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Persson's list of singular fibers for a rational elliptic surface

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0. Introduction

This work is the product of my attempts to understand the list of possible singular fibres which can occur on a rational elliptic surface with section, which has recently been produced by U. Persson [P]. In his work, he constructs all the possible configurations, and proves the impossibility of the ones which cannot exist, by using very geometric arguments; these all boil down to various constructions involving plane curves of low degree, and distinguished points on these curves, having prescribed singularities. It is an impressive illustration of the beauty of the geometry of plane curves, and any interested reader will have a lot of fun studying the necessary constructions.

In this article I will concentrate on more combinatorial criteria for the existence of a rational elliptic surface with prescribed singular fibres. In this way one is able to reproduce Persson's list, and it is hoped that the two approaches complement and reinforce one another. In addition, one obtains a completely different construction for the surfaces which exist, and gives an independent verification for the final list.

One can take a rational elliptic surface with section S, and blow down all components of fibers which do not meet S; one obtains an elliptic surface with a finite number of rational double points. The classification of the rational double point configurations which can be obtained this way has been done: the reader should consult [D], [L], [T], and [U]. This classification ignores the difference between several fiber types: I_0, I_1, and II contribute no rational double point, I_2 and III both give an A_1 singularity, and I_3 and IV both

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give an A_2 singularity. Moreover, from the point of view of the J-function, these fiber types are quite different. Therefore this work is a detailed refinement of these coarser classifications.

1. Some numerical criteria

Every rational elliptic surface with section is the blowup of the plane $\mathbb{P}^2$ at nine points, and therefore has Picard number 10; the section uses up one dimension, and so the number of components of any one singular fiber is at most nine, since these components are always independent in the Neron-Severi group, even modulo the section.

The possible singular fibers with at most nine components are listed in Table (1.1), together with certain numerical characters which will be explained more fully below.

The first column of Table (1.1) is the Kodaira name of the singular fiber. The second column, e, is the Euler number of that fiber.

Each Kodaira fiber consists of a certain number of components, and the third column, r, is one less than the number of components in the fiber. The given section of the elliptic fibration can only meet one component, and so r is the number of components of the fiber which do not meet the section. These components are linearly independent in the Picard group of the surface, and in fact generate a negative definite sublattice of the Picard group of rank r. The fourth column, δ, is the discriminant of this sublattice. The last column is the name for this root lattice.

The fifth column, J, is the modulus of the fiber. All singular fibers except for I_0^* have $J=0$, 1 or ∞; fibers of type I_0^* (and smooth fibers, i.e., type I_0) can have any finite J value. The sixth column, m, is the multiplicity of the J function, thought of as a map from the base curve to $\mathbb{P}^1$.

Every rational elliptic surface with section can be written in Weierstrass form

$$y^2 = x^3 + Ax + B \tag{1.2}$$

where A and B are polynomials in an affine variable t on the base curve $\mathbb{P}^1$ of degrees at most 4 and 6, respectively. This representation exhibits the surface as a double cover of the rational ruled surface $\mathbb{F}_2$, branched over the (-2)-curve and over a trisection $\mathscr{D}$. The curve $\mathscr{D}$ meets the fiber of $\mathbb{F}_2$ in general at three distinct smooth points (where the elliptic surface has a smooth fiber), and, where the elliptic fiber is singular, at 1 or 2 points, which may in fact be singular points of $\mathscr{D}$. (If there are 2 points, at most one is singular on $\mathscr{D}$.) This singularity is a "simple" curve singularity of type A_n, D_n, or E_n, using the notation in [BPV]; note that this is the same notation as for the associated root lattice. The seventh column, γ, is the genus drop contributed by the singularity of $\mathscr{D}$ to the computation of the geometric genus of $\mathscr{D}$, if $\mathscr{D}$ would be irreducible.

This completes the description of the elements of Table (1.1). There are several "easy" numerical criteria which one can apply to a possible configuration of singular fibers on a rational elliptic surface with section, involving simply a computation with the numbers in the above table. These criteria will now be discussed.

Table (1.1)

Fiber	e	r	δ	J	m	γ	Lattice
II^*	10	8	1	0	2 mod 3	4	E_8
I_4^*	10	8	4	∞	4	5	D_8
I_9	9	8	9	∞	9	4	A_8
III^*	9	7	2	1	1 mod 2	4	E_7
I_3^*	9	7	4	∞	3	4	D_7
I_8	8	7	8	∞	8	4	A_7
IV^*	8	6	3	0	1 mod 3	3	E_6
I_2^*	8	6	4	∞	2	4	D_6
I_7	7	6	7	∞	7	3	A_6
I_1^*	7	5	4	∞	1	3	D_5
I_6	6	5	6	∞	6	3	A_5
				0	0 mod 3		
I_0^*	6	4	4	1	0 mod 2	3	D_4
				$\neq 0,1$	–		
I_5	5	4	5	∞	5	2	A_4
I_4	4	3	4	∞	4	2	A_3
IV	4	2	3	0	2 mod 3	1	A_2
I_3	3	2	3	∞	3	1	A_2
III	3	1	2	1	1 mod 2	1	A_1
I_2	2	1	2	∞	2	1	A_1
II	2	0	1	0	1 mod 3	0	0
I_1	1	0	1	∞	1	0	0
				0	0 mod 3		
I_0	0	0	1	1	0 mod 2	0	0
				$\neq 0,1$	–		

The most basic equation which restricts the singular fibers is that the sum of the Euler numbers of the singular fibers must be 12:

$$\Sigma e = 12. \tag{1.3}$$

Since every singular fiber contributes at least one to this sum, this equation effectively bounds the number of possible configurations. In fact, the number of configurations satisfying (1.3) is 379. It is this set of 379 configurations which form our "database", from which we will further discard configurations as they are proved to be impossible.

The sublattices of the Picard group of the surface generated by components of fibers not meeting the section are all negative definite, and mutually orthogonal, so that their direct sum forms a sublattice of rank Σr. This sublattice is perpendicular to the class of the section, and to the class of the fiber; since the Picard group has rank 10, the direct sum can have rank at most 8:

$$\Sigma r \leqq 8. \tag{1.4}$$

There is a refinement of (1.4) which is useful in the extremal case of $\Sigma r = 8$. The Picard group of the surface is unimodular, and the rank 2 sublattice generated by the section and the fiber is unimodular; therefore the orthogonal complement to this rank two sublattice is a unimodular sublattice of rank 8. It must be even, since the canonical class K is minus a fiber, and any class perpendicular to K must have even square. Finally it is negative definite, since the Picard group has signature (1, 9), and the rank 2 sublattice generated by the section and the fiber has signature (1, 1). Therefore this perpendicular lattice must be isomorphic to the E_8 lattice. Therefore, if the direct sum of the root lattices has rank 8, it has a unimodular overlattice, and so must have a perfect square discriminant:

(1.5) If $\Sigma r = 8$, then $\Pi \delta$ is a perfect square.

(See also [MP, Corollary (2.6)

The trisection $\mathscr{D}$ on $\mathbb{F}_2$ which forms part of the branch locus of the double cover representation of the elliptic surface has arithmetic genus 4. If the surface has a fiber of type II, IV, IV^*, or II^*, the curve $\mathscr{D}$ meets the fiber of $\mathbb{F}_2$ at only one point, which is unibranch on $\mathscr{D}$. Therefore, if the surface has one of these singular fibers, $\mathscr{D}$ must be irreducible. In that case, the total genus drop of the singularities of $\mathscr{D}$ is at most 4.

Let i_n denote the number of fibers of type I_n on the surface; similarly define i_n^*, ii, iii, iv, iv^*, iii^*, and ii^*. The above condition can be expressed as follows:

(1.6) If $(ii + iv + iv^* + ii^*) \geqq 1$, then $\Sigma \gamma \leqq 4$.

The remaining criteria involve a study of the J-map from the base curve C (isomorphic to $\mathbb{P}^1$) to $\mathbb{P}^1$. The degree of the J-map is equal to the number of its poles, counted with multiplicity; since every pole of J is associated to a singular fiber of type I_n^* or I_n, this gives

(1.7) $$\text{degree}(J) = \sum_{n \geqq 1} n(i_n + i_n^*).$$

Hence the degree of the J-map can be determined from the singular fibers. If the degree of J is 0, so that J is a constant map, then all singular fibers must have the same J-value; by analyzing the singular fibers with finite J, this can be expressed as follows.

(1.8) If degree$(J) = 0$ then *either* $(ii + iv + iv^* + ii) = 0$
or $(iii + iii^*) = 0$.

If the degree of J is not 0, then there are several requirements. Firstly, the sum of the multiplicities over every point of $\mathbb{P}^1$ must equal the degree of J. In particular, using the J value of 0, we find that

(1.9) If degree$(J) \neq 0$ then
degree$(J) - ii - iv^* - 2iv - 2ii^* \geqq 0$
and is divisible by 3.

Using the J-value of 1, we have

(1.10) If degree$(J) \neq 0$ then
degree$(J) - iii - iii^* \geqq 0$
and is divisible by 2.

Statement (1.9) is obtained by considering the multiplicities of the J-map at the fibers with $J=0$, modulo 3; (1.10) is gotten by considering the multiplicities of J at the fibers with $J=1$, modulo 2.

A more subtle condition is obtained by analyzing the ramification of the J-map. It is easy to see that the ramification of J over 0 is minimized when one assumes that every fiber of type II and IV^* has $m=1$, every fiber of type IV and II^* has $m=2$, and every fiber of type I_0 and I_0^* (with $J=0$) has $m=3$. If $d=\text{degree}(J)$, then there must be $(d-ii-iv^*-2iv-2ii^*)/3$ fibers of type I_0 and I_0^* with $J=0$, giving a minimum contribution of $(iv+ii^*)+2(d-ii-iv^*-2iv-2ii^*)/3$ to the total ramification of J.

Similarly, the ramification of J over $J=1$ is minimized when one assumes that every fiber of type III or III^* has $m=1$, and every fiber of type I_0 and I_0^* (with $J=1$) has $m=2$; then there must be $(d-iii-iii^*)/2$ fibers of type I_0 or I_0^* with $J=1$, giving a minimum contribution of $(d-iii-iii^*)/2$ to the total ramification of J.

Finally, over $J=\infty$, we have the fibers of type I_n and I_n^* with $n\geqq 1$, contributing exactly $\sum_{n\geqq 1}(n-1)(i_n+i_n^*)$ to the total ramification.

From Hurwitz's formula, since the base curve has genus 0, the total ramification of the J-map is $2d-2$. Therefore:

$$\begin{aligned}(iv+ii^*)&+2(d-ii-iv^*-2iv-2ii^*)/3\\ &+(d-iii-iii^*)/2+\sum_{n\geqq 1}(n-1)(i_n+i_n^*)\leqq 2d-2.\end{aligned}$$

By using (1.7) and collecting terms, one finds that

$$(1.11)\quad \tfrac{1}{6}\Big[\sum_{n\geqq 1}(6-n)(i_n+i_n^*)+2(iv+ii^*)+3(iii+iii^*)+4(ii+iv^*)-12\Big]\geqq 0.$$

The quantity in the brackets is divisible by 6, and the entire expression after dividing by 6 represents the "extra" ramification of the J-map, not accounted for by the above considerations. In particular, if the above quantity is 0, then every assumption made above about the multiplicities of J is valid, and every ramification point and multiplicity for the J-map is known. In what follows let x be the left hand side of (1.11).

The formula (1.11) is equivalent to Proposition 3.4 of [MP].

2. The impossible configurations

Of the 379 sets of singular fibers which satisfy (1.3), exactly 100 do not occur. A list follows in Table (2.1), ordered lexicographically as in [P].

Of the above 100 configurations, 86 are ruled out by one or more of the numerical criteria from Sect. 1. The fourteen which are not are numbers 46, 56, 67, 68, 69, 73, 76, 79, 80, 82, 86, 92, 93, and 94.

Probably the easiest of these to dismiss is # 56; assume that such a surface exists. Assume that the I_0^* fiber is over $t=\infty$, and the I_3 fiber is over $t=0$. Then, in the Weierstrass form (1.2) for this surface, the degree of A is at most 2, and the degree of B is at most 3. By replacing A by t^2A and B by t^3B,

Table (2.1). The impossible configurations of singular fibers

# : Fibers	$[\Sigma r, \Pi\delta, d, \Sigma\gamma, x]$	Reasons for non-existence
1: $II^* I_2$	[9, 2, 2, 5, −1]	(1.4), (1.6), (1.11)
2: $I_4^* I_2$	[9, 8, 6, 6, −1]	(1.4), (1.11)
3: $I_4^* II$	[8, 4, 4, 5, −1]	(1.6), (1.11)
4: $I_9 I_3$·	[10, 27, 12, 5, −2]	(1.4), (1.11)
5: $I_9 III$	[9, 18, 9, 5, −2]	(1.4), (1.11)
6: $I_9 I_2 I_1$	[9, 18, 12, 5, −1]	(1.4), (1.11)
7: $I_9 II I_1$	[8, 9, 10, 4, −1]	(1.11)
8: $III^* I_3$	[9, 6, 3, 5, −1]	(1.4), (1.11)
9: $I_3^* I_3$	[9, 12, 6, 5, −1]	(1.4), (1.11)
10: $I_3^* III$	[8, 8, 3, 5, −1]	(1.5), (1.11)
11: $I_3^* I_2 I_1$	[8, 8, 6, 5, 0]	(1.5)
12: $I_8 I_4$	[10, 32, 12, 6, −2]	(1.4), (1.11)
13: $I_8 IV$	[9, 24, 8, 5, −2]	(1.4), (1.6), (1.11)
14: $I_8 I_3 I_1$	[9, 24, 12, 5, −1]	(1.4), (1.11)
15: $I_8 III I_1$	[8, 16, 9, 5, −1]	(1.11)
16: $I_8 I_2 I_2$	[9, 32, 12, 6, −1]	(1.4), (1.11)
17: $I_8 I_2 II$	[8, 16, 10, 5, −1]	(1.6), (1.11)
18: $I_8 II II$	[7, 8, 8, 4, −1]	(1.11)
19: $IV^* I_4$	[9, 12, 4, 5, −1]	(1.4), (1.6), (1.11)
20: $IV^* I_2 I_2$	[8, 12, 4, 5, 0]	(1.5), (1.6)
21: $I_2^* I_4$	[9, 16, 6, 6, −1]	(1.4), (1.11)
22: $I_2^* IV$	[8, 12, 2, 5, −1]	(1.5), (1.6), (1.11)
23: $I_2^* I_3 I_1$	[8, 12, 6, 5, 0]	(1.5)
24: $I_2^* I_2 II$	[7, 8, 4, 5, 0]	(1.6)
25: $I_7 I_5$	[10, 35, 12, 5, −2]	(1.4), (1.11)
26: $I_7 I_4 I_1$	[9, 28, 12, 5, −1]	(1.4), (1.11)
27: $I_7 IV I_1$	[8, 21, 8, 4, −1]	(1.5), (1.11)
28: $I_7 I_3 I_2$	[9, 42, 12, 5, −1]	(1.4), (1.11)
29: $I_7 I_3 II$	[8, 21, 10, 4, −1]	(1.5), (1.11)
30: $I_7 I_3 I_1 I_1$	[8, 21, 12, 4, 0]	(1.5)
31: $I_7 III I_2$	[8, 28, 9, 5, −1]	(1.5), (1.11)
32: $I_7 III II$	[7, 14, 7, 4, −1]	(1.11)
33: $I_7 I_2 I_2 I_1$	[8, 28, 12, 5, 0]	(1.5)
34: $I_1^* I_5$	[9, 20, 6, 5, −1]	(1.4), (1.11)
35: $I_1^* I_3 I_2$	[8, 24, 6, 5, 0]	(1.5)
36: $I_6 I_6$	[10, 36, 12, 6, −2]	(1.4), (1.11)
37: $I_6 I_0^*$	[9, 24, 6, 6, −2]	(1.4), (1.11)
38: $I_6 I_5 I_1$	[9, 30, 12, 5, −1]	(1.4), (1.11)
39: $I_6 I_4 I_2$	[9, 48, 12, 6, −1]	(1.4), (1.11)
40: $I_6 I_4 II$	[8, 24, 10, 5, −1]	(1.5), (1.6), (1.11)
41: $I_6 I_4 I_1 I_1$	[8, 24, 12, 5, 0]	(1.5)
42: $I_6 IV I_2$	[8, 36, 8, 5, −1]	(1.6), (1.11)
43: $I_6 IV II$	[7, 18, 6, 4, −1]	(1.11)
44: $I_6 I_3 I_3$	[9, 54, 12, 5, −1]	(1.4), (1.11)
45: $I_6 I_3 III$	[8, 36, 9, 5, −1]	(1.11)
46: $I_6 I_3 II I_1$	[7, 18, 10, 4, 0]	(2.6.6)
47: $I_6 III III$	[7, 24, 6, 5, −1]	(1.11)
48: $I_6 I_2 I_2 I_2$	[8, 48, 12, 6, 0]	(1.5)
49: $I_6 I_2 I_2 II$	[7, 24, 10, 5, 0]	(1.6)
50: $I_0^* I_5 I_1$	[8, 20, 6, 5, −1]	(1.5), (1.11)

Table (2.1) continued

#: Fibers	$[\Sigma r, \Pi\delta, d, \Sigma\gamma, x]$	Reasons for non-existence
51: $I_0^* \, I_4 \, I_2$	[8, 32, 6, 6, −1]	(1.5), (1.11)
52: $I_0^* \, I_4 \, II$	[7, 16, 4, 5, −1]	(1.6), (1.11)
53: $I_0^* \, IV \, I_2$	[7, 24, 2, 5, −1]	(1.6), (1.11)
54: $I_0^* \, I_3 \, I_3$	[8, 36, 6, 5, −1]	(1.11)
55: $I_0^* \, I_3 \, III$	[7, 24, 3, 5, −1]	(1.11)
56: $I_0^* \, I_3 \, I_2 \, I_1$	[7, 24, 6, 5, 0]	Twist of # 11
57: $I_0^* \, I_2 \, I_2 \, II$	[6, 16, 4, 5, 0]	(1.6)
58: $I_5 \, I_5 \, I_2$	[9, 50, 12, 5, −1]	(1.4), (1.11)
59: $I_5 \, I_5 \, II$	[8, 25, 10, 4, −1]	(1.11)
60: $I_5 \, I_4 \, I_3$	[9, 60, 12, 5, −1]	(1.4), (1.11)
61: $I_5 \, I_4 \, III$	[8, 40, 9, 5, −1]	(1.5), (1.11)
62: $I_5 \, I_4 \, I_2 \, I_1$	[8, 40, 12, 5, 0]	(1.5)
63: $I_5 \, IV \, I_3$	[8, 45, 8, 4, −1]	(1.5), (1.11)
64: $I_5 \, IV \, III$	[7, 30, 5, 4, −1]	(1.11)
65: $I_5 \, I_3 \, I_3 \, I_1$	[8, 45, 12, 4, 0]	(1.5)
66: $I_5 \, I_3 \, I_2 \, I_2$	[8, 60, 12, 5, 0]	(1.5)
67: $I_5 \, I_3 \, II \, II$	[6, 15, 8, 3, 0]	(2.6.3)
68: $I_5 \, III \, I_2 \, I_2$	[7, 40, 9, 5, 0]	(2.5.2)
69: $I_5 \, I_2 \, I_2 \, I_2 \, I_1$	[7, 40, 12, 5, 1]	(2.5.2)
70: $I_4 \, I_4 \, I_4$	[9, 64, 12, 6, −1]	(1.4), (1.11)
71: $I_4 \, I_4 \, IV$	[8, 48, 8, 5, −1]	(1.5), (1.6), (1.11)
72: $I_4 \, I_4 \, I_3 \, I_1$	[8, 48, 12, 5, 0]	(1.5)
73: $I_4 \, I_4 \, III \, I_1$	[7, 32, 9, 5, 0]	(2.6.4)
74: $I_4 \, I_4 \, I_2 \, II$	[7, 32, 10, 5, 0]	(1.6)
75: $I_4 \, IV \, IV$	[7, 36, 4, 4, −1]	(1.11)
76: $I_4 \, IV \, I_3 \, I_1$	[7, 36, 8, 4, 0]	(2.5.1)
77: $I_4 \, IV \, I_2 \, I_2$	[7, 48, 8, 5, 0]	(1.6)
78: $I_4 \, I_3 \, I_3 \, I_2$	[8, 72, 12, 5, 0]	(1.5)
79: $I_4 \, I_3 \, I_3 \, II$	[7, 36, 10, 4, 0]	(2.5.1)
80: $I_4 \, I_3 \, I_3 \, I_1 \, I_1$	[7, 36, 12, 4, 1]	(2.5.1)
81: $I_4 \, I_2 \, I_2 \, I_2 \, II$	[6, 32, 10, 5, 1]	(1.6)
82: $IV \, IV \, I_3 \, I_1$	[6, 27, 4, 3, 0]	(2.6.1)
83: $IV \, IV \, III \, I_1$	[5, 18, 1, 3, –]	(1.9)
84: $IV \, IV \, I_2 \, II$	[5, 18, 2, 3, –]	(1.9)
85: $IV \, IV \, II \, I_1 \, I_1$	[4, 9, 2, 2, –]	(1.9)
86: $IV \, I_3 \, I_3 \, II$	[6, 27, 6, 3, 0]	(2.6.2)
87: $IV \, III \, III \, II$	[4, 12, 0, 3, –]	(1.8)
88: $IV \, III \, II \, II \, I_1$	[3, 6, 1, 2, –]	(1.9)
89: $IV \, I_2 \, I_2 \, I_2 \, I_2$	[6, 48, 8, 5, 1]	(1.6)
90: $IV \, I_2 \, II \, II \, II$	[3, 6, 2, 2, –]	(1.9)
91: $IV \, II \, II \, II \, I_1 \, I_1$	[2, 3, 2, 1, –]	(1.9)
92: $I_3 \, I_3 \, I_3 \, III$	[7, 54, 9, 4, 0]	(2.6.5)
93: $I_3 \, I_3 \, I_3 \, II \, I_1$	[6, 27, 10, 3, 1]	(2.11)
94: $I_3 \, I_3 \, I_2 \, I_2 \, I_2$	[7, 72, 12, 5, 1]	(2.5.3)
95: $III \, III \, III \, II \, I_1$	[3, 8, 1, 3, –]	(1.10)
96: $III \, III \, II \, II \, II$	[2, 4, 0, 2, –]	(1.8)
97: $III \, II \, II \, II \, II \, I_1$	[1, 2, 1, 1, –]	(1.9)
98: $I_2 \, I_2 \, I_2 \, I_2 \, I_2 \, II$	[5, 32, 10, 5, 2]	(1.6)
99: $I_2 \, II \, II \, II \, II \, II$	[1, 2, 2, 1, –]	(1.9)
100: $II \, II \, II \, II \, II \, I_1 \, I_1$	[0, 1, 2, 0, –]	(1.9)

we obtain a surface with an I_3^* fiber over $t=0$, a smooth fiber over $t=\infty$, and otherwise all fibers are left unchanged, i.e., the result is a fibration with singular fibers I_3^*, I_2, and I_1. This is forbidden by (1.5) (# 11 of Table (2.1)). Hence # 56 is impossible. This process of transfering "*"'s from one fiber to another will be referred to as "twisting" the fibration.

Although it is not strictly necessary to rule out any of the forbidden configurations, no discussion of this subject would be complete without mentioning the appropriate lattice conditions more thoroughly. As mentioned above, the direct sum of the root lattices associated to the singular fibers forms a sublattice of the perpendicular space to the rank 2 subspace generated by the section and the fiber inside the Picard group. This perpendicular space is isomorphic to the even negative definite rank 8 unimodular lattice E_8. Therefore, a necessary condition for the existence of the surface is that

(2.3) the direct sum of the associated root lattices must have an embedding into the E_8 lattice.

Although there are several ways to try to decide whether a given direct sum of root lattices embeds into E_8, I will discuss here the method of discriminant-forms. For a complete treatment, one may consult [N].

Given an even nondegenerate lattice L, the adjoint map embeds L into its dual $L^\#$. $L^\#$ inherits the bilinear form from L, with values in $\mathbb{Q}$. The cokernel G_L is a finite abelian group whose order is the discriminant of L, and it also inherits a quadratic form q_L with values in $\mathbb{Q}/\mathbb{Z}$; by definition, $q_L(x \bmod L) = \langle x, x\rangle/2 \bmod \mathbb{Z}$.

There are two facts we need to use. Firstly, if L and K are both embedded in a unimodular lattice U, with $L=K^\perp$ and $K=L^\perp$, then $G_L \cong G_K$. Secondly, the overlattices of a lattice M are classified by the totally isotropic subgroups of G_M. Indeed, if $H \subset G_M$ is a totally isotropic subgroup corresponding to $M \subset N$, then $G_N \cong H^\perp/H$.

Now assume that a lattice R embeds into E_8. Let $L=R^{\perp\perp}$ and $K=R^\perp$. By the first remark, $G_L \cong G_K$. Since $R \subseteq L$, there is a totally isotropic subgroup $H \subset G_R$ such that $G_L \cong H^\perp/H$. Hence $G_K \cong H^\perp/H$. Now L and R have the same rank Σr, and K has complementary rank $8-\Sigma r$, as does the dual $K^\#$. Therefore G_K can be generated by $8-\Sigma r$ elements. If one defines the length of a finite abelian group as the minimum number of generators, we have the following criterion.

(2.4) Proposition. *Let R be the direct sum of the root lattices of the singular fibers of a rational elliptic surface with section. Then there is a totally isotropic subgroup $H \subset G_R$ such that the length of $H^\perp/H$ is at most $8-\Sigma r$.*

This can be thought of as a refinement of the numerical criterion (1.5): if $\Sigma r=8$, then we require a totally isotropic subgroup H of G_R with $H=H^\perp$. Since $|G_G|=|H|\cdot|H^\perp|$, we see that $\Pi\delta=\mathrm{disc}(R)=|G_R|=|H|^2$.

For our purposes we will only need to know the discriminant forms for the lattices A_n. We have that $G_{A_n}\cong\mathbb{Z}/(n+1)\mathbb{Z}$, and $q(x)=-nx^2/(2n+2) \bmod \mathbb{Z}$. This suffices for the following examples.

(2.5) Corollary.

(2.5.1) *The root lattice $A_1 \oplus A_2 \oplus A_3$ does not embed into E_8.*

(2.5.2) *The root lattice* $A_1 \oplus A_1 \oplus A_1 \oplus A_4$ *does not embed into* E_8.
(2.5.3) *The root lattice* $A_1 \oplus A_1 \oplus A_1 \oplus A_2 \oplus A_2$ *does not embed into* E_8.

Proof. In each of these cases, the discriminant form group has no nontrivial isotropic elements at all! Let us check this for $A_2 \oplus A_2 \oplus A_3$. The finite group G is $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, and $q(a \bmod 3,\ b \bmod 3,\ c \bmod 4) = \frac{-1}{3}(a^2+b^2) - \frac{3}{8}c^2 \bmod \mathbb{Z}$. Since there can be no cancellation between the two terms above (the denominators are coprime), for this element (a, b, c) to be isotropic we must have $a^2+b^2 \equiv 0 \bmod 3$ and $c^2 \equiv 0 \bmod 8$. This forces $a=b=c=0$.

In general, a finite abelian group with quadratic form splits (orthogonally) into its p-Sylow subgroups, and no cancellation can occur among the p-Sylow subgroups of the groups above, so it suffices to show that there are no isotropic elements in the p-Sylow subgroup of the remaining cases. For A_4, the group is $\mathbb{Z}/5\mathbb{Z}$, and the result is almost obvious. For $A_2 \oplus A_2$, this is the calculation we have done above. Finally, for $A_1 \oplus A_1 \oplus A_1$, we have $G \cong (\mathbb{Z}/2\mathbb{Z})^3$, with $q(a, b, c) = -\frac{1}{4}(a^2+b^2+c^2)$, which is never integral unless $a=b=c=0 \bmod 2$. □

The above corollary rules out numbers 68, 69, 76, 79, 80, and 94. The remaining 7 to be discussed are numbers 46, 67, 73, 82, 86, 92, and 93; not only do these pass the discriminant-form test (2.4), but in fact the associated root lattice in each of these cases *can* be embedded into E_8. Note that in all but the last case #93, we have $x=0$, i.e., the ramification of the J-map occurs entirely above $J=0$, 1, and ∞, and is completely known. From the general theory of branched coverings of $\mathbb{P}^1$, there must exist three permutations σ_0, σ_1, and σ_∞, whose cycle structures are given by the multiplicities of the preimages over 0, 1, and ∞, and such that their product $\sigma_0 \sigma_1 \sigma_\infty = 1$ in S_d. Moreover they must generate a transitive subgroup of S_d. In the first 6 of the remaining 7 cases, it turns out that no such set of three permutations exist.

(2.6) Lemma. *There are not three permutations in* S_d *satisfying the above conditions with cycle structures*

(2.6.1)	$(2^2)(2^2)(13)$	(*here* $d=4$)
(2.6.2)	$(123)(2^3)(3^2)$	(*here* $d=6$)
(2.6.3)	$(1^2 3^2)(2^4)(35)$	(*here* $d=8$)
(2.6.4)	$(3^3)(12^4)(14^2)$	(*here* $d=9$)
(2.6.5)	$(3^3)(12^4)(3^3)$	(*here* $d=9$)
(2.6.6)	$(13^3)(2^5)(136)$	(*here* $d=10$).

Proof. It is perhaps easier to prove that three permutations α, β, and γ do not exist as above with $\alpha\beta=\gamma$; letting $\sigma_0=\alpha$, $\sigma_1=\beta$, and $\sigma_\infty^{-1}=\gamma$ gives the result. The easiest by far is (2.6.1); in fact, the elements of S_4 with cycle structure (2^2) form a subgroup, so this is impossible. In the above list α tends to have many 3-cycles, and β many 2-cycles. Note the following useful fact:

(2.7) If a 2-cycle of β is contained in a 3-cycle of α, the product must have a fixed point: $(abc)(ab)$ fixes b.

Consider (2.6.2); we may assume $\alpha=(1)(23)(456)$. Since γ has cycle structure (3^2), it has no fixed points, so by (2.7) none of the 2-cycles of β are contained within $\{4, 5, 6\}$. Hence neither (12) nor (13) is a 2-cycle of β, so we may assume that (14) is; by switching 2 and 3 if necessary we may assume that $\beta=(14)(25)(36)$. But then $\alpha\beta=(1534)(26)$, with the wrong cycle structure. This proves (2.6.2).

Consider (2.6.3); we may assume $\alpha=(1)(2)(345)(678)$. Since α and β generate a transitive subgroup of S_8, (12) is not a 2-cycle of β; hence we may assume (13) is. If either (24) or (25) were part of β, γ would have a fixed point by (2.7); hence we may assume (26) is. This leaves only 2 possibilities: $\beta=(13)(26)(47)(58)$ or $(13)(26)(48)(57)$. In the first case $\alpha\beta=(1483)(2756)$ and in the second $\alpha\beta=(146273)(58)$, neither of which is correct for γ.

For both (2.6.4) and (2.6.5), we may assume $\alpha=(123)(456)(789)$, and β fixes 1. If (23) were part of β, then $\{1, 2, 3\}$ would be left stable by α and β, contradicting transitivity; we may therefore assume (24) is part of β. If (35) is a part of β, then $\alpha\beta$ contains the 3-cycle (25), violating (2.6.4); moreover, this would force one of the 2-cycles of β to come from $\{7, 8, 9\}$, giving a fixed point to $\alpha\beta$, violating (2.6.5). Hence (35) cannot be a part of β. If (36) is a part of β, then $\alpha\beta$ contains the 2-cycle (34), violating both (2.6.4) and (2.6.5). Hence we may assume that (37) is a part of β. This leaves 3 possibilities for β: $\beta=(1)(24)(37)(56)(89)$, $\beta=(1)(24)(37)(58)(69)$, and $\beta=(1)(24)(37)(59)(68)$. In the first case the product has two fixed points, so is ruled out. In the second case $\alpha\beta=(125943867)$, and in the third case $\alpha\beta=(1257)(384)(69)$, neither of which are correct for either of (2.6.4) or (2.6.5). This proves that these two cases are impossible.

Finally consider (2.6.6); we may assume that $\alpha=(0)(123)(456)(789)$, and that (01) is part of β. By transitivity (23) cannot be a part of β, so we may assume that (24) is. If β contains (35), then $\alpha\beta$ contains (1025), violating (2.6.6). If β contains (36), then $\alpha\beta$ contains (34), also violating (2.6.6). Hence we may assume (37) is a part of β. Then $\alpha\beta$ sends 7 to 1 to 0 to 2 to 5 and 4 to 3 to 8; therefore (438) must be the 3-cycle of γ, i.e., γ must send 8 to 4, forcing β to send 8 to 6. Therefore (86) is a part of β, and this determines $\beta=(01)(24)(37)(59)(68)$; however now $\alpha\beta=(10257)(384)(69)$, which is not the correct cycle structure for γ. □

Lemma (2.6) serves to rule out #82, #86, #67, #73, #92, and #46 with (2.6.1)–(2.6.6) respectively. This leaves stubborn #93.

The configuration #93 has $x=1$, which means that there is one extra ramification point for the J-map which is not accounted for by the singular fibers. This gives the following possibilities for the ramification of the J-map:

(2.8.1) $(3^2 4)(2^5)(13^3)$ over 0, 1, and ∞
(2.8.2) $(136)(2^5)(13^3)$ over 0, 1 and ∞
(2.8.3) $(13^3)(2^3 4)(13^3)$ over 0, 1 and ∞
(2.8.4) $(13^3)(2^5)(13^3)(1^8 2)$ over 0, 1, ∞, and a fourth branch point λ.

The first can be ruled out most easily:

(2.9) Lemma. *There are no three permutations in S_{10} with cycle structure as in* (2.8.1) *whose product is the identity.*

Proof. We again use the notation used in the proof of (2.6). We may assume $\alpha=(0)(123)(456)(789)$ and that β has cycle structure (2^5). I'll show that the product γ cannot have cycle structure $(3^2 4)$. We may assume (01) is a part of β. If (23) is a part of β, then γ sends 3 to 3. Therefore we may assume (24) is a part of β. Then γ sends 1 to 0, 0 to 2, and 2 to 5, so that (1025) must be the 4-cycle of γ; in particular, γ sends 5 to 1, so β must send 5 to 3. Hence (35) is a part of β. In this case γ sends 4 to 3 and 3 to 6, so (436) is a part of γ; hence γ sends 6 to 6, forcing β to fix 6; this is a contradiction. □

We have already seen (2.8.2): this is case (2.6.6) in a different context. Let us turn to (2.8.3).

(2.10) Lemma. *There are not three permutations in S_{10} with cycle structure as in* (2.8.3) *whose product is the identity.*

Proof. Write $\alpha=(0)\,(123)\,(456)\,(789)$ as above, and let β have cycle structure $(2^3 4)$. Let us first assume that 0 is a part of the 4-cycle of β, which in full is $(0abc)$ for some a, b, and c. Write $\delta=(0b)$; since $(0abc)=\delta(0a)\,(bc)$, we may rewrite $\alpha\beta$ as $\alpha\delta(0a)\,(bc)$ (rest of β). Reassociating, we see that $\alpha\delta$ must have cycle structure $(3^2 4)$. If $\alpha\beta$ had cycle structure (13^3) as required, then this would imply the existence of three permutations with cycle structure as in (2.8.1) whose product is the identity, namely $\alpha\delta$, $(0a)\,(bc)$ (rest of β), and $(\alpha\beta)^{-1}$, contradicting Lemma (2.9). We may therefore assume that 0 is not a part of the 4-cycle of β, and that (01) is part of β.

Note that (102) must be a part of γ, so γ sends 2 to 1, hence β sends 2 to 3. Let us assume first that (23) is a part of β. Then 3 is fixed by γ, so γ has no other fixed points; by (2.7) we may assume (47) is a part of β. In the 4-cycle of β, 5 must occur; assume that 5 is sent to 6. Then γ sends 7 to 5 to 4 to 8, a contradiction. Assume that 5 is sent to 8 by β; then γ must contain (759), so γ sends 9 to 7, so β must send 9 to 9, also a contradiction. Hence β must send 5 to 9, implying that (57) is a part of γ; this contradiction shows that (23) cannot be a part of β, and that 2 and 3 occur in the 4-cycle of β.

Now neither 0, 1, 2, nor 3 can be fixed by γ, so we may assume that 4 is; hence β must send 4 to 6. If 46 completes the 4-cycle of β, which would then be (2346), then (635) would be part of γ, so γ send 5 to 6, and β would send 5 to 5, a contradiction. Therefore (46) is a 2-cycle of β. The remaining 2-cycle of β cannot have both elements from $\{7, 8, 9\}$ by (2.7); hence 5 must occur in the 2-cycle, and we can assume it is (57). Then γ sends 6 to 5 to 8, and so must send 8 to 6; hence β must send 8 to 4, a contradiction. □

This leaves us with the case (2.8.4). Assume that there exists 4 permutations as in (2.8.4) whose product is 1. Let δ be the 2-cycle, and let σ be the permutation with cycle structure (2^5). The product $\delta\sigma$ either has cycle structure $(1^2 2^4)$ (if δ is part of σ) or $(2^3 4)$ (if not). The second case cannot happen, by the previous lemma. Therefore we need only deal with the following case:

(2.11) Lemma. *The only* 3 *permutations in S_{10} with cycle structures* (13^3), $(1^2 2^4)$, *and* (13^3) *whose product is the identity is*

$$[(0)(123)(456)(789)]\cdot[(1)(4)(25)(36)(07)(89)]\cdot[(162)(354)(078)(9)]=1,$$

up to conjugacy.

Proof. Again we will let α and γ have cycle structure (13^3), β have cycle structure $(1^2 2^4)$, and prove that $\alpha\beta=\gamma$ implies the above for α, β, and γ^{-1}. We may assume as usual that $\alpha=(0)(123)(456)(789)$.

Assume that 0 is fixed by β, so that 0 is the fixed point of γ. Another must also be fixed by β, and we can assume that it is 1; by (2.7), (23) cannot be part of β, so we may assume that (24) is. Then γ sends 1 to 2 to 5, and so sends 5 to 1; hence β must send 5 to 3, so that (35) is part of β. Then γ sends 4 to 3 to 6, and so 6 to 4, forcing β to fix 6, a contradiction. Hence 0 is not fixed by β, and we may assume 1 is.

If the other fixed point of β is 2, then (123) is a part of γ, so 3 must be fixed by β; if the other fixed point of β is 3, then (312) is part of γ, and 2 must be fixed by β. Either way we have a contradiction, so we may assume that the other fixed point of β is 4.

Where is 0 sent to by β? If (02) is in β, then (120) is in γ, so (03) is in γ; if (03) is in β, then (3012 ...) is part of γ. Similar contradictions occur if (05) or (06) are in β. Hence we may assume (07) is in β.

Where is 2 sent by β? If (23) is in β, then (12) is in γ. If (26) is in β, then γ contains (1245 ...). If (28) is in β, then (129) is in γ, so (39) is in β, forcing (0837) in γ. If (29) is in β, then (12708 ...) is in γ. This leaves only (25) as a possibility.

So far $\beta=(1)(4)(07)(25)(\ldots)(\ldots)$, and so γ is now determined: $\gamma=(126)(453)(708)(9)$. This forces the rest of β to be (36) (89), giving the result above. □

This finally serves to rule out case (2.8.4), and # 93. The four permutations in this case must be [(0) (123) (456) (789)], [(14) (25) (36) (07) (89)], [(14)], and [(162) (354) (078) (9)], up to conjugacy, by Lemma (2.11). In this case {0, 7, 8, 9} is left stable by each of these, so the subgroup generated by them is not transitive.

3. The sufficiency of the existence of the *J*-map

The last several cases considered in the previous section were ruled out essentially because the J-map from the base curve $\mathbb{P}^1$ to the moduli space $\mathbb{P}^1$ could not exist. In this section I will indicate that a converse to the arguments used above exists: if one can construct an appropriate J-map, then a rational elliptic surface with the prescribed singular fibers exists.

What does an "appropriate" J-map mean? One can take a hint from the 'm' column of Table (1.1). Suppose a list of singular fibers is given, and the task is to construct a rational elliptic surface with exactly those singular fibers. Assume that the list of fibers satisfies the various numerical criteria of Sect. 1. Let $d=\text{degree}(J)$, which is computed using (1.7). Let us say that a map $J:\mathbb{P}^1\to\mathbb{P}^1$ *belongs to* the list of singular fibers if the multiplicities over 0, 1, and ∞ are as follows:

over 0: $(ii+iv^*)$ points of multiplicity 1,
$(iv+ii^*)$ points of multiplicity 2, and
$(d-ii-iv^*-2iv-2ii^*)/3$ points of multiplicity 3;
over 1: $(iii+iii^*)$ points of multiplicity 1, and
$(d-iii-iii^*)/2$ points of multiplicity 2;
over ∞: $(i_n+i_n^*)$ points of multiplicity $n\,(n\geqq 1)$.

The construction of the surface proceeds in two steps. First, we pull back via an appropriate J-map one of the surfaces with $J=$ the identity. Second, we "twist" extraneous fibers away. Let us describe these steps in turn.

The rational elliptic surface with Weierstrass equation

$$y^2=x^3-3t(t-1)^3x+2t(t-1)^5 \tag{3.1}$$

has $J=t$, and has exactly three singular fibers: a fiber of type II over $t=0$, a fiber of type III^* over $t=1$, and a fiber of type I_1 over $t=\infty$. Upon a base change from this surface, one will have singular fibers over the points going to 0, 1, and ∞, and the types of the singular fibers are determined merely by the multiplicity of the base change map at these points. In particular, we have the following (see [MP, Table (7.1)):

(3.2) over a point above 0:
if $m=1$: II; if $m=2$: IV; if $m=3$: I_0^*
over a point above 1:
if $m=1$: III^*; if $m=2$: I_0^*
over a point above ∞:
I_m if multiplicity m.

After the pull-back, one makes a "twist" of the resulting surface to adjust the fibers of type IV^*, III^*, II^*, and I_n^*. There are really two processes going on here. One was described earlier in the discussion of # 56; it will be referred to as "transfer of *". Assume that the surface is given in Weierstrass form as $y^2=x^3+Ax+B$, where A and B are forms in s and t. Suppose that s^2 divides A and s^3 divides B, but either t^2 does not divide A or t^3 does not divide B. Then over $s=0$ there is a singular fiber of the surface, of type I_n^*, II^*, III^*, or IV^*, and over $t=0$ there is a fiber of type I_m, II, III, or IV. The "transfer of *" is effected by replacing A by t^2A/s^2 and B by t^3B/s^3. After making this replacement, over $s=0$ there is a singular fiber of type I_n, II, III, or IV, and over $t=0$ there is a fiber of type I_m^*, II^*, III^*, or IV^*; all other singular fibers remain the same, and the J-map of the surface is unaffected. One can be more precise: the fibers are switched according to the following schedule:

$$\begin{aligned} I_n &\leftrightarrow I_n^* \quad (n\geqq 0)\\ II &\leftrightarrow IV^*\\ III &\leftrightarrow III^*\\ IV &\leftrightarrow II^* \end{aligned} \tag{3.3}$$

Note that "transfer of *" preserves the number of "*" fibers, and keeps the p_g of the surface invariant. The second process, which will be called "deflation of *'s", simultaneously "deflates" two "*" fibers as in (3.3), and so the number of these drops by two; the p_g drops by 1. Suppose that over $t=0$ and over $s=0$ we have "*" fibers; then in the Weierstrass equation, s^2t^2 divides A and s^3t^3 divides B. Replace A by A/s^2t^2 and B by B/s^3t^3; this deflates the fibers over $s=0$ and $t=0$ as in (3.3) (the fiber on the right is replaced by that on the left). All other singular fibers are unaffected, and the J-map remains the same.

The main result for constructing surfaces with prescribed singular fibers can now be stated.

(3.4) Proposition. *Suppose that a list of singular fibers for the rational elliptic surface is given, satisfying the numerical criteria of Sect.* 1, *with* $d = \mathrm{degree}(J) \neq 0$. *Suppose further that a map* $J: \mathbb{P}^1 \to \mathbb{P}^1$ *exists, which belong to the list of singular fibers. Then a rational elliptic surface with section can be constructed with exactly the singular fibers of the list, by first pulling back the surface given by* (3.1) *via* J, *then by applying a suitable number of "deflation of *'s", and finally by applying at most one "transfer of *".*

Proof. Write
$$\begin{aligned} d &= \textstyle\sum n(i_n + i_n^*) \\ &= ii + 2iv + iv^* + 2ii^* + 3a \\ &= iii + iii^* + 2b. \end{aligned}$$

Let Y be the pullback of the surface (3.1) via the J-map which belongs to the given list of singular fibers. By (3.2), we have the following singular fibers on Y:

over the points over $J=0$:
- $(ii + iv^*)$ fibers of type II
- $(iv + ii^*)$ fibers of type IV
- a fibers of type I_0^*;

over points over $J=1$:
- $(iii + iii^*)$ fibers of type III^*
- b fibers of type I_0^*;

over points over $J=\infty$:
- $(i_n + i_n^*)$ fibers of type $I_n (n \geqq 1)$.

The total number of "*" fibers here is $(a + b + iii + iii^*)$. Let e be the number of "*" fibers in the given list; $e=0$ or 1. Then by (1.3) we have $2(ii + iv^*) + 3(iii + iii^*) + 4(iv + ii^*) + 6e + d = 12$. Therefore

$$\begin{aligned} &(a + b + iii + iii^*) \\ &\quad = ((12 - 2(ii + iv^*) - 3(iii + iii^*) - 4(iv + ii^*) - 6e) - ii - iv^* - 2iv - 2ii^*)/3 \\ &\qquad + ((12 - 2(ii + iv^*) - 3(iii + iii^*) - 4(iv + ii^*) - 6e) - iii - iii^*)/2 + iii + iii^* \\ &\quad = 10 - 5e - 2(ii + iv^*) - 2(iii + iii^*) - 4(iv + ii^*) \end{aligned}$$

which is even if $e=0$ and odd if $e=1$. Therefore, after a suitable number of "deflation of *'s", we can arrange exactly e "*" fibers. If $e=0$ we are done. If $e=1$, then after one "transfer of *" operation, we arrive at a surface with the prescribed singular fibers.

This surface has the correct fiber types to be our desired rational elliptic surface, and the only point left to check is that it is indeed rational, and not Enriques (which is the only other serious possibility. However our surface has a section, which an Enriques surface does not; this section is pulled back from the section of the surface given by (3.1), and both the "transfer of *" and "deflation of *" operations preserve the existence of this section. □

The above Proposition reduces the construction of an elliptic surface with prescribed singular fibers to the construction of an appropriate J-map. Suppose first that we have a list of singular fibers with $x=0$, so that if the J-map exists, it is ramified only over 0, 1, and ∞, and the ramification is determined. Then the existence of the J-map is equivalent to the existence of the three permutations σ_0, σ_1, and σ_∞ in S_d, with the appropriate cycle structure, generating a transitive subgroup of S_d, whose product is the identity. Therefore:

(3.5) Corollary. *Suppose a list of singular fibers is given with $d\geqq 1$ and $x=0$, satisfying the numerical criteria of Sect. 1. Then a rational elliptic surface with those singular fibers exist if and only if there are three permutations σ_0, σ_1, and σ_∞ in S_d satisfying the following conditions:*

(3.5.1) *the cycle structure of σ_0 is:*
$(ii+iv^*)$ *1-cycles*
$(iv+ii^*)$ *2-cycles*
$(d-ii-iv^*-2iv-2ii^*)/3$ *3-cycles;*

(3.5.2) *the cycle structure of σ_1 is:*
$(iii+iii^*)$ *1-cycles*
$(d-iii-iii^*)/2$ *2-cycles;*

(3.5.3) *the cycle structure of σ_∞ is:*
$(i_n+i_n^*)$ *n-cycles, for each $n\geqq 1$;*

(3.5.4) $\sigma_0\,\sigma_1\,\sigma_\infty=$ *the identity in S_d;*

(3.5.5) *the subgroup of S_d generated by $\{\sigma_0\,\sigma_1, \sigma_\infty\}$ is transitive.*

It is by exhibiting the three permutations that the existence of the rational elliptic surfaces with $x=0$ will be demonstrated.

I claim that examples of rational elliptic surfaces with configurations of singular fibers with $x\geqq 1$ can be constructed by suitably deforming the J-map of a surface with $x=0$, and in fact all configurations which occur can be found this way. The basic observation is the identity

$$(1,2,\ldots,n)=(1,2,\ldots,k)\,(k+1,k+2,\ldots,n)\,(k,n).$$

Suppose a J-map exists which belongs to a configuration of singular fibers. This J-map may be ramified over 0, 1, and ∞, and possibly elsewhere. Therefore we have permutations σ_0, σ_1, σ_∞, etc., satisfying the conditions of (3.5) (generalized to more than three permutations). Suppose that an n-cycle occurs in one of the permutations. Then, by using the above identity, this n-cycle can be replaced by a product of a k-cycle and an $(n-k)$-cycle in the given permutation, at the cost of introducing an extra permutation which is a transposition. This new set of permutations can then be used to construct a J-map, which belongs to an altered list of singular fibers. Using this method, one sees immediately that certain singular fibers can be "deformed" into two singular fibers, leaving all others alone. The resulting fibers are given in the following table.

Table (3.6)

Fiber	Deforms into
I_n	I_k+I_{n-k}, $1\leqq k\leqq n-1$
I_0^*	$II+IV$ if over $J=0$
I_0^*	$III+III$ if over $J=1$
IV	$II+II$

Note that I_n cannot deform to $I_k + I_{n-k}^*$, since the total Euler number must be preserved.

These techniques suffice to demonstrate the existence of rational elliptic surfaces with every possible configuration of singular fibers; there are exactly 279 configurations which exist. These are listed in the next section, together with the permutations if $x=0$ which "construct" the J-map belonging to the list of singular fibers.

4. The configurations of singular fibers which exist

Using the techniques described in the previous section, it is natural for us to stratify the 279 possible configurations by the degree of the J-map.

In the special case of $d=0$, there are almost no restrictions, except for of course (1.3) and (1.8). Furthermore, all of these can be easily constructed simply by writing down the Weierstrass equations; see for example [MP] for more details.

Degree $(J)=0$

$I_0^*\ IV\ II;\ II^*\ II;\ IV^*\ IV$
$I_0^*\ III\ III;\ III^*\ III$
$I_0^*\ II\ II\ II;\ IV^*\ II\ II$
$I_0^*\ I_0^*$
$IV\ IV\ IV$
$IV\ IV\ II\ II$
$IV\ II\ II\ II\ II$
$III\ III\ III\ III$
$II\ II\ II\ II\ II\ II$

The ones on the same line are "twists" of each other; they can be obtained one from the other by the "transfer of *" process.

Degree $(J)=1$

$$I_0^*\ III\ II\ I_1;\ III^*\ II\ I_1;\ IV^*\ III\ I_1;\ I_1^*\ III\ II$$

In every case above the J-map is the identity.

In the cases where the degree is 2 or more, I will employ a slight abbreviation for the configurations of singular fibers. A "*" will denote I_0^*; "n" will denote I_n: II, III, and IV will still be used for these. Exponents will be used to indicate repetition. The other "*" fibers will not be listed as such; all can be twisted, via the "transfer of *" process, to a configuration with only I_0^*. For example, the notation "$*II^2 2$" stands for all three configurations $I_2^*\ II\ II\ I_2$, $IV^*\ II\ I_2$, and $I_2^*\ II\ II$. As another example, the four configurations given above with degree$(J)=1$ can be denoted by "$*II\ III\ 1$".

In what follows, the configurations with $x=0$ will have printed next to them the cycle structure of σ_0, σ_1, and σ_∞; following that, the three permutations in S_d whose product is the identity will be given. The configurations with $x \geqq 1$ will simply be listed; we will leave it to the reader to check that each one can be obtained from an existing configuration with one less x using a "deformation" indicated in Table (3.6). With the new notation, these deformations are realized as follows. A "*" can be replaced by $II\ IV$ if the I_0^* can exist over $J=0$; a "*" can be replaced by III^2 if the I_0^* can exist over $J=1$; a IV can be replaced by II^2; an "n" can be replaced by "$k(n-k)$" for any k between 1 and $n-1$.

Degree$(J)=2$

$x=0$:
$*IV\,1^2$: $(2)(2)(1^2)$. $[(12)]\cdot[(12)]\cdot[(1)(1)]$
$*II^2\,2$: $(1^2)(2)(2)$. $[(1)(1)]\cdot[(12)]\cdot[(12)]$
$IV\,III^2\,2$: $(2)(11)(2)$. $[(12)]\cdot[(1)(1)]\cdot[(12)]$
$x=1$: $*II^2\,1^2; IV\,III^2\,1^2; III^2\,II^2\,2$.
$x=2$: $III^2\,II^2\,1^2$.

Degree$(J)=3$

$x=0$:
$*III\,21$: $(3)(12)(12)$. $[(123)]\cdot[(1)(23)]\cdot[(3)(12)]$
$IV\,III\,II\,3$: $(12)(12)(3)$. $[(1)(23)]\cdot[(2)(13)]\cdot[(132)]$
$III^3\,3$: $(3)(111)(3)$. $[(123)]\cdot[(1)(2)(3)]\cdot[(132)]$
$x=1$: $*III\,1^3; IV\,III\,II\,21; III\,II^3\,3; III^3\,21$.
$x=2$: $IV\,III\,II\,1^3; III^3\,1^3; III\,II^3\,21$.
$x=3$: $III\,II^3\,1^3$.

Degree$(J)=4$

$x=0$:
$*II\,31$: $(13)(22)(13)$. $[(1)(234)]\cdot[(12)(34)]\cdot[(4)(123)]$
$IV\,II^2\,4$: $(112)(22)(4)$. $[(1)(2)(34)]\cdot[(13)(24)]\cdot[(1324)]$
$III^2\,II\,4$: $(13)(112)(4)$. $[(1)(234)]\cdot[(2)(3)(14)]\cdot[(1432)]$
$IV^2\,2^2$: $(22)(22)(22)$. $[(12)(34)]\cdot[(13)(24)]\cdot[(14)(23)]$
$x=1$: $*II\,21^2; II^4\,4; IV^2\,21^2; IV\,II^2\,31; IV\,II^2\,2^2; III^2\,II\,31; III^2\,II\,2^2$.
$x=2$: $*II\,1^4; IV^2\,1^4; IV\,II^2\,21^2; II^4\,31; III^2\,II\,21^2; II^4\,2^2$.
$x=3$: $IV\,II^2\,1^4; III^2\,II\,1^4; II^4\,21^2$.
$x=4$: $II^4\,1^4$.

$Degree(J)=5$

$x=0$:

$III\,II^2\,5$: (113)(122)(5). [(1)(2)(345)]·[(3)(14)(25)]·[(14325)]

$IV\,III\,41$: (23)(122)(14). [(12)(345)]·[(1)(34)(25)]·[(4)(2153)]

$IV\,III\,32$: (23)(122)(23). [(12)(345)]·[(3)(14)(25)]·[(15)(243)]

$x=1$: $III\,II^2\,41; IV\,III\,31^2; IV\,III\,2^2\,1; III\,II^2\,32.$

$x=2$: $IV\,III\,21^3; III\,II^2\,31^2; III\,II^2\,2^2\,1.$

$x=3$: $IV\,III\,1^5; III\,II^2\,21^3.$

$x=4$: $III\,II^2\,1^5.$

$Degree(J)=6$

$x=0$:

$*41^2$: (33)(222)(114). [(123)(456)]·[(13)(25)(46)]·[(1)(4)(2356)]

$*2^3$: (33)(222)(222). [(123)(456)]·[(14)(26)(35)]·[(15)(24)(36)]

$II^3\,6$: (1113)(222)(6). [(1)(3)(5)(246)]·[(16)(23)(45)]·[(165432)]

$IV\,II\,51$: (123)(222)(15). [(1)(23)(456)]·[(12)(34)(56)]·[(6)(12453)]

$III^2\,51$: (33)(1122)(15). [(123)(456)]·[(5)(6)(13)(24)]·[(1)(23465)]

$IV\,II\,42$: (123)(222)(24). [(1)(23)(456)]·[(14)(25)(36)]·[(26)(1435)]

$III^2\,42$: (33)(1122)(24). [(123)(456)]·[(1)(4)(26)(35)]·[(36)(1542)]

$III^2\,3^2$: (33)(1122)(33). [(123)(456)]·[(1)(4)(25)(36)]·[(162)(354)]

$x=1$: $*31^3; *2^2\,1^2; II^3\,51; IV\,II\,41^2; III^2\,41^2;$
$II^3\,42; IV\,II\,321; IV\,II\,2^3; II^3\,3^2; III^2\,321; III^2\,2^3.$

$x=2$: $*21^4; II^3\,41^2; IV\,II\,31^3; IV\,II\,2^2\,1^2;$
$III^2\,31^3; II^3\,321; III^2\,2^2\,1^2; II^3\,2^3.$

$x=3$: $*1^6; IV\,II\,21^4; II^3\,31^3; III^2\,21^4; II^3\,2^2\,1^2.$

$x=4$: $IV\,II\,1^6; III^2\,1^6; II^3\,21^4.$

$x=5$: $II^3\,1^6.$

$Degree(J)=7$

$x=0$:

$III\,II\,61$: (133)(1222)(16). [(1)(234)(567)]·[(2)(13)(45)(67)]·[(7)(132564)]

$III\,II\,52$: (133)(1222)(25). [(1)(234)(567)]·[(2)(15)(46)(37)]·[(47)(26153)]

$III\,II\,43$: (133)(1222)(34). [(1)(234)(567)]·[(2)(15)(36)(47)]·[(273)(1546)]

$x=1$: $III\,II\,51^2; III\,II\,421; III\,II\,3^2\,1; III\,II\,32^2.$

$x=2$: $III\,II\,41^3; III\,II\,321^2; III\,II\,2^3\,1.$

$x=3$: $III\,II\,31^4; III\,II\,2^2\,1^3.$

$x=4$: $III\,II\,21^5.$

$x=5$: $III\,II\,1^7.$

$Degree(J)=8$

$x=0$:

$II^2 71$: (1133) (2222) (17).

[(1) (2) (345) (678)] · [(17) (28) (35) (46)] · [(3) (1745628)]

$IV 61^2$: (233) (2222) (116).

[(12) (345) (678)] · [(13) (26) (45) (78)] · [(5) (8) (167234)]

$II^2 62$: (1133) (2222) (26).

[(1) (2) (345) (678)] · [(13) (26) (48) (57)] · [(58) (137264)]

$IV 521$: (233) (2222) (125).

[(12) (345) (678)] · [(13) (24) (56) (78)] · [(8) (14) (23675)]

$II^2 4^2$: (1133) (2222) (44).

[(1) (2) (345) (678)] · [(13) (26) (47) (58)] · [(8413) (5726)]

$IV 3^2 2$: (233) (2222) (233).

[(12) (345) (678)] · [(13) (26) (48) (57)] · [(58) (164) (237)]

$x=1$: $II^2 61^2; IV 51^3; II^2 521; IV 421^2; II^2 431;$
$II^2 42^2; IV 3^2 1^2; IV 32^2 1; II^2 3^2 2$.

$x=2$: $II^2 51^3; IV 41^4; II^2 421; IV 321^3; IV 2^3 1^2; II^2 3^2 1^2; II^2 32^2 1; II^2 2^4$.

$x=3$: $II^2 41^4; IV 31^5; IV 2^2 1^4; II^2 321^3; II^2 2^3 1^2$.

$x=4$: $IV 21^6; II^2 31^5; II^2 2^2 1^4$.

$x=5$: $IV 1^8; II^2 21^6$.

$x=6$: $II^2 1^8$.

$Degree(J)=9$

$x=0$:

$III 71^2$: (333) (12222) (117).

[(123) (456) (789)] · [(1) (24) (37) (56) (89)] · [(6) (9) (1783452)]

$III 621$: (333) (12222) (126).

[(123) (456) (789)] · [(1) (24) (36) (57) (89)] · [(9) (34) (167852)]

$III 531$: (333) (12222) (135).

[(123) (456) (789)] · [(1) (24) (35) (67) (89)] · [(9) (152) (34786)]

$III 432$: (333) (12222) (234).

[(123) (456) (789)] · [(1) (24) (37) (59) (68)] · [(69) (348) (1752)]

$x=1$: $III 61^3; III 521^2; III 431^2; III 42^2 1; III 3^2 21; III 32^3$.

$x=2$: $III 51^4; III 421^3; III 3^2 1^3; III 32^2 1^2; III 2^4 1$.

$x=3$: $III 41^5; III 321^4; III 2^3 1^3$

$x=4$: $III 31^6; III 2^2 1^5$.

$x=5$: $III 21^7$.

$x=6$: $III 1^9$.

$Degree(J)=10$

$x=0$:
$II\ 81^2$: (1333) (22222) (118).
[(0) (123) (456) (789)] · [(13) (46) (27) (80) (59)] · [(1) (4) (23756908)]
$II\ 721$: (1333) (22222) (127).
[(0) (123) (456) (789)] · [(13) (24) (57) (69) (80)] · [(1) (67) (2349085)]
$II\ 541$: (1333) (22222) (145).
[(0) (123) (456) (789)] · [(01) (24) (35) (67) (89)] · [(9) (0152) (34786)]
$II\ 532$: (1333) (22222) (235).
[(0) (123) (456) (789)] · [(01) (24) (37) (59) (68)] · [(69) (348) (01752)]
$x=1$: $II\ 71^2$; $II\ 621^2$; $II\ 531^2$; $II\ 52^2 1$; $II\ 4^2 1^2$; $II\ 4321$; $II\ 3^2 2^2$.
$x=2$: $II\ 61^4$; $II\ 521^3$; $II\ 431^3$; $II\ 42^2 1^2$; $II\ 3^2 21^2$; $II\ 32^3 1$.
$x=3$: $II\ 51^5$; $II\ 421^4$; $II\ 3^2 1^4$; $II\ 32^2 1^3$; $II\ 2^4 1^2$.
$x=4$: $II\ 41^6$; $II\ 321^5$; $II\ 2^3 1^4$.
$x=5$: $II\ 31^7$; $II\ 2^2 1^6$.
$x=6$: $II\ 21^8$.
$x=7$: $II\ 1^{10}$.

$Degree(J)=12$

$x=0$:
91^3: (3333) (222222) (1119).
[(123) (456) (789) (abc))] · [(13) $(2a)$ (46) $(5b)$ (79) $(8c)$] · [(1) (4) (7) $(23a89c56b)$]
821^2: (3333) (222222) (1128).
[(123) (456) (789) (abc)] · [(13) (46) (27) $(5a)$ $(8c)$ $(9b)$] · [(1) (4) $(9c)$ $(237b56a8)$]
6321: (3333) (222222) (1236).
[(123) (456) (789) (abc)] · [(13) (24) (57) $(6a)$ $(8c)$ $(9b)$] · [(1) $(9c)$ $(67b)$ $(234a85)$]
$5^2 1^2$: (3333) (222222) (1155).
[(123) (456) (789) (abc)] · [(13) (46) (27) $(8a)$ $(9b)$ $(5c)$] · [(1) (4) $(37b82)$ $(6c9a5)$]
$4^2 2^2$: (3333) (222222) (2244).
[(123) (456) (789) (abc)] · [(14) (26) (37) $(5a)$ $(9b)$ $(8c)$] · [(24) $(9c)$ $(17b5)$ $(36a8)$]
3^4: (3333) (222222) (3333).
[(123) (456) (789) (abc)] · [(14) (27) (68) $(3a)$ $(9b)$ $(5c)$] · [$(1a5)$ (248) $(37b)$ $(6c9)$]
$x=1$: 81^4; 721^3; 631^3; $62^2 1^2$; 541^3; 5321^2; $4^2 21^2$; $432^2 1$; 42^4; $3^3 21$.
$x=2$: 71^5; 621^4; 531^4; $52^2 1^3$; $4^2 1^4$; 4321^3; $42^3 1^2$; $3^3 1^3$; $3^2 2^2 1^2$; $32^4 1$; 2^6.
$x=3$: 61^6; 521^5; 431^5; $42^2 1^4$; $3^2 21^4$; $32^3 1^3$; $2^5 1^2$.
$x=4$: 51^7; 421^6; $3^2 1^6$; $32^2 1^5$; $2^4 1^4$.
$x=5$: 41^8; 321^7; $2^3 1^6$.
$x=6$: 31^9; $2^2 1^8$.
$x=7$: 21^{10}.
$x=8$: 1^{12}.

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