as not properly stable is* $-1/2$, 0, *and* 1 *respectively, using Proposition* 4.2.

Remark. Propositions 4.3 and 4.4 actually give the criteria for stability and semi-stability for *any* point of $P(\Lambda^2 S^3 V)$ in terms of the Plücker coordinates, not just for cubic pencils (which correspond to decomposable 2-forms).

5. The Stability condition in terms of generators of a pencil. Having computed the criteria for instability and non-proper stability in terms of the Plücker coordinates, one is still left with the question of a geometric characterization: what property or properties do the unstable (respectively, non-properly stable) pencils have? We claim that the vanishing conditions of Propositions 4.3 and 4.4 translate directly to various base conditions on the pencil. To see this, we assume that two cubics A and B generate an unstable (or non-properly stable) pencil P. Choose coordinates $[x, y, z]$ of $\mathbf{P}^2$ so that one of the vanishing conditions of Proposition 4.3 (or Proposition 4.4) are satisfied. In these coordinates we write

$$A : \Sigma\, a_{ij} x^i y^j z^{3-i-j} = 0$$

and

$$B : \Sigma\, b_{ij} x^i y^j z^{3-i-j} = 0.$$

Since the Plücker coordinates $m_{ijk\ell}$ equal $a_{ij}b_{k\ell} - a_{k\ell}b_{ij}$, each set of vanishing conditions gives equations involving the coefficients a_{ij} and $b_{k\ell}$. After some algebraic manipulation, these equations are easily seen to be equivalent to the vanishing of certain of the coefficients of *some pair* of cubics A', B' in the pencil P (not necessarily the original pair A, B). I will omit this part of the analysis and present the result of these calculations below. We use the notation $\langle M_1, M_2, \ldots, M_K\rangle$ to denote the subspace of the vector space of cubic forms in x, y, and z generated by the monomials M_i.

PROPOSITION 5.1. *A pencil P is unstable if and only if there exist coordinates $[x, y, z]$ of $\mathbf{P}^2$ and two generators A, B of P with equations $F_A(x, y, z) = 0$, $F_B(x, y, z) = 0$, respectively, satisfying one of the following five conditions:*

(1) $F_A \in \langle x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2z, y^3, xz^2, xyz, xy^2, x^2z, x^2y, x^3\rangle$
(2) $F_A \in \langle x^2y, x^3\rangle$
$F_B \in \langle y^2z, y^3, xz^2, xyz, xy^2, x^2z, x^2y, x^3\rangle$
(3) $F_A \in \langle x^2z, x^2y, x^3\rangle$
$F_B \in \langle y^3, xz^2, xyz, xy^2, x^2z, x^2y, x^3\rangle$
(4) $F_A \in \langle y^3, xy^2, x^2y, x^3\rangle$
$F_B \in \langle y^2z, y^3, xyz, xy^2, x^2z, x^2y, x^3\rangle$
(5) $F_A \in \langle xyz, xy^2, x^2z, x^2y, x^3\rangle$
$F_B \in \langle y^3, xyz, xy^2, x^2z, x^2y, x^3\rangle$

A pencil with generators A, B, having the form of cases (1) or (3) has Plücker coordinates $(m_{ijk\ell})$ which satisfy the vanishing conditions of Case 1 of Proposition 4.3. A pencil with generators in cases (4) or (5) has Plücker coordinates which satisfy Case 2 of Proposition 4.3. A pencil with generators in case (2) has Plücker coordinates which satisfy both cases of Proposition 4.3.

PROPOSITION 5.2. *A pencil P is not properly stable if and only if there exist coordinates $[x, y, z]$ of $\mathbf{P}^2$ and two generators A, B of P with equations $F_A = 0$, $F_B = 0$ respectively, satisfying one of the following five conditions:*

(6) $F_A \in \langle x^2z, x^2y, x^3\rangle$
$F_B \in \langle z^3, yz^2, y^2z, y^3, xz^2, xyz, xy^2, x^2z, x^2y, x^3\rangle$
(7) $F_A \in \langle xz^2, xyz, xy^2, x^2z, x^2y, x^3\rangle$
$F_B \in \langle xz^2, xyz, xy^2, x^2z, x^2y, x^3\rangle$

(8) $F_A \in \langle xy^2, x^2z, x^2y, x^3 \rangle$
$F_B \in \langle y^2z, y^3, xz^2, xyz, xy^2, x^2z, x^2y, x^3 \rangle$
(9) $F_A \in \langle y^3, xy^2, x^2y, x^3 \rangle$
$F_B \in \langle yz^2, y^2z, y^3, xz^2, xyz, xy^2, x^2z, x^2y, x^3 \rangle$
(10) $F_A \in \langle y^3, xyz, xy^2, x^2z, x^2y, x^3 \rangle$
$F_B \in \langle y^2z, y^3, xyz, xy^2, x^2z, x^2y, x^3 \rangle$

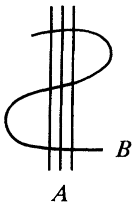
A pencil with generators in cases (6) and (7) has Plücker coordinates which satisfy Case 3 of Proposition 4.4. A pencil with generators in case (8) has Plücker coordinates which satisfy Case 4 of Proposition 4.4. A pencil with generators in cases (9) and (10) has Plücker coordinates which satisfy Case 5 of Proposition 4.4.

Notice that in all the cases (1)–(10), the subspace in which F_B is constrained to lie contains the subspace in which F_A is constrained to lie. Thus any linear combination of F_A and F_B will lie in the subspace containing F_B. One can therefore think of B as a 'general' member of the pencil P and of A as the 'special' member. A quick inspection of the geometric configuration of the two cubics A and B in each of the 10 cases bears this out; if one assumes that the generators A and B are general enough, one can easily draw a representative graph in each case, illustrating this property. See Figures 5.3 and 5.4.

6. The associated elliptic surface and instability. Let us call a pencil P of cubics *smooth* if some member of the pencil P is a smooth cubic. If P is a smooth pencil, then the general member of P is a smooth cubic and by blowing up the nine base points of the pencil (some possibly infinitely near) we obtain in a natural way an elliptic surface $\tilde{X}$ fibred over $\mathbf{P}^1$. This elliptic surface has a section; if we perform the nine blow-ups of $\mathbf{P}^2$ in some order, the final exceptional curve will be a section of the elliptic structure. (There will in general be many others, also.) Finally, note that $\tilde{X}$ is a smooth rational surface. Conversely, every smooth rational elliptic surface with a section may be obtained in this way; the proof of this is elementary and since we do not require it, we will leave it to the reader. Let us call the rational elliptic surface $\tilde{X}$ obtained from P as above the *induced elliptic surface*; $\tilde{X}$ is determined up to isomorphism, including the elliptic fibration, but no section is canonically given.

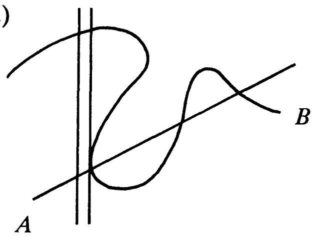
We wish to use the induced elliptic surface to characterize the stability of cubic pencils in a more geometric way; although the classification of generators for unstable pencils given in section 5 is complete, the list is not

(1)



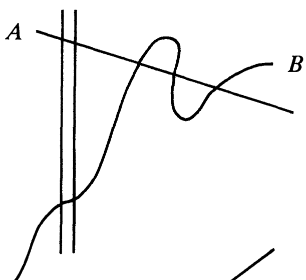
A = triple line
B = arbitrary cubic

(2)



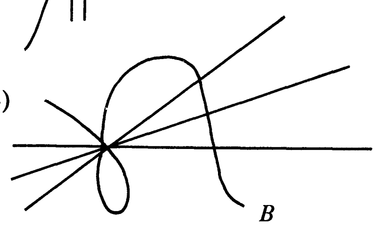
A = double line ∪ other line
B = tangent to double line at the point of intersection

(3)



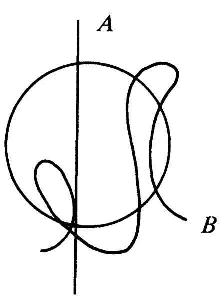
A = double line ∪ other line
B meets double line at only one point

(4)



A = three concurrent lines
B is double at the point of concurrency

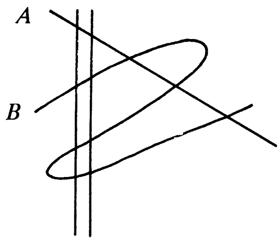
(5)



A = conic ∪ line
B is double at a point of intersection of the conic and the line, with one tangent equal to the line.

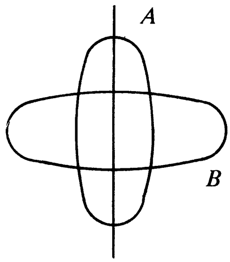
Figure 5.3 *Graphs of 'general' unstable pencils*

(6)



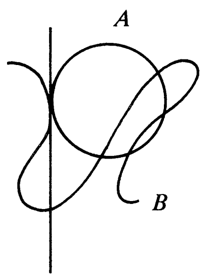
A = double line ∪ other lines
B is arbitrary

(7)



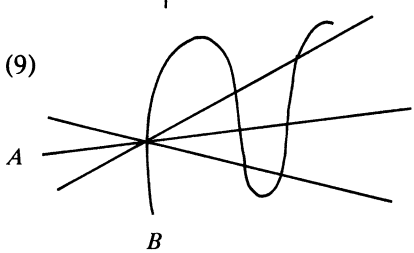
A and B share a line

(8)



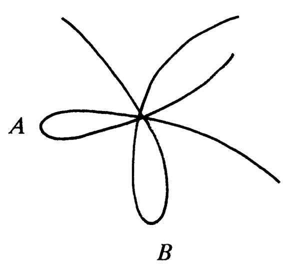
A = conic ∪ tangent line
B is also tangent to the line at the same point.

(9)



A = three concurrent lines
B passes through the point of concurrency

(10)



A and B share a singular point.

Figure 5.4 *Graphs of 'general' non-properly stable pencils* very informative as it stands. We claim that instability for smooth pencils is characterized however by the existence of certain singular fibres on the induced elliptic surface $\tilde{X}$.

THEOREM 6.1. *Let P be a smooth pencil of cubic curves in $\mathbf{P}^2$, inducing the elliptic surface $\tilde{X}$. Then P is unstable*

$\Leftrightarrow$ *$\tilde{X}$ contains a singular fibre $X_0 = \Sigma n_i C_i$ with $n_i \geq 3$ for some i*
$\Leftrightarrow$ *$\tilde{X}$ contains a fibre of type II*, III*, or IV* in Kodaira's notation of singular fibres of elliptic surfaces* [Ko].

Proof. Note that the second equivalence is an immediate consequence of Kodaira's classification of possible singular fibres on $\tilde{X}$; these three singular fibres are the only ones with a component of multiplicity larger than 2.

To prove the first equivalence, assume first that the pencil P is unstable. By the classification given by Proposition 5.1, there must exist generators A, B of P such that A and B are of the form demanded by either case (1), (2), or (3). We shall address each of these cases in turn.

Case 1. Note that in this case the member A of the pencil is a triple line; the proper transform of this triple line on the elliptic surface X will be a component of multiplicity 3 in its fibre.

Case 2. In this case we have A given by an equation $a_{21}x^2y + a_{30}x^3 = 0$. If $a_{21} = 0$ in fact we see that A is a triple line and proceed to argue as in case 1. If $a_{21} \neq 0$ then the member A is the double line $x^2 = 0$ plus another line through (0, 0, 1). Since the general member of P is smooth, B must be smooth at (0, 0, 1). Thus the coefficient b_{10} of xz^2 in the equation of B must be non-zero. Also one of the coefficients b_{01}, b_{03} of y^2z, y^3 respectively must be non-zero; otherwise B would contain the line $x = 0$ as well as A. In either case, B passes thru the point (0, 0, 1) and is tangent to the line $x = 0$ there. Thus to separate A and B we must blow up the point (0, 0, 1) and also the infinitely near point to (0, 0, 1) in the direction of the line $x = 0$. It is easily seen that the exceptional curve for this second blow-up has multiplicity 3 in the fibre over A after further blowing up to obtain $\tilde{X}$.

Case 3. In this case, A is given by the equation $a_{20}x^2z + a_{21}x^2y + a_{30}x^3 = 0$ and is thus composed of the double line $x = 0$ and some other line $a_{20}z + a_{21}y + a_{30}x = 0$. The member B must again be smooth at (0, 0, 1), and so $b_{10} \neq 0$. In this case also the coefficient b_{03} of y^3 must be non-zero; otherwise B would contain the line $x = 0$. We thus see that B has a flex at (0, 0, 1) with tangent line $x = 0$. To separate the members A and B we must blow up the point (0, 0, 1), the infinitely near point in the direction of the line $x = 0$, and the doubly infinitely near point in the direction of the line $x = 0$. It is again easily seen that the exceptional curve for this third blow-up has multiplicity 3 in its fibre in $\tilde{X}$.

The converse is slightly more complicated. Assume P is a pencil of plane cubics such that the rational elliptic surface F has a fibre of type II*, III*, or IV*. The problem is to find generators A and B of P satisfying the conditions of one of the unstable cases 1, 2, or 3. We shall proceed using a series of reductions.

LEMMA 6.2. *P must contain a member A which is singular at one of the base points.*

Proof. We need only note that given a pencil $\{C_\lambda\}$, the induced pencil $\{\tilde{C}_\lambda\}$ on the surface R obtained by blowing up the base point p of $\{C_\lambda\}$ is $\{\pi^* C_\lambda - E\}$ where π is the blow up map and E is the exceptional curve for π. Thus given any member A of the pencil, the induced member on R is $\overline{A} + (m_p(A) - 1)E$ where $\overline{A}$ is the proper transform of A on R and $m_p(A)$ is the multiplicity of the point p on A. Thus if, at every base point, every member of the pencil P is smooth, $m_p(A) = 1$ for every p and A. Thus the induced pencil on R consists of only the proper transforms of the curves in the original pencil. This argument can be applied to each blow-up, and in particular the fibres of X must be only proper transforms of the members of the pencil. But each of the fibres II*, III*, and IV* have at least seven components and hence cannot be the proper transform of a cubic in the plane. This contradiction proves the lemma. Q.E.D.

LEMMA 6.3. *P must contain a member A which is either*

(a) *a cuspidal cubic, with cusp at a base point,*
(b) *3 concurrent lines, whose common point is a base point,*
(c) *a line and a conic tangent to the line, the point of tangency a base point,*
(d) *a double line and another line, or*
(e) *a triple line.*

Proof. Using Lemma 6.2, some member of P must be singular at some base point. The list above contains all singular cubics except

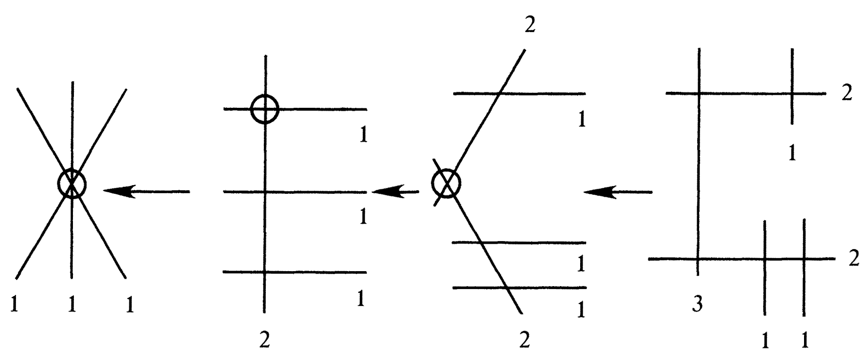
(1) a nodal cubic,
(2) three non-concurrent lines,
(3) a conic and a non-tangent line.

It is elementary to see that none of the above members of a pencil P can induce a fibre of type II*, III*, or IV* on X. Let us only examine case (1) of a nodal cubic A. Upon 1 blow-up at the node, the pencil on the blow up has corresponding member $\overline{A} + E$ where $\overline{A}$ is the proper transform of the nodal cubic A and E is the exceptional curve. Notice that the singularities of this member are still only double points with distinct tangents, and there is no component of multiplicity larger than 1. This situation remains true after each blow-up; thus on the surface X no component will have multiplicity greater than 1. Cases (2) and (3) above are handled identically; again the crucial fact to be noticed is that at each blow-up, the corresponding member has only "normal crossings" and reduced components. Q.E.D.

LEMMA 6.4. *P must contain a member A which is either*

(d) *a double line and another line, or*
(e) *a triple line.*

Proof. We must eliminate cases (a), (b), and (c) of Lemma 6.3. The argument is slightly different from these reductions than as before; we can conceivably obtain a component of multiplicity three in the fibre if we blow up enough times at the singular point, as is illustrated below for case (b)



(The numbers are the multiplicity of the component in the pencil; the circled points are the points blown up at each stage.)

To see that this cannot occur, we will show that in order to separate the member A from a smooth cubic B through the singular point p, we need never apply the above sequence of blow-ups (or any other sequence leading to a multiplicity 3 component). Let us argue first in case (b). There are several cases to consider for the smooth cubic B:

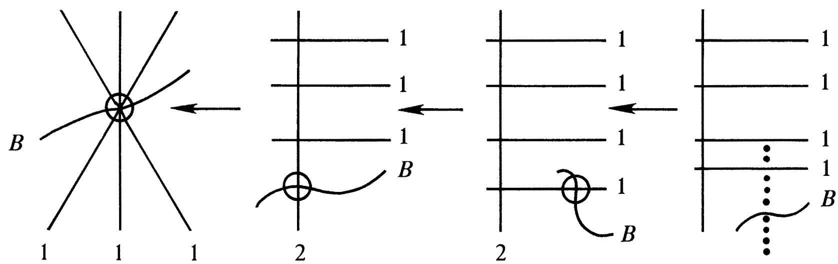
(Case 1): B passes through the singular point p with tangent different from the three lines

(Case 2): B is simply tangent to one of the lines at p.

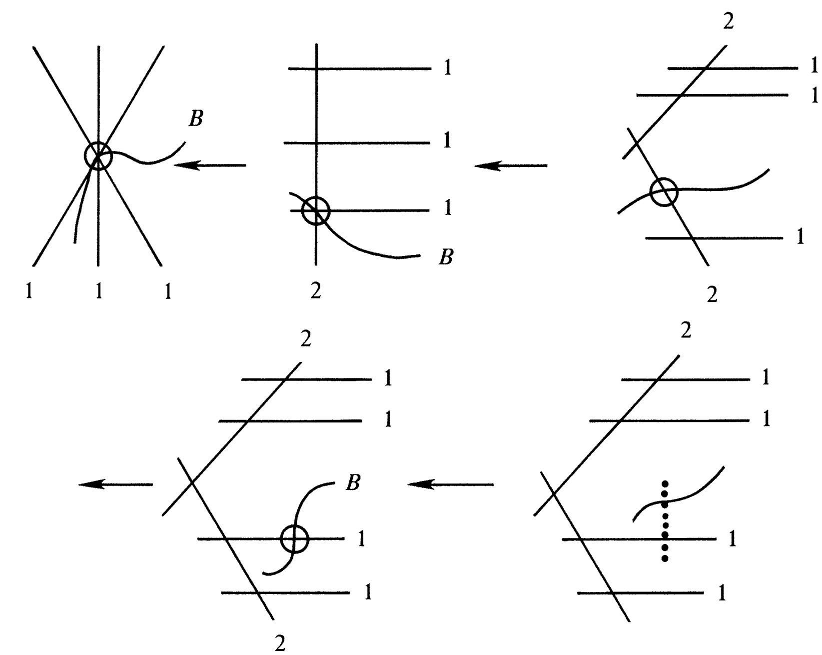
(Case 2): B is tangent, to order 3, to one of the lines at p.

In each of these cases we separate A from B explicitly and see what results:

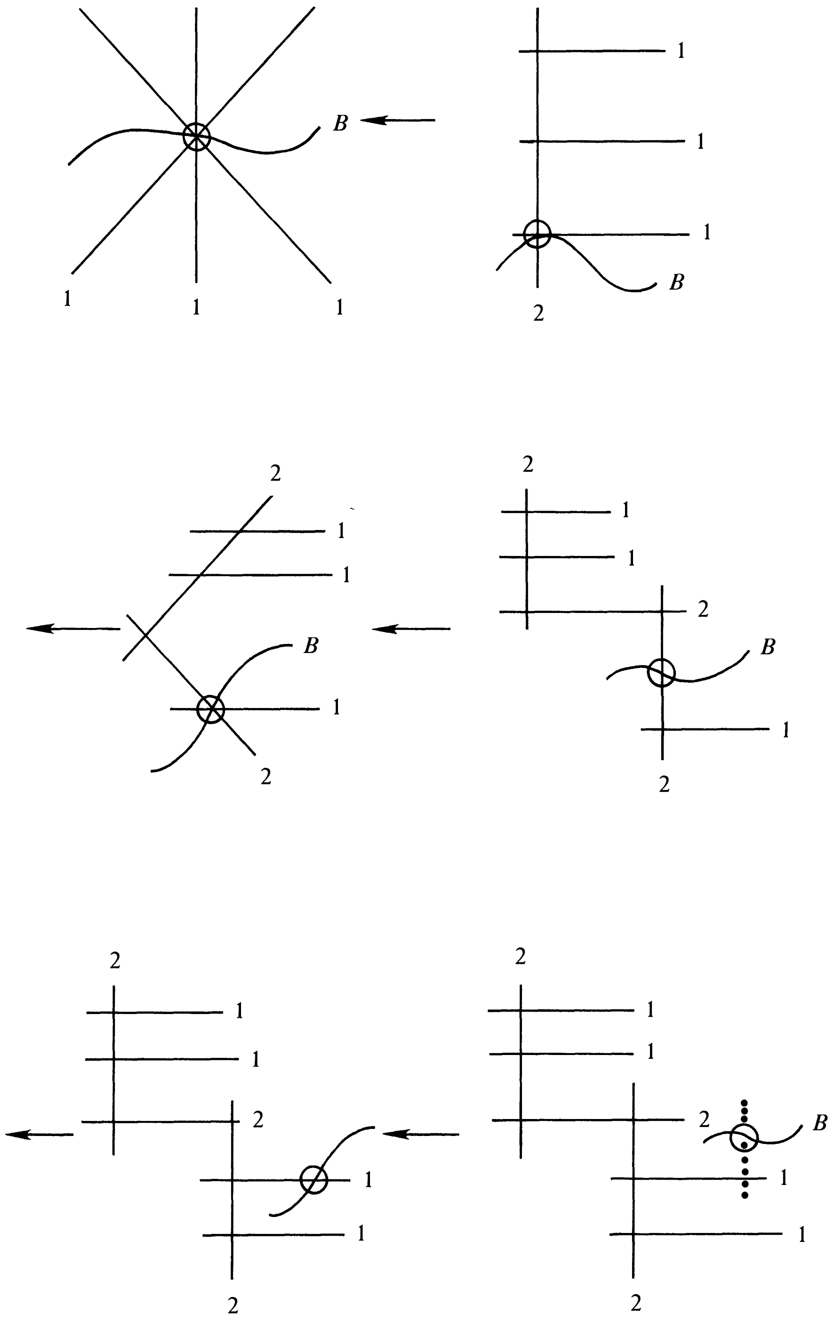
(Case 1):



(Case 2):

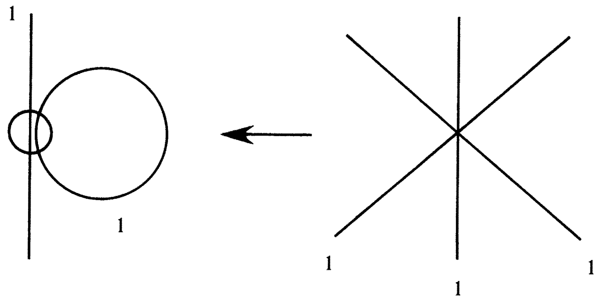


(Case 3):

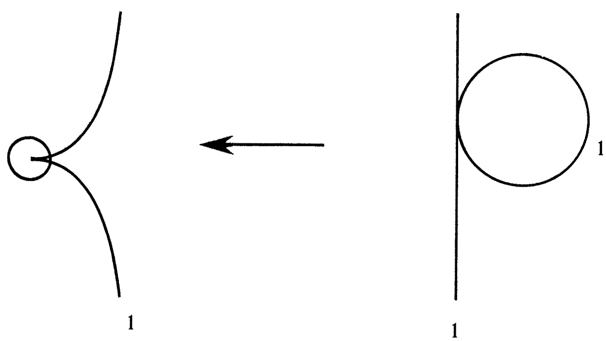


We see by inspecting these three cases that the result follows for the case (b).

To complete the proof we note that the situation of case (b) is obtained as soon as one blow up is performed on the cubic of case (c):



So any smooth cubic B passing through the base point in case (c) will, after one blow-up, look like a cubic satisfying one of the 3 cases of base (b). The argument for case (b) above then applies. In case (a), one blow-up yields the situation of case (c):



So, arguing by reduction to case (b), the lemma is proved. Q.E.D.

Lemma 6.4 allows us to complete the proof of Theorem 6.1. Suppose we are in case (e) and there exists a member A of the pencil P which is a triple line. Then if B is any other member of P, the pair A, B of generators satisfy the conditions of case 1 for instability. Thus the pencil P is unstable.

Suppose we are in case (d) and there exists a member A of the pencil consisting of a double line L and another line M. The possibilities for a second smooth member B are the following.

(1) B passes thru $p = L \cap M$, and has a flex there with tangent L.
(2) B passes thru p, and is tangent to L at p, and meets L in one other point q.
(3) B passes thru p, with tangent distinct from L, and meets L in two other points q, r.
(3) B passes thru p, with tangent distinct from L, and meets L in one other point q, with tangent L.
(4) B meets L in three distinct points, and $p \notin B$.
(5) B is tangent to L at a point $q \neq p$, and passes thru another point $r \neq p$, on L.
(6) B is tangent to L to order 3 at a point $q \neq p$.

Now in cases (1) and (2) the pair A, B satisfies the conditions of case 2 for instability. In case (6), the pair A, B satisfies the conditions of case 3. By explicitly separating the two members A, B by a sequence of blow-ups one can compute, as in the proof of Lemma 3.4, that in cases (3), (4), and (5) no component of multiplicity greater than two ever appears in the fibres of $\tilde{X}$. I shall omit the tedious verification of this fact. The theorem is proved. Q.E.D.

There are a few remarks to be made. Note first that for smooth pencils, instability is completely determined by the rational elliptic surface X. Thus if two distinct pencils give rise to the same rational elliptic surface, they are either both unstable or both semi-stable. This offers evidence that the moduli of cubic pencils is closely related to that of rational elliptic surfaces with section.

Secondly, note that certain pencils are semi-stable even though they have no smooth members! For example, the pencil generated by the cubic $yz^2 = x^3$ and the cubic $x^2z = y^2z$ is semi-stable. However, *all properly stable* pencils have smooth members. It is the computation of the properly stable pencils which we shall now present.

7. The characterization of properly stable pencils. Since we have a complete classification of properly stable pencils given by Proposition 5.2, we would like some characterization of them similar to the characterization of unstable pencils given by Theorem 6.1. In particular, if P is a properly stable pencil having a smooth member, then P will induce a rational elliptic surface $\tilde{X}$ and we can ask what properties of $\tilde{X}$ are implied by the stability of P. Conversely, given the surface $\tilde{X}$ induced by a pencil P, we can ask what properties of $\tilde{X}$ imply the stability of P. Again the answer to these questions can be expressed in terms of the singular fibres of $\tilde{X}$:

THEOREM 7.1. *Let P be a smooth pencil of cubics in $\mathbf{P}^2$, inducing the rational elliptic surface $\tilde{X}$. Then*

(i) *P is not properly stable* $\Leftrightarrow$ *$\tilde{X}$ contains a fibre $X_0 = \Sigma\, n_i C_i$ with $n_i \geq 2$ for some i.*
$\Leftrightarrow$ *$\tilde{X}$ contains a fibre of type I_N^*, IV^*, III^*, or II^* in Kodaira's notation for singular fibres of elliptic surfaces.*

(ii) *P is properly stable* $\Leftrightarrow$ *all fibres of X are reduced.*

Proof. (ii) is simply a restatement of the first equivalence of (i), and the second equivalence of (i) is an immediate consequence of Kodaira's classification of singular fibres of $\tilde{X}$ and the first equivalence. Hence it suffices to demonstrate the first equivalence.

Assume first that the pencil P is not properly stable. Since P has a smooth member, there must be coordinates in $\mathbf{P}^2$ and generators A, B of P satisfying either case 6, case 8, or case 9 of Proposition 5.2. In case 6, the generator A consists of a double line and one other line, or is a triple line. In either case the fibre it induces on $\tilde{X}$ will have a multiple component.

In case 9, A is either three concurrent lines, a double line plus another line, or a triple line. As above if A is one of the latter two, A clearly induces on $\tilde{X}$ a fibre with a multiple component. If A is three concurrent lines, meeting at the point Q, we see that in case 9, Q is a base point of the pencil P. To resolve the pencil we must then blow up the point Q at least once; the exceptional curve for this blow-up will have multiplicity two in the pencil, and so its proper transform on $\tilde{X}$ will have multiplicity two in its fibre.

In case 8, A is either a conic plus a tangent line, a double line plus another line or a triple line. If A is one of the latter two, we are done as above. If A is a conic plus a tangent line L, tangent at the point Q, and B is a cubic smooth at Q, and also tangent to the line L, (as is the situation in case 8), then to resolve the pencil we must blow up the point Q and the point infinitely near to Q in the direction of L. The exceptional curve for the second blow-up will have multiplicity two in the pencil, and its proper transform on $\tilde{X}$ will then have multiplicity two in its fibre.

This establishes one direction of the equivalence.

The converse is more complicated, as in the proof of Theorem 6.1. However, some of the intermediate results obtained in the course of that proof are still valid in our situation. Assume now that P induces the elliptic surface $\tilde{X}$ which has a fibre with a multiple component.

LEMMA 7.2. *P must contain a member A which is either*

(a) *a cuspidal cubic, with cusp at a base point,*
(b) *three concurrent lines, whose common point is a base point,*
(c) *a line and a conic tangent to the line, the point of tangency a base point,*
(d) *a double line and another line, or*
(e) *a triple line.*

The proof is identical to that of Lemma 6.3 and will not be repeated.

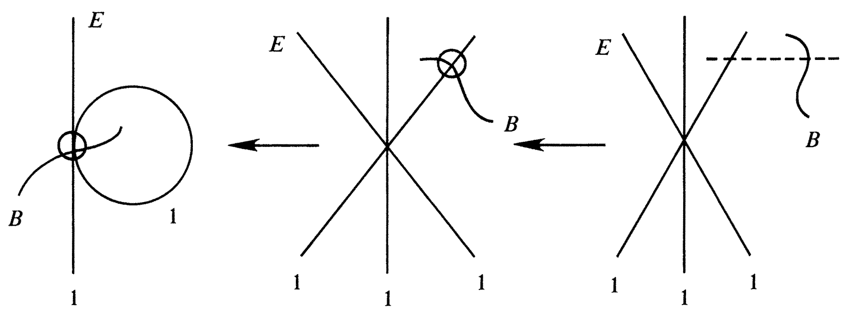
To prove that P is not properly stable, we will use the classification in Table 5.2 of non-properly stable pencils. Note that if P has a member A which is one of cases (b), (d), or (e), then P is not properly stable in cases 9, 6, and 6 respectively. We will take up cases (a) and (c) in turn. We will show that in these cases there are further restrictions on the pencil P, forcing P to be non-properly stable.

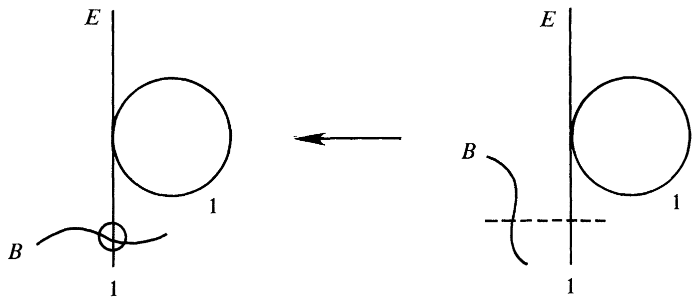
Assume P contains a member A which is a cuspidal cubic, with cusp at a base point Q of P. P has a smooth member by hypothesis so every other member of P is smooth at Q. To resolve the pencil we must blow up Q at least once; the exceptional curve E for this blow-up will have multiplicity 1 in its fibre on X since the multiplicity of Q on A is 2. The proper transform of any other member B of P will intersect E transversally since B must be smooth at Q. The proper transform of A is a smooth rational curve tangent to E. There are two cases to consider:

Case 1. $B \cap E \cap Q \neq \emptyset$.

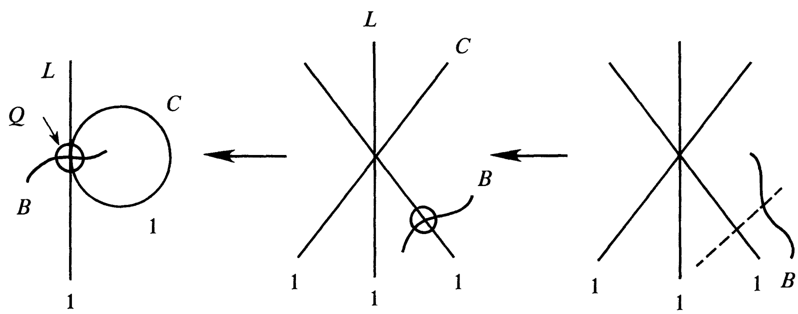
Case 2. $B \cap E \cap Q = \emptyset$.

In either case, it is easy to see that no multiple component is introduced in resolving the pencil.





Assume finally that P contains a member A which consists of line L and a conic C tangent to the line, the point Q of tangency a base point. If any other member B of P is either singular at Q or tangent to L at Q, then P is not properly stable in case 8. Thus assume all other members B of P pass thru Q meeting L transversally. It is then easy to see as above that no multiple component is introduced in resolving the pencil:



This completes the argument and finishes the proof of the theorem. Q.E.D.

As a final remark, let us observe that if P is a properly stable pencil then P always induces a rational elliptic surface.

THEOREM 7.3. *Let P be a properly stable pencil of cubics in* $\mathbf{P}^2$. *Then P has a smooth member and induces a rational elliptic surface X containing no fibres with non-reduced components.*

Proof. We will assume P has no smooth members and show it is not properly stable using the classification of Proposition 5.2 of non-properly stable pencils.

Assume then that P has no smooth members. It follows from Bertini's theorem [Sh] that one of the following conditions must hold:

(i) There is a base point of P at which all members of P are singular.
(ii) The base locus of P contains a line.
(iii) The base locus of P contains a conic Q and P is the pencil $Q + \overline{P}$ where $\overline{P}$ is a pencil of lines through a point R of Q.
(iv) The base locus of P contains a conic Q and P is the pencil $Q + \overline{P}$ where $\overline{P}$ is a pencil of lines through a point R not in Q.

We shall show that in all of these cases, P is not properly stable.

In case (i), P is not properly stable in case 10 of Proposition 5.2. In case (ii), P is not properly stable in case 7 of Proposition 5.2. In case (iii), P is not properly stable in case 10 again; the point R is a singular point of every member of P. In case (iv), we note that there is a member of P which consists of the conic Q and one of the two lines through the point R tangent to Q. Take this member as the generator A of P. Let B be any other member of P. We see then that A and B generate P, A consists of a conic Q and a line L tangent to the conic and B is also tangent to the line L at the point $Q \cap L$. Thus P is unstable in case 8 of Proposition 5.2. Q.E.D.

This completes the analysis of proper stability for pencils of cubics in $\mathbf{P}^2$. We note that the stability or instability of a pencil P depends only on the induced elliptic surface $\tilde{X}$; if two pencils induce isomorphic elliptic surfaces, then the pencils are either both unstable, both strictly semi-stable, or both stable.

8. The characterization of strictly semi-stable smooth pencils. Let P be a strictly semi-stable smooth pencil; recall that then P is not properly stable, but not unstable (Definition 3.1). Combining Theorems 6.1 and 7.1, we have

THEOREM 8.1. *Let P be a smooth pencil of cubics in $\mathbf{P}^2$, inducing the rational elliptic surface $\tilde{X}$. Then*

P is strictly semi-stable $\Leftrightarrow$ $\tilde{X}$ contains a fibre of type $I_N{}^*$ (using Kodaira's notation).

Proof. We must show that if $\tilde{X}$ contains a fibre of type $I_N{}^*$, then X does not contain a fibre of type II*, III*, or IV*. For this we use the 'canonical bundle formula' of Kodaira: let $M(T)$ be the number of singular fibres of type T or an elliptic surface X with section. Then

$$12\chi(\mathcal{O}_{\tilde{X}}) = \sum_{N=1}^{\infty} NM(I_N) + \sum_{N=1}^{\infty} (N+6)M(I_N{}^*)$$

$$+ 2M(\mathrm{II}) + 3M(\mathrm{III}) + 4M(\mathrm{IV})$$

$$+ 10M(\mathrm{II}^*) + 9M(\mathrm{III}^*) + 8M(\mathrm{IV}^*). \quad \text{[Ko]}$$

In our situation, $\tilde{X}$ is rational so $\chi(\mathcal{O}_{\tilde{X}}) = 1$. However, if $\tilde{X}$ contains a fibre of type $I_N{}^*$ and one of type II*, III*, or IV*, the right side of the equation is bigger than 12. Q.E.D.

We will now describe the strictly semi-stable smooth pencils whose orbits are closed. By using the numerical criterion (Theorem 3.7(e)) and by applying arguments similar to those used in section 4, one can easily derive the following vanishing criterion for such pencils.

PROPOSITION 8.2. *A strictly semi-stable pencil P has a closed orbit if and only if there exist coordinates in* $\mathbf{P}^2$ *such that, if* $(m_{ijk\ell})$ *are the induced coordinates of P, then either*

Case 1. All $m_{ijk\ell}$ vanish except possibly

$$m_{0020},\ m_{0021},\ m_{0120},\ m_{0121},\ m_{0220},\ m_{0221},\ m_{0320},$$
$$m_{0321},\ m_{1011},\ m_{1012},\ \textit{and}\ m_{1112},$$

or

Case 2. All $m_{ijk\ell}$ vanish except possibly

$$m_{0030},\ m_{0121},\ m_{0212},\ m_{0220},\ m_{0311},\ m_{1012},\ \textit{and}\ m_{1020}.$$

As is the case for instability, one can express these vanishing criteria in terms of 'standard' generators for the pencil P. The result of this calculation is given below.

PROPOSITION 8.3. *A strictly semi-stable smooth pencil P has a closed orbit if and only if there exist coordinates* $[x, y, z]$ *of* $\mathbf{P}^2$ *and two* *generators A, B of P with equations* $F_A = 0$, $F_B = 0$, *respectively, satisfying either*

Case 1. $F_A = x^2z$, $F_B = b_{00}z^3 + b_{01}yz^2 + b_{02}y^2z + b_{03}y^3$, $b_{00} \neq 0$, F_B factors into distinct linear factors,

or

Case 2. $F_A = x(a_{12}y^2 + a_{20}xz)$, $F_B = z(b_{02}y^2 + b_{10}xz)$, $a_{12}a_{20}b_{02}b_{10} \neq 0$, $a_{12}b_{10} - a_{20}b_{02} \neq 0$.

In Case 1, A is a double line plus a single line, and B is three concurrent lines meeting at a point on the single line of A. In Case 2, both A and B are smooth conics plus tangent lines; moreover, the two conics are tangent to each other at the points of tangency with the lines. See Figure 8.4.

(Case 1)

(Case 2)

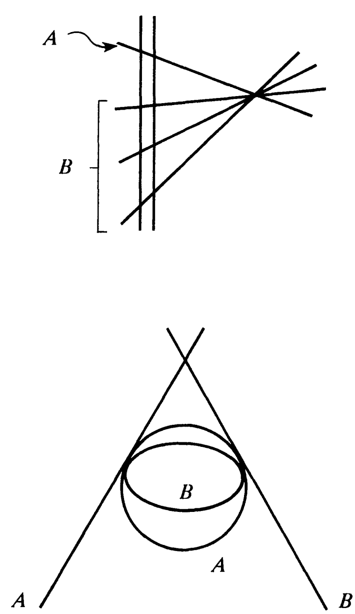


Figure 8.4 *Graphs of smooth strictly semi-stable pencils with closed orbits*

The characterization of smooth strictly semi-stable pencils with closed orbits in terms of the induced rational elliptic surface $\tilde{X}$ is as follows:

THEOREM 8.5. *Let P be a smooth strictly semi-stable pencil of cubic curves in* $\mathbf{P}^2$ *inducing the rational elliptic surface* $\tilde{X}$. *Then the orbit of P is closed* $\Leftrightarrow$ $\tilde{X}$ *contains two singular fibres of type* I_0^*.

The implication ($\Rightarrow$) can be proved using Proposition 8.3; the two singular fibres are precisely the transforms on $\tilde{X}$ of the given generators A and B. The converse can be verified using arguments identical to those of sections 6 and 7 and this will be omitted.

Note that by the canonical bundle formula, a rational elliptic surface $\tilde{X}$ with section, which has two fibres of type I_0^*, has no other singular fibres. Moreover, since the j-function (associating to every point p of the base curve $\mathbf{P}^1$ of $\tilde{X}$, the j-invariant of the fibre over p) is a rational function on $\mathbf{P}^1$, and since this j-function is finite at smooth fibres and at fibres of type I_0^* (see [Ko]), this j-function in our case must be a constant. In the situation of Case 1 for example, this j-value corresponds to the cross-ratio of the four points (on the double line of the generator A) obtained by intersection with the three lines of B and the other single line of A.

Remark. To produce rational elliptic surfaces X with section as above, we may start with a pencil P of cubics with generators satisfying Case 1 or Case 2. The two cases are clearly *not* projectively equivalent. However, they are birationally equivalent, i.e., any rational elliptic surface as above can be blown down to $\mathbf{P}^2$ in (at least) two ways, one of which leads to a pencil with generators in case (1), the other to one with generators in case (2).

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