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ON THE SURJECTIVITY OF THE WAHL MAP

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CONTENTS

1. Introduction. In this paper we will prove a theorem (stated at the end of this introduction) describing the rank of the Wahl map of a general curve of large genus. We begin here by describing this map and some aspects of its significance.

To begin with, consider a smooth curve C, a line bundle L on C, and a linear system $V \subseteq H^0(C, L)$. Given a section $\sigma \in V$ of L, we can try to define a "differential" dσ of σ, which will be a section of the tensor product $K \otimes L$ of the canonical bundle $K = K_C$ with L, by choosing a trivializing section σ_0 of C, writing σ locally as

$$\sigma(z) = f(z) \cdot \sigma_0$$

and setting

$$d\sigma = df \otimes \sigma_0.$$

This clearly doesn't work: if τ_0 is another trivializing section on L, with $\sigma_0(z) = g(z) \cdot \tau_0$, we would have

$$\sigma(z) = f(z) \cdot g(z) \cdot \tau_0,$$

so that the "differential" would be

$$d\sigma = (f \cdot dg + g \cdot df) \otimes \tau_0 = df \otimes \sigma_0 + f \cdot dg \otimes \tau_0,$$

i.e., it would differ from the earlier differential dσ by the addition of $f \cdot dg \otimes \tau_0$. The expression $d\sigma$ is thus only well defined at the points where σ is zero!

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This, however, also suggests a way to obtain a well-defined map, albeit not between the same spaces. The key observation here is that the difference between the two differentials we obtain above in terms of the two trivializations of L is *linear* in σ. This means that, given a pair of section $\sigma, \tau \in V$, the difference

$$\mathbb{W}_V(\sigma, \tau) = d\sigma \otimes \tau - d\tau \otimes \sigma$$

will be a well-defined section of the tensor product $K \otimes L^2$, independent of the choice of local coordinate. Since $\mathbb{W}_V$ is obviously skew-symmetric in σ and τ, we thus have a natural map

$$\mathbb{W}_V \colon \Lambda^2 V \to H^0(C, K \otimes L^2).$$

By way of notation, we will denote by $\mathbb{W}_L$ the map associated to the complete linear series $V = H^0(C, L)$.

There are a number of alternative ways of viewing this map. One is to write

$$\mathbb{W}_V(\sigma \wedge \tau) = d(\sigma/\tau) \otimes \tau^2,$$

where σ/τ is interpreted as a meromorphic function. From this we obtain the description of $\mathbb{W}_V$ as the map that associates to a decomposable tensor $\sigma \wedge \tau$ in $\Lambda^2 V$ the ramification points of the pencil spanned by σ and τ (whenever σ and τ have no common zeroes); since the decomposable tensors form a copy of the Grassmannian $G(2, V) \subset \mathbb{P}(\Lambda^2 V)$, which spans $\mathbb{P}(\Lambda^2 V)$, this characterizes $\mathbb{W}_V$.

Another way of expressing this last is in terms of the map

$$\phi_V \colon C \to \mathbb{P}(V^*)$$

associated to the linear series V, and its *first associated map*. This is the map

$$\phi_V^{(1)} \colon C \to G(2, V^*) \subset \mathbb{P}(\Lambda^2 V^*)$$

that sends a point $p \in C$ to the tangent line to the image curve $\phi(C)$ at the point $\phi(p)$, viewed as a point in the Grassmannian $G(2, V^*)$, which is in turn mapped by its Plücker embedding to $\mathbb{P} = \mathbb{P}(\Lambda^2 V^*)$. Now, the pull-back to C of the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}$ is the line bundle $K \otimes L^2$, and the corresponding pull-back map on sections

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) = \Lambda^2 V \to H^0(C, K \otimes L^2)$$

is just the map $\mathbb{W}_V$ above.

The image of $\mathbb{W}_V$ is called in this context the *first associated linear series* of V and is denoted by $V^{(1)}$. Observe in particular that if L is sufficiently special, then just by comparing dimensions we see that this map cannot be surjective; indeed, it's not immediately clear when a sublinear series of a complete linear series $H^0(C, M)$ is of the form $V^{(1)}$ for some other series V. On the other hand, we will also see below that when the degree of L is large enough compared to the genus, the first associated series of the complete series $H^0(C, L)$ will be complete.

Another way to view the map $\mathbb{W}_V$ is via a generalization of it. Specifically, suppose that L and M are any line bundles on the curve C, $V \subseteq H^0(C, L)$ and $V' \subseteq H^0(C, M)$ two linear series, and denote by $\mathscr{R}(V, V')$ the kernel of the multiplication map

$$V \otimes V' \to H^0(C, L \otimes M).$$

We then have a map

$$\mathbb{W}_{V, V'}: \mathscr{R}(V, V') \to H^0(C, L \otimes M \otimes K)$$

obtained by setting

$$\mathbb{W}_{V, V'}\left(\sum \sigma_i \otimes \tau_i\right) = \sum (\sigma_i \otimes d\tau_i - d\sigma_i \otimes \tau_i),$$

where the sections $d\sigma_i$ and $d\tau_i$ are defined locally as above; as before, we can check that this is well defined. (In case $V = H^0(C, L)$ and $V' = H^0(C, M)$, we will write $\mathscr{R}(L, M)$ for $\mathscr{R}(V, V')$ and $\mathbb{W}_{L, M}$ for $\mathbb{W}_{V, V'}$.) Now, in the special case where $L = M$ and $V = V'$, we can describe $\mathscr{R}(V, V')$ in terms of the decomposition of $V \otimes V$ into alternating and symmetric parts: we have

$$\mathscr{R}(V, V) = \wedge^2 V \oplus \mathbb{I}_2,$$

where $\mathbb{I}_2$ is the degree 2 part of the ideal of the image $\phi_V(C) \subseteq \mathbb{P}V^*$ of the curve C under the map associated to the linear series V, viewed as a subspace of $\mathrm{Symm}^2(V) \subseteq V \otimes V$. It's clear now that the map $\mathbb{W}_{V, V}$ is identically zero on $\mathbb{I}_2$ and is the map $\mathbb{W}_V$ above on $\wedge^2 V$.

The more general map $\mathbb{W}_{V, V'}$ can be interpreted as given by a restriction of line bundles on the product $C \times C$. Specifically, let π_1, π_2: $C \times C \to C$ be the projection maps, set

$$N = \pi_1^* L \otimes \pi_2^* M,$$

and consider the restriction

$$N \to N \otimes \mathscr{O}_\Delta = L \otimes M.$$

The corresponding map on global sections is just the multiplication map $H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ above; so we can make the identification

$$\mathscr{R}(L, M) = H^0(C \times C, N(-\Delta)).$$

Now, the restriction map

$$N(-\Delta) \to N(-\Delta) \otimes \mathcal{O}_\Delta = L \otimes M \otimes K_C$$

likewise induces a map on global sections

$$\mathcal{R}(L, M) \to H^0(C, L \otimes M \otimes K_C);$$

and this is the map $\mathbb{W}_{L,M}$ above.

As one consequence of the last description of the map $\mathbb{W}_{L,M}$, we see that the map $\mathbb{W}_L$ associated to a complete linear series $V = H^0(C, L)$ will be surjective whenever

$$H^1(C \times C, N(-2\Delta)) = 0.$$

This immediately implies the statement above, that the map $\mathbb{W}_L$ will be surjective whenever the degree of the series is sufficiently large compared to the genus. Indeed, we can see that this will be the case whenever $d = \deg(L) \geqslant 4g + 6$ (see the appendix); we do not know what the best possible estimate is.

One further general observation about the map $\mathbb{W}_V$ is that its definition may be readily extended to singular curves C, as long as we replace the canonical bundle K_C by the sheaf Ω_C^1 of Kähler differentials on C (although L has to remain invertible).

Clearly, the behavior of the map $\mathbb{W}_V$ depends very much on the linear series V as well as on the geometry of the curve C. However, if we take V to be the complete canonical series, we arrive at a map

$$\mathbb{W}: \wedge^2 H^0(C, K) \to H^0(C, K^3)$$

that depends only on the curve C, and on no further choices; it is this map that we shall call the *Wahl map* of C. Obviously, any invariant of this map is an invariant of C; in particular, the rank of the Wahl map is a basic numerical invariant of C itself.

What geometric interpretation may be given to this rank? A more detailed discussion is to be found in [W], but we will mention just one fact here: *If the curve C may be embedded on a $K - 3$ surface, the Wahl map of C cannot be surjective* (see also [BM]). Note that this ties in nicely with naive dimension counts: on the one hand, the moduli of pairs (C, S), with C a smooth curve of genus g and S a $K - 3$ surface, have dimension $19 + g$, so that the general curve C might a priori appear on a $K - 3$ surface only when $19 + g \geqslant 3g - 3$, i.e., when $g \leqslant 11$; on the other hand, the Wahl map can be surjective only when $g(g - 1)/2 \geqslant 5g - 5$, i.e., when $g \geqslant 10$. In fact, it is the case ([MM], [Mu]) that a general curve of genus g may be realized as the hyperplane section of a $K - 3$ surface if and only if $g \leqslant 9$ or $g = 11$.

The main result of this paper is to describe the behavior of the Wahl map on a general curve. Specifically, we shall prove the

THEOREM. *If C is a general curve of genus $g \geqslant 10$ with $g \neq 11$, then the Wahl map for C is surjective.*

This, in conjunction with the result of Mori and Mukai [MM] in the case genus 11, settles the surjectivity of the Wahl map for a general curve of any genus. Clearly it raises further questions: What can we say about the stratification of the moduli space $\mathscr{M}_g$ given by the rank of the Wahl map? In particular, is it the case that the closure of any of these strata—for example, the locus of curves whose Wahl map fails to be surjective—actually coincides with the closure of the locus of curves embeddable in $K-3$ surfaces?

A word about technique. As is often the case when a theorem is to be proved about the geometry of the general curve of a given genus, the basic technique used in this paper is degeneration to a singular (but still stable) curve. On a family of curves specializing to a stable curve, the canonical bundle of the general fiber specializes to the dualizing sheaf ω of the special fiber, and the Wahl map of the general fiber C likewise specializes to what we may call the Wahl map of the special fiber X:

$$\mathbb{W}\colon \Lambda^2 H^0(X, \omega_X) \mapsto H^0\big(X, \Omega_X^1 \otimes \omega_X^2\big).$$

The surjectivity of $\mathbb{W}$ is a Zariski open condition on the moduli space of stable curves; thus, to prove the theorem above it is sufficient to exhibit a single stable curve of genus g for which the map $\mathbb{W}$ is surjective. One novelty about the present circumstance, however, is this: while it is often useful in other contexts to look at specializations to curves of compact type—that is, curves in the complement of Δ_0 in $\overline{\mathscr{M}}_g$, or, equivalently, curves all of whose nodes are disconnecting —we see here that *if the curve X possesses any disconnecting nodes at all, then the Wahl map of X cannot be surjective*. Instead, for our present purposes, we want to look at stable curves with a maximum number $3g-3$ of nodes, none of which are disconnecting. Such curves will consist of a configuration of $2g-2$ copies of $\mathbb{P}^1$, each of which will meet the others at exactly three nodes, and so will be completely determined by its dual graph (which will be trivalent and at least 2-connected); they are correspondingly called *graph curves* and are the subject of a monograph by Bayer and Eisenbud [BE]. There are finitely many graph curves of each genus (the exact number is not known), which behave differently in many respects. In particular, it is not the case for any g that the Wahl map is surjective for all graph curves of genus g; the ones we use here were suggested by Eisenbud.

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2. The graph curve $X_{n,p}$. Fix a genus g and let $n = g - 1$. Choose $p \in \{2,\ldots,n-1\}$ relatively prime to n, and let $q \in \{2,\ldots,n-1\}$ be the inverse of p mod n, i.e., $pq \equiv 1 \pmod n$.

Define a graph $G_{n,p}$ with $2n$ vertices $(a_1,\ldots,a_n,b_1,\ldots,b_n\}$ whose edges connect the vertices as follows:

(2.1) a_i is connected to $a_{i+1(\text{mod } n)}$ for every i,
b_j is connected to $b_{j+1(\text{mod } n)}$ for every j, and
a_k is connected to $b_{pk(\text{mod } n)}$ for every k.
Hence b_j is connected to $a_{qj(\text{mod } n)}$ for every j, also.

The graph $G_{n,p}$ is trivalent and 3-connected as long as $g \geqslant 3$.

Let $X_{n,p} = X(G_{n,p})$ be the associated "graph curve" [BE]; $X_{n,p}$ consists of $2n$ smooth rational components $(A_1,\ldots,A_n,B_1,\ldots,B_n\}$ having dual graph $G_{n,p}$, i.e., each vertex a_i (respectively b_j) corresponds to the component A_i (respectively B_j) $\cong \mathbb{P}^1$, and the nodes of $X_{n,p}$, where two distinct components meet transversely, are determined by the edges of $G_{n,p}$. For $g \geq 4$, $X_{n,p}$ is a stable curve of genus g whose canonical map embeds $X_{n,p}$ into $\mathbb{P}^{g-1}$, with the components of $X_{n,p}$ going to straight lines [BE].

Let us label the nodes of $X_{n,p}$ as follows: Define

$$P_i = A_i \cap A_{i+1(\text{mod } n)},$$

$$Q_j = B_j \cap B_{j+1(\text{mod } n)}, \quad \text{and}$$

$$R_k = A_k \cap B_{pk(\text{mod } n)},$$

for every i, j, and k. We will from now on drop the *mod n* from the subscripts of the components A_i and B_j and the nodes P_i, Q_j, and R_k.

3. Sections of the dualizing sheaf on $X_{n,p}$. Let $X = X_{n,p}$ and denote by ω_X the dualizing sheaf of X. A section ω of ω_X over X is determined by its restrictions $\omega|_{A_i}$ and $\omega|_{B_j}$, which are meromorphic 1-forms subject to the following conditions:

(3.1.1) *$\omega|_{A_i}$ and $\omega|_{B_j}$ have only simple poles at the nodes of A_i and B_j, respectively, and no other poles;*

$$\operatorname{res}_{P_{i-1}}(\omega|_{A_i}) + \operatorname{res}_{R_i}(\omega|_{A_i}) + \operatorname{res}_{P_i}(\omega|_{A_i}) = 0; \tag{3.1.2}$$

$$\operatorname{res}_{Q_{j-1}}(\omega|_{B_j}) + \operatorname{res}_{R_{qj}}(\omega|_{B_j}) + \operatorname{res}_{Q_j}(\omega|_{B_j}) = 0; \tag{3.1.3}$$

$$\operatorname{res}_{P_i}(\omega|_{A_i}) + \operatorname{res}_{P_i}(\omega|_{A_{i+1}}) = 0; \tag{3.1.4}$$

$$\operatorname{res}_{Q_j}(\omega|_{B_j}) + \operatorname{res}_{Q_j}(\omega|_{B_{j+1}}) = 0; \tag{3.1.5}$$

$$\operatorname{res}_{R_k}(\omega|_{A_k}) + \operatorname{res}_{R_k}(\omega|_{B_{pk}}) = 0. \tag{3.1.6}$$

(3.1.1) is from the definition of ω_X, (3.1.2) and (3.1.3) are the "sum of the residues = 0" condition on each restricted 1-form, and (3.1.4)–(3.1.6) are the compatibility conditions at the nodes of X.

Conversely, given the restricted 1-forms $\omega|_{A_i}$ and $\omega|_{B_j}$ for every i and j, satisfying (3.1.1)–(3.1.6), there is a unique section ω of ω_X over X with those restrictions to each component of X [BE].

Fix three distinct points x_1, x_2, and x_3 on $\mathbb{P}^1$. Given any three complex numbers r_1, r_2, and r_3, with $r_1 + r_2 + r_3 = 0$, there is a unique meromorphic 1-form ω on $\mathbb{P}^1$ whose only poles are at x_1, x_2, and x_3, such that these poles are simple, and such that $\mathrm{res}_{x_i}(\omega) = r_i$ for each i. Since each component of X has exactly three nodes on it, a global section of ω_X is determined by the $6n$ residues

$$\mathrm{res}_{P_{i-1}}(\omega|_{A_i}), \mathrm{res}_{R_i}(\omega|_{A_i}), \mathrm{res}_{P_i}(\omega|_{A_i}), \quad \text{for every } i, \text{ and}$$

$$\mathrm{res}_{Q_{j-1}}(\omega|_{B_j}), \mathrm{res}_{R_{qj}}(\omega|_{B_j}), \mathrm{res}_{Q_j}(\omega|_{B_j}), \quad \text{for every } j,$$

subject to the conditions (3.1.2)–(3.1.6). The reader can check that these equations impose $5n - 1$ linearly independent conditions on the $6n$ complex numbers; hence the dimension of the space of global sections of ω_C is $6n - (5n - 1) = 6g - 6 - (5g - 5 - 1) = g$, as expected.

It will be convenient to introduce the following notation.

(3.2) *Definition.* Let ω be a global section of ω_X. The *residue triple of ω on A_i* is

$$\mathrm{restriple}(\omega|_{A_i}) = \left(\mathrm{res}_{P_{i-1}}(\omega|_{A_i}), \mathrm{res}_{R_i}(\omega|_{A_i}), \mathrm{res}_{P_i}(\omega|_{A_i})\right);$$

similarly, the *residue triple of ω on B_j* is

$$\mathrm{restriple}(\omega|_{B_j}) = \left(\mathrm{res}_{Q_{j-1}}(\omega|_{B_j}), \mathrm{res}_{R_{qj}}(\omega|_{B_j}), \mathrm{res}_{Q_j}(\omega|_{B_j})\right).$$

Let us define certain global sections of ω_X which will be used in the sequel, by their residues; we will use residue triples to define them efficiently.

(3.3) *Definition.* Fix an integer α. Define σ_α by declaring

$$\mathrm{restriple}(\sigma_\alpha|_{A_i}) = \begin{cases} (0,0,0) & \text{if } i \neq \alpha, \alpha+1 \\ (0,-1,1) & \text{if } i = \alpha \\ (-1,1,0) & \text{if } i = \alpha+1 \end{cases} \qquad \text{and}$$

$$\mathrm{restriple}(\sigma_\alpha|_{B_j}) = \begin{cases} (0,0,0) & \text{if } j \neq p\alpha, p\alpha+1, \ldots, p\alpha+p \\ (0,1,-1) & \text{if } j = p\alpha \\ (1,0,-1) & \text{if } j = p\alpha+1, \ldots, p\alpha+p-1 \\ (1,-1,0) & \text{if } j = p\alpha+p \end{cases}$$

Define η_B by declaring

$$\text{restriple}(\eta_B|_{A_i}) = (0,0,0), \quad \text{for every } i, \quad \text{and}$$

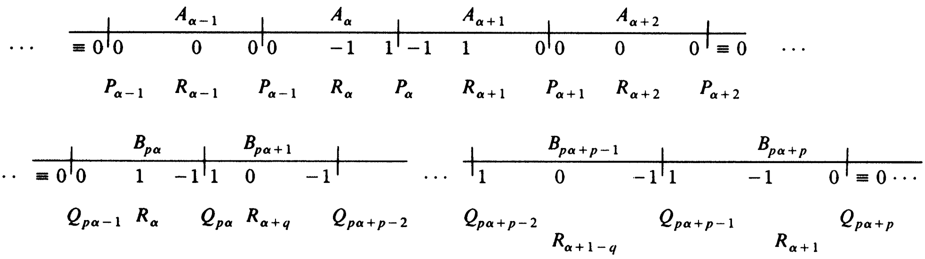
$$\text{restriple}(\eta_B|_{B_j}) = (-1,0,1), \quad \text{for every } j.$$

Note that the residue triple is (0, 0, 0) if and only if the restriction of the 1-form to that component is identically 0.

The reader may verify that the above residues satisfy (3.1.2)–(3.1.6) and so define global sections σ_α and η_B of ω_X.

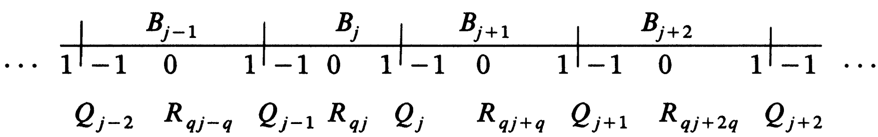
It may be useful to envision these 1-forms with the following diagrams, indicating their residues at the points P_i, Q_j, and R_k on the various components.

σ_α:



η_B:

$\eta_B|_{A_i} \equiv 0$ for every i;



These are in fact all the sections of ω_X that are needed:

(3.4) PROPOSITION. *The sections* $\{\sigma_i\}$ *and* η_B *form a basis for* $H^0(X, \omega_X)$.

Proof. It is enough to show that they are linearly independent, since there are $n + 1 = g = \dim H^0(X, \omega_X)$ of them. Let $\omega = a\eta_B + \Sigma c_i\sigma_i$ for some constants a and c_i. Assume $\omega = 0$; then $\text{restriple}(\omega|_{A_m}) = (0,0,0)$ for every m. Since restriple is linear, we have

$$\begin{aligned}(0,0,0) &= \text{restriple}(\omega|_{A_m})\\ &= a(\text{restriple}(\eta_B|_{A_m})) + \textstyle\sum c_i(\text{restriple}(\sigma_i|_{A_m}))\\ &= c_{m-1}\,\text{restriple}(\sigma_{m-1}|_{A_m}) + c_m\,\text{restriple}(\sigma_m|_{A_m})\\ &= (-c_{m-1}, c_{m-1}, 0) + (0, -c_m, c_m) = (-c_{m-1}, c_{m-1} - c_m, c_m).\end{aligned}$$

This forces $c_m = 0$ for every m, so $\omega = a\eta_B$; hence $a = 0$ and $\omega = 0$. Q.E.D.

Let z_i be the coordinate on A_i such that $z_i = 0$ at P_i, $z_i = 1$ at R_i, and $z_i = \infty$ at P_{i-1}; z_i is a local coordinate on A_i at P_i. Similarly, let w_j be the coordinate on B_j such that $w_j = 0$ at Q_j, $w_j = 1$ at R_{qj}, and $w_j = \infty$ at Q_{j-1}; w_j is a local coordinate on B_j at Q_j. Let $x_i = 1 - z_i$ and $y_i = 1/z_i$; on A_i, x_i is a local coordinate at P_{i-1}. Similarly, let $u_j = 1 - w_j$ and $v_j = 1/w_j$; on B_j, u_j is a local coordinate at R_{qj} and v_j is a local coordinate at Q_{j-1}.

The 1-forms $\omega = \sigma_\alpha$ and η_B we have defined above in (3.3) all have the form

$$(3.5) \qquad \omega|_{A_i} = \frac{az_i + b}{z_i(z_i - 1)}\, dz_i = \frac{ax_i + (-a-b)}{x_i(x_i - 1)}\, dx_i = \frac{by_i + a}{y_i(y_i - 1)}\, dy_i$$

for some constants a and b, and similarly,

$$\omega|_{B_j} = \frac{cw_j + d}{w_j(w_j - 1)}\, dw_j = \frac{cu_j + (-c-d)}{u_j(u_j - 1)}\, du_j = \frac{dv_j + c}{v_j(v_j - 1)}\, dv_j$$

for some constants c and d. These constants can be expressed in terms of the residues of ω (restricted to those components), and it is easily checked that in fact

$$(3.6) \qquad \omega|_{A_i} = \frac{-\operatorname{res}_{P_{i-1}}(\omega|_{A_i})z_i - \operatorname{res}_{P_i}(\omega|_{A_i})}{z_i(z_i - 1)}\, dz_i \quad \text{and}$$

$$\omega|_{B_j} = \frac{-\operatorname{res}_{Q_{j-1}}(\omega|_{B_j})w_j - \operatorname{res}_{Q_j}(\omega|_{B_j})}{w_j(w_j - 1)}\, dw_j.$$

Using (3.5) and (3.6), we can express any ω, restricted to any component, locally at any of the nodes of X, in terms of the given residues.

4. The Wahl map. The purpose of this article is to prove the surjectivity of the Wahl map $\mathbb{W}\colon \Lambda^2 H^0(\omega_X) \to H^0(\omega_X^{\otimes 2} \otimes \Omega_X^1)$, which is defined as follows: Let σ and τ be two global sections of ω_X. At any $p \in X$, choose a local generator g for the sheaf ω_X, and write $\sigma = sg$ and $\tau = tg$, where s and t are in $O_{X,p}$. Define $\mathbb{W}(\sigma \wedge \tau) = (tds - sdt)g^2$. $\mathbb{W}$ is independent of the choice of local generator for ω_X and is therefore a well-defined map from $\Lambda^2 H^0(\omega_X)$ to $H^0(\omega_X^{\otimes 2} \otimes \Omega_X^1)$, as desired.

On our graph curve X, Ω_X^1 is not locally free; the torsion part of Ω_X^1 is supported at the nodes of X, and at each node is a 1-dimensional skyscraper sheaf. Modulo torsion, Ω_X^1 is the direct sum over the components C of X of the sheaves Ω_C^1. Since the torsion in Ω_X^1 is supported in dimension 0, it has no H^1, and, after tensoring with $\omega_X^{\otimes 2}$, we obtain a short exact sequence

$$(4.1) \qquad 0 \to H^0(\operatorname{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1)) \to H^0(\omega_X^{\otimes 2} \otimes \Omega_X^1) \to \bigoplus_C H^0(\omega_X^{\otimes 2}|_C \otimes \Omega_C^1) \to 0.$$

For any component C ($= A_i$ or B_j) of X, $\omega_X|_C$ is a line bundle of degree 1; hence $\omega_X^{\otimes 2}|_C \otimes \Omega_C^1$ has degree 0 on any C, and the last term of the sequence (4.1) is a complex vector space of dimension $2n$.

Denote by Φ the composition of the Wahl map $\mathbb{W}$ with the projection $H^0(\omega_X^{\otimes 2} \otimes \Omega_X^1) \to \bigoplus_C H^0(\omega_X^{\otimes 2}|_C \otimes \Omega_C^1)$ of (4.1). Our strategy to show that $\mathbb{W}$ is surjective is first to show that Φ is surjective, and then to show that $H^0(\mathrm{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1))$ is in the image of $\mathbb{W}$.

5. The surjectivity modulo torsion. Let $\Phi\colon \wedge^2 H^0(\omega_X) \to [\bigoplus_i H^0(\omega_X^{\otimes 2}|_{A_i} \otimes \Omega_{A_i}^1)] \oplus [\bigoplus_j H^0(\omega_X^{\otimes 2}|_{B_j} \otimes \Omega_{B_j}^1)]$ be defined as in section 4, and let Φ_{A_i} and Φ_{B_j} be the composition of Φ with the $2n$ projections onto the direct summands of the range of Φ. At a general point of A_i, we can trivialize $\omega_X|_{A_i}$ by using the local generator $(dz_i)/z_i(z_i - 1)$. Let

$$\omega_1|_{A_i} = \frac{a_1 z_i + b_1}{z_i(z_i - 1)}\, dz_i \quad \text{and} \quad \omega_2|_{A_i} = \frac{a_2 z_i + b_2}{z_i(z_i - 1)}\, dz_i;$$

by the definition of the Wahl map, we have that

$$\Phi_{A_i}(\omega_1 \wedge \omega_2) = [(a_2 z_i + b_2) a_1\, dz_i - (a_1 z_i + b_1) a_2\, dz_i]\left[\frac{dz_i}{z_i(z_i - 1)}\right]^2$$
$$= [a_1 b_2 - a_2 b_1]\, dz_i \left[\frac{dz_i}{z_i(z_i - 1)}\right]^2.$$

The choice of local coordinate z_i on A_i gives an explicit isomorphism of $H^0(\omega_X^{\otimes 2}|_{A_i} \otimes \Omega_{A_i}^1)$ with $\mathbb{C}$, taking $c \cdot dz_i[(dz_i)/z_i(z_i - 1)]^2$ to c. We will abuse notation and refer to Φ_{A_i} as the map to $\mathbb{C}$, composing with this isomorphism. With this notation, $\Phi_{A_i}(\omega_1 \wedge \omega_2) = a_1 b_2 - a_2 b_1$. Similar remarks apply to the components Φ_{B_j} of Φ; using the coordinate w_j on B_j, if

$$\omega_1|_{B_j} = \frac{c_1 w_j + d_1}{w_j(w_j - 1)}\, dw_j \quad \text{and} \quad \omega_2|_{B_j} = \frac{c_2 w_j + d_2}{w_j(w_j - 1)}\, dw_j$$

for some constants c_1, d_1, c_2, d_2, then $\Phi_{B_j}(\omega_1 \wedge \omega_2) = c_1 d_2 - c_2 d_1$. These calculations immediately give the following:

(5.1) Lemma. *Let ω_1 and ω_2 be global sections of ω_X. Then*

$$(5.1.1) \qquad \Phi_{A_i}(\omega_1 \wedge \omega_2) = \det\begin{pmatrix} \mathrm{res}_{P_{i-1}}(\omega_1|_{A_i}) & \mathrm{res}_{P_i}(\omega_1|_{A_i}) \\ \mathrm{res}_{P_{i-1}}(\omega_2|_{A_i}) & \mathrm{res}_{P_i}(\omega_2|_{A_i}) \end{pmatrix}$$

and

$$\Phi_{B_j}(\omega_1 \wedge \omega_2) = \det\begin{pmatrix} \operatorname{res}_{Q_{j-1}}(\omega_1|_{B_j}) & \operatorname{res}_{Q_j}(\omega_1|_{B_j}) \\ \operatorname{res}_{Q_{j-1}}(\omega_2|_{B_j}) & \operatorname{res}_{Q_j}(\omega_2|_{B_j}) \end{pmatrix}. \tag{5.1.2}$$

Proof. From (3.5), the above calculations give that

$$\Phi_{A_i}(\omega_1 \wedge \omega_2) = \left(-\operatorname{res}_{P_{i-1}}(\omega_1|_{A_i})\right)\left(-\operatorname{res}_{P_i}(\omega_2|_{A_i})\right) - \left(-\operatorname{res}_{P_i}(\omega_1|_{A_i})\right)\left(-\operatorname{res}_{P_{i-1}}(\omega_2|_{A_i})\right),$$

which is the determinant given in (5.1.1). (5.2.2) is identical. Q.E.D.

Note that the determinant of (5.1.1) (respectively (5.1.2)) is simply the $1-3$ minor of the 2×3 matrix whose rows are the residue triples of the sections ω_1 and ω_2 on the component A_i (respectively B_j).

We will use these formulas to compute the coordinates of Φ for various pure wedges of the global sections of ω_X defined in (3.3). Note that if *either* ω_1 or ω_2 is not supported on a component C of X, then $\Phi_C(\omega_1 \wedge \omega_2) = 0$.

(5.2) **Proposition.** *Assume $g \geqslant 5$, so that $n \geqslant 4$.*

(5.2.1) *Fix any α. Then*

$$\Phi_{A_i}(\eta_B \wedge \sigma_\alpha) = 0 \text{ for every } i, \text{ and}$$

$$\Phi_{B_j}(\eta_B \wedge \sigma_\alpha) = \begin{cases} 0 & \text{if } j \neq p\alpha \text{ or } p\alpha + p \\ 1 & \text{if } j = p\alpha \\ -1 & \text{if } j = p\alpha + p \end{cases}$$

(5.2.2) *Assume $2p + 2 \leqslant n$. If $2 \leqslant m \leqslant n - 2$ and $m \neq \pm q, \pm 2q, \dots, \pm(p-1)q$, then $\sigma_\alpha \wedge \sigma_{\alpha+m}$ is in the kernel of Φ.*

(5.2.3) *Assume $2p + 1 \leqslant n$. Then*

$$\Phi_{A_i}(\sigma_\alpha \wedge \sigma_{\alpha-1}) = \begin{cases} 0 & \text{if } i \neq \alpha \\ 1 & \text{if } i = \alpha \end{cases} \quad \text{and}$$

$$\Phi_{B_j}(\sigma_\alpha \wedge \sigma_{\alpha-1}) = \begin{cases} 0 & \text{if } j \neq p\alpha \\ 1 & \text{if } j = p\alpha \end{cases}.$$

(5.2.4) *Assume $2p \leqslant n$. Fix any α, and any $r - 1, \dots, p - 1$. Then*

$$\Phi_{A_i}(\sigma_\alpha \wedge \sigma_{\alpha+qr}) = 0 \quad \text{for every } i, \text{ and}$$

$$\Phi_{B_j}(\sigma_\alpha \wedge \sigma_{\alpha+qr}) = \begin{cases} 0 & \text{if } j \neq p\alpha + r \text{ or } p\alpha + p \\ -1 & \text{if } j = p\alpha + r \text{ or } p\alpha + p \end{cases}.$$

(5.2.5) *Assume* $2p \leqslant n$. *Then*

$$\Phi_{A_i}(\sigma_\alpha \wedge \sigma_{\alpha+1-q}) = 0 \quad \textit{for every } i, \textit{ and}$$

$$\Phi_{B_j}(\sigma_\alpha \wedge \sigma_{\alpha+1-q}) = \begin{cases} 0 & \textit{if } j \neq p\alpha + p - 1 \textit{ or } p\alpha + p \\ -1 & \textit{if } j = p\alpha + p - 1 \textit{ or } p\alpha + p \end{cases}.$$

Proof. Since η_B is not supported on any A_i, $\Phi_{A_i}(\eta_B \wedge \sigma_\alpha) = 0$ for every i. The common support on the "B" components of σ_α and η_B are $B_{p\alpha}, B_{p\alpha+1}, \ldots, B_{p\alpha+p}$. Using (5.1.2), we have

$$\Phi_{B_{p\alpha}}(\eta_B \wedge \sigma_\alpha) = \det\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = 1,$$

$$\Phi_{B_{p\alpha+p}}(\eta_B \wedge \sigma_\alpha) = \det\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = -1,$$

and

$$\Phi_{B_{p\alpha+m}}(\eta_B \wedge \sigma_\alpha) = \det\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = 0$$

for $m = 1, \ldots, p-1$; this proves (5.2.1).

Statement (5.2.2) is true simply because those sections have no common support along any components of X.

If $n \geqslant 3$, then the common support on the "A" components of X of $\sigma_{\alpha-1}$ and σ_α is only the component A_α, and

$$\Phi_{A_\alpha}(\sigma_\alpha \wedge \sigma_{\alpha-1}) = \det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1$$

by (5.1.1). If in addition $2p + 1 \leqslant n$, then the common support on the "B" components is only the component $B_{p\alpha}$, and

$$\Phi_{B_{p\alpha}}(\sigma_\alpha \wedge \sigma_{\alpha-1}) = \det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1$$

by (5.1.2). This proves (5.2.3).

Since $n \geqslant 4$, if $r \neq 0$ and $r \neq \pm p \pmod n$, then σ_α and $\sigma_{\alpha+qr}$ have no common "A" component in their support. If $1 \leqslant r \leqslant p-1$, the common support on the "B" components of σ_α and $\sigma_{\alpha+qr}$ are $B_{p\alpha+r}, \ldots, B_{p\alpha+p}$, since $r + p + 1 \leqslant 2p \leqslant n$. Again using (5.1.2), we have

$$\Phi_{B_{p\alpha+r}}(\sigma_\alpha \wedge \sigma_{\alpha+qr}) = \det\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = -1,$$

$$\Phi_{B_{p\alpha+p}}(\sigma_\alpha \wedge \sigma_{\alpha+qr}) = \det\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = -1,$$

and

$$\Phi_{B_{p\alpha+m}}(\sigma_\alpha \wedge \sigma_{\alpha+qr}) = \det\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 0$$

if $m = r + 1, \ldots, p - 1$. This proves (5.2.4).

Statement (5.2.5) is simply (5.2.4) applied with $r = p - 1$. Q.E.D.

Note that we have in fact computed Φ for all pure wedges of the elements of the basis $\{\sigma_i\} \cup \{\eta_B\}$. The "missing" ones, $\sigma_\alpha \wedge \sigma_{\alpha+1}$, can be computed using (5.2.3). The rest, namely, $\sigma_\alpha \wedge \sigma_{\alpha-qr}$ for $1 \leqslant r \leqslant p-1$, can be computed using (5.2.4): $\sigma_\alpha \wedge \sigma_{\alpha-qr} = -\sigma_{\alpha-qr} \wedge \sigma_\alpha = -\sigma_{\alpha-qr} \wedge \sigma_{(\alpha-qr)+qr}$.

(5.3) COROLLARY. *Assume that p is odd and $2p \leqslant n$. Let*

$$\Gamma_m = -\frac{1}{2}\left[\eta_B \wedge \sigma_{qm-1} - \sum_{k=0}^{p-1}(-1)^k \sigma_{qm-1-kq} \wedge \sigma_{qm-(k+1)q}\right].$$

Then $\Phi_{A_i}(\Gamma_m) = 0$ for every i, and

$$\Phi_{B_j}(\Gamma_m) = \begin{cases} 0 & \text{if } j \neq m \\ 1 & \text{if } j = m \end{cases}.$$

Proof. It is clear from (5.2.1) and (5.2.5) that $\Phi_{A_i} = 0$; it is zero on each term. From (5.2.5),

$$\Phi_{B_j}(\sigma_{qm-1-kq} \wedge \sigma_{qm-(k+1)q}) = \begin{cases} 0 & \text{if } j \neq m-k-1 \text{ or } m-k \\ -1 & \text{if } j = m-k-1 \text{ or } m-k \end{cases},$$

so that the alternating sum telescopes, giving

$$\Phi_{B_j}\left(\sum_{k=0}^{p-1}(-1)^k \sigma_{qm-1-kq} \wedge \sigma_{qm-(k+1)q}\right) = \begin{cases} 0 & \text{if } j \neq m \text{ or } m-p \\ -1 & \text{if } j = m \\ (-1)^p & \text{if } j = m-p \end{cases}.$$

If p is odd, then $(-1)^p = -1$ and the result follows from (5.2.1). Q.E.D.

The Γ_j's are "dual" to the B_j's with respect to the map Φ, for odd p.

(5.4) COROLLARY. *Assume that $p = 2$ and $n \geqslant 5$. Let*

$$H_m = -\frac{1}{2}\sum_{k=0}^{n-1}(-1)^k \sigma_{qm-1-kq} \wedge \sigma_{qm-(k+1)q}.$$

Then

$$\Phi_{A_i}(H_m) = 0 \quad \textit{for every } i, \quad \textit{and}$$

$$\Phi_{B_j}(H_m) = \begin{cases} 0 & \text{if } j \neq m \\ 1 & \text{if } j = m \end{cases}.$$

Proof. The assumptions imply that n is odd and $2p \leqslant n$. Again $\Phi_{A_i} = 0$ for every i as in (5.3). For the Φ_{B_j}'s, the sum telescopes as above, and this time comes all the way around the cycle; the only nonzero Φ_{B_j} is Φ_{B_m}, and the contribution to Φ_{B_m} is -1 (from the $k = 0$ term) and $(-1)^n$ (from the $k = n - 1$ term). Since n is odd, the result follows. Q.E.D.

The elements H_m are "dual" to the B_j's in this case.

We can now state the main result of this section.

(5.5) Proposition. *Assume $g \geqslant 5$, so that $n \geqslant 4$. If either p is odd and $2p \leqslant n$, or $p = 2$ and $n \geqslant 5$, then $\Phi: \Lambda^2 H^0(\omega_X) \to [\bigoplus_i H^0(\omega_X^{\otimes 2}|_{A_i} \otimes \Omega^1_{A_i})] \oplus [\bigoplus_j H^0(\omega_X^{\otimes 2}|_{B_j} \otimes \Omega^1_{B_j})]$ is surjective.*

Proof. We claim that we can find $2n$ elements $\{\gamma_k\}$ of $\Lambda^2 H^0(\omega_X)$ such that the matrix with entries $\Phi_C(\gamma_k)$, as C ranges over the $2n$ components of X, has rank $2n$. This will suffice, since the target space has dimension $2n$. In fact we can achieve the matrix

$$\begin{pmatrix} I_n & 0 \\ * & I_n \end{pmatrix}.$$

Order the rows by $A_1, \ldots, A_n, B_1, \ldots, B_n$. Use for the first n γ's the elements $\sigma_i \wedge \sigma_{i-1}$; by (5.2.3), this gives the left half of the desired matrix. To get the right half, if p is odd and $2p \leqslant n$, use the elements Γ_m of (5.3), suitably ordered; if $p = 2$ and $n \geqslant 5$, use the elements H_m of (5.4), suitably ordered. Q.E.D.

We will require an understanding of the kernel of Φ.

(5.6) Proposition. *Assume $p = 2$ and $n \geqslant 7$. Then* $\ker(\Phi)$ *is spanned by the following elements*:

(5.6.1) $$\sigma_\alpha \wedge \sigma_{\alpha+m}, \quad \text{where } 2 \leqslant m \leqslant n-2 \text{ and } m \neq \pm(q-1) = \{q, q-1\}$$

(5.6.2) $$\eta_B \wedge \sigma_\alpha - \sigma_\alpha \wedge \sigma_{\alpha-q} - \sigma_\alpha \wedge \sigma_{\alpha+q}.$$

Proof. The elements of (5.6.1) are those with no common support along any of the components of X, so they are certainly in $\ker(\Phi)$. By (5.2.3), no term of the form $\sigma_i \wedge \sigma_{i\pm 1}$ can appear in any element of the kernel of Φ: the Φ_{A_i} (or $\Phi_{A_{i+1}}$) component will be nonzero and cannot be canceled by a contribution from any other term by (5.2.4). We are left with finding the linear combinations of the three terms appearing in (5.6.2) which are in the kernel of Φ; the reader can check that (5.6.2) is correct. Q.E.D.

(5.7) Proposition. *Assume p is odd and $2p + 2 \leqslant n$. Then* $\ker(\Phi)$ *is spanned by the following elements*:

(5.7.1) $$\sigma_\alpha \wedge \sigma_{\alpha+m}, \quad \text{for } 2 \leqslant m \leqslant n-2 \text{ and } m \neq \pm q, \pm 2q, \ldots, \pm(p-1)q$$

(5.7.2) $$\sigma_\alpha \wedge \sigma_{\alpha+qr} + \Gamma_{p\alpha+r} + \Gamma_{p\alpha+p}, \quad \text{for } r = 1, \ldots, p-1.$$

Proof. The elements of (5.7.1) are simply those of (5.2.2) again. Those of (5.7.2) are obtained from (5.2.4) and (5.3). That they span $\ker(\Phi)$ follows from the computations of (5.2) and the fact that the Γ_m's are dual to the B_j's. Q.E.D.

This completes our analysis of the map Φ. We now proceed to show that $H^0(\mathrm{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1))$ is in the image of $\mathbb{W}$.

6. The torsion part. Let p ($= P_i$, Q_j, or R_k) be one of the nodes of X, and assume that the components meeting at p are C and D. Locally near p, X is defined by $st = 0$, where we will assume that $s = 0$ defines C and $t = 0$ defines D. The dualizing sheaf ω_X is generated at p by the form $ds/s - dt/t$, and the Kähler differentials Ω_X^1 are generated at p by ds and dt, with the relation $t\,ds + s\,dt = 0$. The torsion in $\omega_X^{\otimes 2} \otimes \Omega_X^1$ is 1-dimensional at p, generated by $(ds/s - dt/t)^2 \otimes (t\,ds - s\,dt)$.

Our main tool will be the following:

(6.1) LEMMA. *Let ω_1 and ω_2 be sections of ω_X. Let p be a node of X, the intersection of components C and D. Assume that, using the local coordinates described above,*

$$\omega_1 = (c_1 + sf_1(s) + tg_1(t))\left(\frac{ds}{s} - \frac{dt}{t}\right) \quad \textit{and}$$

$$\omega_2 = (c_2 + sf_2(s) + tg_2(t))\left(\frac{ds}{s} - \frac{dt}{t}\right),$$

where the f_i's and g_i's are regular at 0. Then, locally at p,

$$\mathbb{W}(\omega_1 \wedge \omega_2) = \begin{vmatrix} f_1(s) & f_2(s) \\ g_1(t) & g_2(t) \end{vmatrix} (t\,ds - s\,dt)$$

$$- \left(\begin{vmatrix} c_1 & c_2 \\ f_1(s) & f_2(s) \end{vmatrix} + s \begin{vmatrix} c_1 & c_2 \\ f_1'(s) & f_2'(s) \end{vmatrix} + s^2 \begin{vmatrix} f_1(s) & f_2(s) \\ f_1'(s) & f_2'(s) \end{vmatrix} \right) ds$$

$$- \left(\begin{vmatrix} c_1 & c_2 \\ g_1(t) & g_2(t) \end{vmatrix} + t \begin{vmatrix} c_1 & c_2 \\ g_1'(t) & g_2'(t) \end{vmatrix} + t^2 \begin{vmatrix} g_1(t) & g_2(t) \\ g_1'(t) & g_2'(t) \end{vmatrix} \right) dt$$

times $\left(\dfrac{ds}{s} - \dfrac{dt}{t}\right)^2$.

Proof. Simply calculate (we'll ignore the $\left(\frac{ds}{s} - \frac{dt}{t}\right)^2$ in the following)

$$\begin{aligned} \mathbb{W}(\omega_1 \wedge \omega_2) &= [c_1 + sf_2(s) + tg_2(t)] \\ &\quad \times [(f_1(s) + sf_1'(s))\,ds + (g_1(t) + tg_1'(t))\,dt] \\ &\quad - [c_1 + sf_1(s) + tg_1(t)] \\ &\quad \times [(f_2(s) + sf_2'(s))\,ds + (g_2(t) + tg_2'(t))\,dt] \\ &= [(f_1c_2 - f_2c_1) + s(f_1'c_2 - f_2'c_1) + s^2(f_1'f_2 - f_2'f_1)]\,ds \\ &\quad + [g_1c_2 - g_2c_1) + t(g_1'c_2 - g_2'c_1) + t^2(g_1'g_2 - g_2'g_1)]\,dt \\ &\quad + [f_1g_2 - f_2g_1](t\,ds - s\,dt), \end{aligned}$$

which is the desired result. Q.E.D.

(6.2) COROLLARY. *If $\omega_1 \wedge \omega_2 \in \ker(\Phi)$, and p is a node of X, then (using the notation of* (6.1))

$$\mathbb{W}(\omega_1 \wedge \omega_2) = (f_1(0)g_2(0) - f_2(0)g_1(0))\left(\frac{ds}{s} - \frac{dt}{t}\right)^2 \otimes (t\,ds - s\,dt)$$

locally near p.

Proof. If $\omega_1 \wedge \omega_2 \in \ker(\Phi)$, then $\mathbb{W}(\omega_1 \wedge \omega_2) \in H^0(\mathrm{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1))$, which is locally generated by $t\,ds - s\,dt$ at p; the term above is the torsion part of (6.1). Q.E.D.

The choice of local coordinates s and t on the two components C and D meeting at a node p gives an isomorphism of $H^0(\mathrm{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1))$ at p with $\mathbb{C}$, using the generator

$$\frac{ds}{s} - \frac{dt}{t}$$

described above. It also provides us with a natural projection onto the torsion part: the "torsion part" of $[c + sf(s) + tg(t)]\,ds + [d + sh(s) + tk(t)]\,dt$ is

$$\frac{[g(0) - h(0)]}{2}(t\,ds - s\,dt).$$

We have chosen coordinates at all of the nodes in section 3, and we will use these to identify $H^0(\mathrm{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1))$ with $\mathbb{C}^{3n}$. In particular, at P_i, the coordinates are z_i and y_{i+1}; at Q_j, the coordinates are w_j and v_{j+1}; at R_k, the coordinates are x_i and u_{pk}. Note that the order is important: we will use as local generator for ω_X the sections (ignoring subscripts for the moment)

$$\frac{dz}{z} - \frac{dy}{y} \quad (\text{at } P), \qquad \frac{dw}{w} - \frac{dv}{v} \quad (\text{at } Q), \quad \text{and} \quad \frac{dx}{x} - \frac{du}{u} \quad (\text{at } R).$$

We will denote the coordinates of the Wahl map $\mathbb{W}$ by subscripting the node: $\mathbb{W}_{P_i}$, $\mathbb{W}_{Q_j}$, and $\mathbb{W}_{R_k}$ will be the $3n$ coordinates of $\mathbb{W}$ on the kernel of Φ. When applied to an element not in the kernel of Φ, the notation $\mathbb{W}_{P_i}$, $\mathbb{W}_{Q_j}$, and $\mathbb{W}_{R_k}$ will denote the local Wahl map followed by projection onto the torsion part.

With this notation, we have the following:

(6.3) PROPOSITION. *Let ω_1 and ω_2 be two sections of ω_X. Then*

(6.3.1) $$\mathbb{W}_{P_i}(\omega_1 \wedge \omega_2) = \begin{vmatrix} \operatorname{res}_{R_{i+1}}(\omega_1|_{A_{i+1}}) & \operatorname{res}_{R_i}(\omega_1|_{A_i}) \\ \operatorname{res}_{R_{i+1}}(\omega_2|_{A_{i+1}}) & \operatorname{res}_{R_i}(\omega_2|_{A_i}) \end{vmatrix},$$

(6.3.2) $$\mathbb{W}_{Q_j}(\omega_1 \wedge \omega_2) = \begin{vmatrix} \operatorname{res}_{R_{qj+q}}(\omega_1|_{B_{j+1}}) & \operatorname{res}_{R_{qj}}(\omega_1|_{B_j}) \\ \operatorname{res}_{R_{qj+q}}(\omega_2|_{B_{j+1}}) & \operatorname{res}_{R_{qj}}(\omega_2|_{B_j}) \end{vmatrix}, \quad \textit{and}$$

(6.3.3) $$\mathbb{W}_{R_k}(\omega_1 \wedge \omega_2) = \begin{vmatrix} \operatorname{res}_{Q_{pk}}(\omega_1|_{B_{pk}}) & \operatorname{res}_{P_k}(\omega_1|_{A_k}) \\ \operatorname{res}_{Q_{pk}}(\omega_2|_{B_{pk}}) & \operatorname{res}_{P_k}(\omega_2|_{A_k}) \end{vmatrix}.$$

Proof. We'll only check (6.3.1); the others follow from an identical calculation. Fix a section ω of ω_X. At $P_i = A_i \cap A_{i+1}$, we have coordinates z_i and y_{i+1}, with local generator

$$\frac{dz_i}{z_i} - \frac{dy_{i+1}}{y_{i+1}} \quad \text{for } \omega_X.$$

From (3.6), we have

$$\begin{aligned} \omega|_{A_i} &= \frac{-\operatorname{res}_{P_{i-1}}(\omega|_{A_i})z_i - \operatorname{res}_{P_i}(\omega|_{A_i})}{z_i(z_i - 1)} dz_i \\ &= \left[\operatorname{res}_{P_i}(\omega|_{A_i}) + z_i\left(\frac{\operatorname{res}_{P_{i-1}}(\omega|_{A_i}) + \operatorname{res}_{P_i}(\omega|_{A_i})}{1 - z_i}\right)\right]\frac{dz_i}{z_i} \\ &= \left[\operatorname{res}_{P_i}(\omega|_{A_i}) = z_i\left(\frac{\operatorname{res}_{R_i}(\omega|_{A_i})}{z_i - 1}\right)\right]\frac{dz_i}{z_i}. \end{aligned}$$

Using (3.5), we have

$$\begin{aligned}
\omega|_{A_{i+1}} &= \frac{-\operatorname{res}_{P_i}(\omega|_{A_{i+1}})z_{i+1} - \operatorname{res}_{P_{i+1}}(\omega|_{A_{i-1}})}{z_{i+1}(z_{i+1}-1)}\,dz_{i+1} \\
&= \frac{-\operatorname{res}_{P_{i+1}}(\omega|_{A_{i+1}})y_{i+1} - \operatorname{res}_{P_i}(\omega|_{A_{i+1}})}{y_{i+1}(y_{i+1}-1)}\,dy_{i+1} \\
&= \frac{\operatorname{res}_{P_{i+1}}(\omega|_{A_{i-1}})y_{i+1} + \operatorname{res}_{P_i}(\omega|_{A_{i+1}})}{(1-y_{i+1})}\,\frac{dy_{i+1}}{y_{i+1}} \\
&= \left[\operatorname{res}_{P_i}(\omega|_{A_{i+1}}) + y_{i+1}\left(\frac{\operatorname{res}_{P_{i+1}}(\omega|_{A_{i+1}}) + \operatorname{res}_{P_i}(\omega|_{A_{i-1}})}{1-y_{i+1}}\right)\right]\frac{dY_{i+1}}{y_{i+1}} \\
&= \left[\operatorname{res}_{P_i}(\omega|_{A_{i+1}}) - y_{i+1}\left(\frac{\operatorname{res}_{R_{i+1}}(\omega|_{A_{i-1}})}{1-y_{i+1}}\right)\right]\frac{dy_{i+1}}{y_{i+1}}.
\end{aligned}$$

Hence, locally at P_i,

$$\omega = \left[\operatorname{res}_{P_i}(\omega|_{A_i}) + z_i\left(\frac{\operatorname{res}_{R_i}(\omega|_{A_i})}{z_i-1}\right) + y_{i+1}\left(\frac{\operatorname{res}_{R_{i+1}}(\omega|_{A_{i+1}})}{1-y_{i+1}}\right)\right]$$

$$\text{times } \frac{dz_i}{z_i} - \frac{dy_{i+1}}{y_{i+1}}.$$

We are now in a position to use (6.2):

$$\begin{aligned}
\mathbb{W}_{P_i}(\omega_1 \wedge \omega_2) &= \left(-\operatorname{res}_{R_i}(\omega_1|_{A_i})\right)\left(\operatorname{res}_{R_{i+1}}(\omega_2|_{A_{i+1}})\right) \\
&\quad - \left(-\operatorname{res}_{R_i}(\omega_2|_{A_i})\right)\left(\operatorname{res}_{R_{i-1}}(\omega_1|_{A_{i+1}})\right),
\end{aligned}$$

which is the determinant of (6.3.1). Q.E.D.

Our first use for these formulas is to show that the torsion part at each P_i is in the image of the Wahl map.

(6.4) PROPOSITION. *Assume that* $3p + 2 \leqslant n$. *Then* $\sigma_{m-1} \wedge \sigma_{m+1}$ *is in the kernel of* Φ *and*

$$\mathbb{W}_{P_i}(\sigma_{m-1} \wedge \sigma_{m+1}) = \begin{cases} 1 & \text{if } i = m \\ 0 & \text{if } i \neq m \end{cases}.$$

Moreover, $\mathbb{W}_{Q_j}(\sigma_{m-1} \wedge \sigma_{m+1}) = 0$ *for every* j, *and* $\mathbb{W}_{R_k}(\sigma_{m-1} \wedge \sigma_{m+1}) = 0$ *for every* k.

Proof. The assumption on p insures that the only common support of σ_{m-1} and σ_{m+1} is P_m. Hence $\sigma_{m-1} \wedge \sigma_{m+1} \in \ker(\Phi)$, all $\mathbb{W}_{P_i} = 0$ for $i \neq m$, and every $\mathbb{W}_{Q_j}$ and $\mathbb{W}_{R_k} = 0$ also. The computation of $\mathbb{W}_{P_m}(\sigma_{m-1} \wedge \sigma_{m+1})$ is from (6.3.1):

$$\mathbb{W}_{P_m}(\sigma_{m-1} \wedge \sigma_{m+1}) = \det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1. \qquad \text{Q.E.D.}$$

The torsion part at the Q_j's is just as easy to handle:

(6.5) PROPOSITION. *Assume that* $2p + 3 \leqslant n$ *and* $q + 4 \leqslant n$. *Then* $\sigma_{qm-1} \wedge \sigma_{qm+q}$ *is in the kernel of* Φ *and*

$$\mathbb{W}_{Q_j}(\sigma_{qm-1} \wedge \sigma_{qm+q}) = \begin{cases} 1 & \text{if } j = m \\ 0 & \text{if } j \neq m \end{cases}.$$

Moreover, $\mathbb{W}_{P_i}(\sigma_{qm-1} \wedge \sigma_{qm+q}) = 0$ *for every* i *and* $\mathbb{W}_{R_k}(\sigma_{qm-1} \wedge \sigma_{qm+q}) = 0$ *for every* k.

Proof. Again the assumptions imply that the only common support of σ_{qm-1} and σ_{qm+q} is the point Q_m, so that $\sigma_{qm-1} \wedge \sigma_{qm+q} \in \ker(\Phi)$ and the only nonzero value for $\mathbb{W}$ is $\mathbb{W}_{Q_m}$. Here (6.3.2) is used:

$$\mathbb{W}_{Q_m}(\sigma_{qm-1} \wedge \sigma_{qm+q}) = \det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1. \qquad \text{Q.E.D.}$$

It remains to analyze the image of the $\mathbb{W}_{R_k}$'s. Unfortunately, we have not been able to find sections ω_1 and ω_2 of ω_X which have only one R_k in their common support, so that we could use the techniques of the previous two propositions. We fall back on brute force.

(6.6) LEMMA.

(6.6.1) $$\mathbb{W}_{R_k}(\eta_B \wedge \sigma_\alpha) = \begin{cases} 0 & \text{if } k \neq \alpha \\ 1 & \text{if } k = \alpha \end{cases}.$$

(6.6.2) $$\mathbb{W}_{R_k}(\sigma_\alpha \wedge \sigma_{\alpha-q}) = \begin{cases} 0 & \text{if } k \neq \alpha \\ 1 & \text{if } k = \alpha \end{cases}.$$

(6.6.3) $$\mathbb{W}_{R_k}(\sigma_\alpha \wedge \sigma_{\alpha+q}) = \begin{cases} 0 & \text{if } k \neq \alpha + q \\ -1 & \text{if } k = \alpha + q \end{cases}.$$

Proof. (6.6.1) is a straightforward application of (6.3.3):

$$\mathbb{W}_{R_k}(\eta_B \wedge \sigma_\alpha) = \det\begin{pmatrix} 1 & 0 \\ * & \operatorname{res}_{P_k}(\sigma_\alpha|_{A_k}) \end{pmatrix} = \operatorname{res}_{P_k}(\sigma_\alpha|_{A_k}) = \begin{cases} 0 & \text{if } k \neq \alpha \\ 1 & \text{if } k = \alpha \end{cases}.$$

To prove (6.6.2), note that $\operatorname{res}_{P_k}(\sigma_\alpha|_{A_k}) \neq 0$ only if $k = \alpha$, so that the second column of the matrix used to compute $\mathbb{W}_{R_k}(\sigma_\alpha \wedge \sigma_{\alpha-q})$ is nonzero only if $k = \alpha$ or $k = \alpha - q$. We have

$$\mathbb{W}_{R_\alpha}(\sigma_\alpha \wedge \sigma_{\alpha-q}) = \det\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = 1$$

and

$$\mathbb{W}_{R_{\alpha-q}}(\sigma_\alpha \wedge \sigma_{\alpha-q}) = \det\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} = 0,$$

proving (6.6.2). The final statement follows from (6.6.2). Q.E.D.

(6.7) COROLLARY. *Assume $p = 2$ and $n \geqslant 7$. Then*

$$\eta_B \wedge \sigma_\alpha - \sigma_\alpha \wedge \sigma_{\alpha-q} - \sigma_\alpha \wedge \sigma_{\alpha+q} \in \ker(\Phi) \quad \textit{and}$$

$$\mathbb{W}_{R_k}(\eta_B \wedge \sigma_\alpha - \sigma_\alpha \wedge \sigma_{\alpha-q} - \sigma_\alpha \wedge \sigma_{\alpha+q}) = \begin{cases} 0 & \textit{if } k \neq \alpha \\ 1 & \textit{if } k = \alpha \end{cases}.$$

Proof. The first statement follows from (5.6.2), and the calculation of $\mathbb{W}_{R_k}$ is obtained from the previous lemma. Q.E.D.

Let us proceed to the case when g is odd, so that n is even and p cannot be equal to 2.

(6.8) LEMMA. *Assume p is odd and $2p + 3 \leqslant n$.*

(6.8.1) $$\mathbb{W}_{R_k}(\sigma_\alpha \wedge \sigma_{\alpha+1-q}) = \begin{cases} 0 & \textit{if } k \neq \alpha + 1 - q \\ -1 & \textit{if } k = \alpha + 1 - q \end{cases}.$$

(6.8.2) $$\mathbb{W}_{R_k}(\Gamma_m) = \begin{cases} -1 & \textit{if } k = qm - 1 \\ -\frac{1}{2} & \textit{if } k = qm - q, qm - 3q, qm - 5q, \dots, qm - (p-2)q \\ \frac{1}{2} & \textit{if } k = qm - 2q, qm - 4q, \dots, qm - (p-1)q \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. Unless $k = \alpha$ or $k = \alpha + 1 - q$, the second column of the matrix whose determinant computes (6.8.1) is zero. For $k = \alpha$, the second row of the matrix is zero; here we use $2p + 3 \leqslant n$. For $k = \alpha + 1 - q$,

$$\mathbb{W}_{R_{\alpha+1-q}}(\sigma_\alpha \wedge \sigma_{\alpha+1-q}) = \det\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = -1,$$

proving (6.8.1).

The second statement is a straightforward computation:

$$\mathbb{W}_{R_k}(\Gamma_m) = -\frac{1}{2}\left[\mathbb{W}_{R_k}(\eta_B \wedge \sigma_{qm-1}) - \sum_{x=0}^{p-1}(-1)^x \mathbb{W}_{R_k}(\sigma_{qm-1-xq} \wedge \sigma_{qm-(x+1)q})\right]$$

$$= \begin{Bmatrix} 0 & \text{if } k \neq qm-1 \\ -\frac{1}{2} & \text{if } k = qm-1 \end{Bmatrix}$$

$$-\sum_{x=0}^{p-1}(-1)^x \begin{Bmatrix} 0 & \text{if } k \neq qm-(x+1)q \\ \frac{1}{2} & \text{if } k = qm-(x+1)q \end{Bmatrix},$$

which is zero unless $k = qm - q, qm - 2q, qm - 3q, \ldots, qm - (p-1)q = qm - 1 + q$, or $qm - pq = qm - 1$. If $k = qm - 1$, then $\mathbb{W}_{R_k}(\Gamma_m) = -1/2 - (-1)^{p-1}(1/2) = -1$, since p is odd. If $k = qm - yq$ with $1 \leqslant y \leqslant p-2$ and y odd, then $\mathbb{W}_{R_k}(\Gamma_m) = -(-1)^{y-1}(1/2) = -1/2$. If $k = qm - yq$ with $2 \leqslant y \leqslant p - 1$ and y even, then $\mathbb{W}_{R_k}(\Gamma_m) = -(-1)^{y-1}(1/2) = 1/2$. This is (6.8.2). Q.E.D.

(6.9) COROLLARY. *Assume p is odd and $2p + 3 \leqslant n$. Then*

$$K_\alpha = -2(\alpha_\alpha \wedge \sigma_{\alpha+1-q} + \Gamma_{p\alpha+p-1} + \Gamma_{p\alpha+p}) \in \ker(\Phi) \quad \textit{and}$$

$$\mathbb{W}_{R_k}(K_\alpha) = \begin{cases} 0 & \textit{if } k \neq \alpha, \alpha - q, \alpha + 1 - q \\ 1 & \textit{if } k = \alpha \\ 2 & \textit{if } k = \alpha - q \\ 3 & \textit{if } k = \alpha + 1 - q \end{cases}$$

Proof. The first statement is (5.7.2), with $r = p - 1$. To finish, again just compute:

$$\mathbb{W}_{R_k}(-2(\sigma_\alpha \wedge \sigma_{\alpha+1-q} + \Gamma_{p\alpha+p-1} - \Gamma_{p\alpha+p}))$$

$$= \begin{Bmatrix} 0 & \text{if } k \neq \alpha+1-q \\ 2 & \text{if } k = \alpha+1-q \end{Bmatrix}$$

$$+ \begin{Bmatrix} 2 & \text{if } k = \alpha - q \\ 1 & \text{if } k = \alpha + 1 - 2q, \alpha + 1 - 4q, \ldots, \alpha + 1 - (p-1)q \\ -1 & \text{if } k = \alpha + 1 - 3q, \alpha + 1 - 5q, \ldots, \alpha + 1 - pq \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$+ \begin{Bmatrix} 2 & \text{if } k = \alpha \\ 1 & \text{if } k = \alpha + 1 - q, \alpha + 1 - 3q, \ldots, \alpha + 1 - (p-2)q \\ -1 & \text{if } k = \alpha + 1 - 2q, \alpha + 1 - 4q, \ldots, \alpha + 1 - (p-1)q \\ 0 & \text{otherwise} \end{Bmatrix}.$$

The terms of the last two parts telescope, leaving as the only nonzero terms $k = \alpha + 1 - q$, $\alpha - q$, and α. If $k = \alpha$, $\mathbb{W}_{R_k} = -1$ (from the second part) + 2 (from the third part) = 1. If $k = \alpha - q$, $\mathbb{W}_{R_k} = 2$ (from the second part). If $k = \alpha + 1 - q$, then $\mathbb{W}_{R_k} = 2$ (from the first part) + 1 (from the third part) = 3. Q.E.D.

Let $U \subset \ker(\Phi)$ be the subspace spanned by the K_α's, and let $V \subset H^0(\mathrm{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1))$ be the subspace consisting of sections supported at the R_k's. From the previous corollary, the projection of the Wahl map from U to V has as image the row space of an $n \times n$ "circulant" matrix C whose first row is $(2, 3, 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$ (where the 1 is the $(q+1)$st entry), and whose succeeding rows are obtained from the first by a cyclic shift. (This first row is the image of K_{q+1}.)

(6.10) LEMMA. *The circulant matrix C is nonsingular if q is odd.*

Proof. In general, the determinant of a circulant matrix whose first row is $(a_0, \ldots, a_{n-1})$ is

$$\prod_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} \zeta_n^{ij} a_j \right),$$

where $\zeta_n = e^{2\pi i/n}$ (see [G], section 34, exercise 19, for example). Therefore we must show that for each i, $\zeta_n^{iq} + 3\zeta_n^i + 2 \neq 0$. Assume that there exists an i such that $\zeta_n^{iq} + 3\zeta_n^i + 2 = 0$; then $3\zeta_n^i + 2 = -\zeta_n^{iq}$, which is an nth root of unity and hence is on the unit circle. Write $\zeta_n^i = x + iy$; then $3\zeta_n^i + 2 = (3x + 2) = 3iy$, and so $9x^2 + 12x + 4 + 9y^2 = 1$. Since $x^2 + y^2 = 1$, we have $12x + 4 + 9 = 1$, or $x = -1$, forcing $y = 0$ and $\zeta_n^i = -1$. However, in this case $\zeta_n^{iq} + 3\zeta_n^i + 2 = (-1)^q - 3 + 2 = -2$, since q is odd. This contradiction proves the lemma. Q.E.D.

7. The cases of genera $\neq$ 13. The analysis of the torsion part in the previous section puts us in position to prove our main theorems. We begin with the even genus case.

(7.1) THEOREM. *Assume g is even and $g \geqslant 10$. Then the Wahl map for the graph curve $X_{g-1,2}$ of genus g is surjective.*

Proof. By hypothesis $n = g - 1 \geqslant 9$ and $p = 2$, so that $3p + 2 = 8 \leqslant n$. By Proposition (5.5), Φ is surjective, so that we need only check the surjectivity onto the torsion part. This follows from (6.4), (6.5), and (6.7) once we check that $q + 4 \leqslant n$. Since n is odd, $q = g/2$, so we require $(g/2) \leqslant g - 5$, or $g \geqslant 10$. Q.E.D.

The odd genus case is trickier.

(7.2) THEOREM. *Assume that g is odd and $g \geqslant 15$. Let p be the smallest prime number not dividing $g - 1$. Then the Wahl map for the graph curve $X_{g-1,p}$ of genus g is surjective.*

Proof. As above, let $n = g - 1$ and $q = p^{-1} \pmod{n}$. Assume for the moment that $3p + 2 \leqslant n$ and $q \leqslant n - 4$. By (5.5), Φ is surjective, and we must show that $\mathbb{W}$: $\ker(\Phi) \to H^0(\mathrm{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1))$ is surjective. Since n is even, p and q are odd, so that (6.4), (6.5), (6.9), and (6.10) apply. Hence the column space of a matrix of the form

$$\begin{pmatrix} I_n & 0 & * \\ 0 & I_n & * \\ 0 & 0 & C^T \end{pmatrix}$$

is in the image of $\mathbb{W}|_{\ker(\Phi)}$, where the rows are ordered by the P_i's, Q_j's, and R_k's, in that order, and the elements generating the columns are those of (6.4), (6.5), and (6.9), in a suitable order. By (6.10), this matrix has rank $3n$; since this is the dimension of $H^0(\mathrm{Tors}(\omega_X^{\otimes 2} \otimes \Omega_X^1))$, $\mathbb{W}|_{\ker(\Phi)}$ is surjective.

It remains to show that $3p + 2 \leqslant n$ and $q \leqslant n - 4$. Assume that $3p + 2 \geqslant n + 1$, or $n \leqslant 3p + 1$. If $p = 3$, then $n \leqslant 10$, or $g \leqslant 11$, a contradiction. If $p = 5$, then $n \leqslant 16$; however, if $n = 14$ or 16, $p = 3$, and $n \geqslant 14$. We may therefore assume $p \geqslant 7$. Let p_m denote the mth prime number, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. Assume that $p = p_m$; hence $2 \cdot 3 \cdot \cdots \cdot p_{m-1} | n$. But $3p_m + 1 \geqslant n \geqslant 2 \cdot 3 \cdot \cdots \cdot p_{m-1} \geqslant 6p_{m-1} \geqslant 3p_m + 6$, since there is always a prime number between r and $2r - 2$ for any $r > 7/2$ (Bertrand's postulate, which we apply with $r = p_{m-1}$). This contradiction proves that $3p + 2 \leqslant n$.

Finally, we must show that $q \leqslant n - 4$. Assume not; then, since n is even and q is odd, we must have either $q = n - 1$ or $q = n - 3$. If $q = n - 1$, then $p = q^{-1} \pmod{n} = n - 1$, violating $3p + 2 \leqslant n$. If $q = n - 3$, then 3 is relatively prime to n, so $p = 3$. Therefore $1 \pmod{n} = pq \pmod{n} = -9 \pmod{n}$, so $n = 10$ and $g = 11$, violating $g \geqslant 15$. Q.E.D.

8. The genus 13 case and the main theorem. Although the specific results of the previous sections do not suffice to prove the surjectivity of the Wahl map for genus 13, the graph curve $X_{12,5}$ ($n = 12$, $p = q = 5$) of genus 13 is a reasonable candidate to try the calculations out on. What we will show in this section is that in fact the Wahl map for $X_{12,5}$ *is* surjective.

By Proposition (5.5), since $p = 5$ is odd and $2p \leqslant n$, we have that Φ: $\bigwedge^2 H^0(\omega_{X_{12,5}}) \to \bigoplus_C H^0(\omega_{X_{12,5}}^{\otimes 2}|_C \otimes \Omega_C^1)$ is surjective; this is the surjectivity of the Wahl map modulo torsion. Hence we need only demonstrate the surjectivity onto the torsion.

Note also that $2p + 2 \leqslant n$; therefore Proposition (5.7) applies and $\ker(\Phi)$ is spanned by the following elements:

(8.1) $$\sigma_\alpha \wedge \sigma_{\alpha+6}, \quad \text{for any } \alpha;$$

(8.2) $$K_{r,\alpha} = \sigma_\alpha \wedge \sigma_{\alpha+5r} + \Gamma_{5\alpha+r} + \Gamma_{5\alpha+5}, \quad \text{for } r = 1,2,3,4, \text{ and any } \alpha.$$

Note that in this case

$$\Gamma_m = -\frac{1}{2}\left[\eta_B \wedge \sigma_{5m-1} - \sum_{k=0}^{4} (-1)^k \sigma_{5m-1-5k} \wedge \sigma_{5m-5k-5}\right]$$

$$= -\frac{1}{2}[\eta_B \wedge \sigma_{5m+11} - \sigma_{5m+11} \wedge \sigma_{5m+7} + \sigma_{5m+6} \wedge \sigma_{5m+2} - \sigma_{5m+1} \wedge \sigma_{5m+9}$$

$$+ \sigma_{5m+8} \wedge \sigma_{5m+4} - \sigma_{5m+3} \wedge \sigma_{5m+11}].$$

First some preliminary calculations, which are immediate consequences of Proposition (6.3).

(8.3) LEMMA. *Assume that $n = 12$ and $p = q = 5$, and fix any α.*

(8.3.1) $$\mathbb{W}_{P_i}(\eta_B \wedge \sigma_\alpha) = 0 \quad \textit{for every } 1,$$

$$\mathbb{W}_{Q_j}(\eta_B \wedge \sigma_\alpha) = 0 \quad \textit{for every } j, \quad \textit{and}$$

$$\mathbb{W}_{R_k}(\eta_B \wedge \sigma_\alpha) = \begin{cases} 1 & \textit{if } k = \alpha \\ 0 & \textit{if } k \neq \alpha \end{cases}.$$

(8.3.2) $$\mathbb{W}_{P_i}(\sigma_\alpha \wedge \sigma_{\alpha+3}) = 0 \quad \textit{for every } i,$$

$$\mathbb{W}_{Q_j}(\sigma_\alpha \wedge \sigma_{\alpha+3}) = 0 \quad \textit{for every } j, \quad \textit{and}$$

$$\mathbb{W}_{R_k}(\sigma_\alpha \wedge \sigma_{\alpha+3}) = \begin{cases} -1 & \textit{if } k = \alpha + 3 \\ 0 & \textit{if } k \neq \alpha + 3 \end{cases}.$$

(8.3.3) $$\mathbb{W}_{P_i}(\sigma_\alpha \wedge \sigma_{\alpha+5}) = 0 \quad \textit{for every } i,$$

$$\mathbb{W}_{Q_j}(\sigma_\alpha \wedge \sigma_{\alpha+5}) = \begin{cases} -1 & \textit{if } j = 5\alpha \textit{ or } 5\alpha + 5 \\ 0 & \textit{otherwise} \end{cases}, \quad \textit{and}$$

$$\mathbb{W}_{R_k}(\sigma_\alpha \wedge \sigma_{\alpha+5}) = \begin{cases} -1 & \textit{if } k = \alpha + 5 \\ 0 & \textit{if } k \neq \alpha + 5 \end{cases}.$$

(8.3.4) $$\mathbb{W}_{P_i}(\sigma_\alpha \wedge \sigma_{\alpha+6}) = 0 \quad \textit{for every } i,$$

$$\mathbb{W}_{Q_j}(\sigma_\alpha \wedge \sigma_{\alpha+6}) = \begin{cases} 1 & \textit{if } j = 5\alpha + 5 \\ -1 & \textit{if } j = 5\alpha + 11 \\ 0 & \textit{otherwise} \end{cases}, \quad \textit{and}$$

$$\mathbb{W}_{R_k}(\sigma_\alpha \wedge \sigma_{\alpha+6}) = 0 \quad \textit{for every } k.$$

(8.3.5) $$\mathbb{W}_{P_i}(\sigma_\alpha \wedge \sigma_{\alpha+8}) = 0 \quad \textit{for every } i,$$

$$\mathbb{W}_{Q_j}(\sigma_\alpha \wedge \sigma_{\alpha+8}) = \begin{cases} -1 & \textit{if } j = 5\alpha + 4 \\ 0 & \textit{if } j \neq 5\alpha + 4 \end{cases}, \quad \textit{and}$$

$$\mathbb{W}_{R_k}(\sigma_\alpha \wedge \sigma_{\alpha+8}) = \begin{cases} -1 & \textit{if } k = \alpha + 8 \\ 0 & \textit{if } k \neq \alpha + 8 \end{cases}.$$

(8.3.6) $$\mathbb{W}_{P_i}(\sigma_\alpha \wedge \sigma_{\alpha+10}) = \begin{cases} -1 & \textit{if } i = \alpha - 1 \\ 0 & \textit{if } i \neq \alpha - 1 \end{cases}.$$

These are the basic tools for the rest of the calculation. Using (8.3.1) and (8.3.5), we have

(8.4) COROLLARY. *Fix any* m. *Then*

$$\mathbb{W}_{P_i}(\Gamma_m) = 0 \quad \textit{for every } i,$$

$$\mathbb{W}_{Q_j}(\Gamma_m) = \begin{cases} -\frac{1}{2} & \textit{if } j = m-1,\ m-3, \textit{ or } m-5 \\ +\frac{1}{2} & \textit{if } j = m-2 \textit{ or } m-4 \\ 0 & \textit{otherwise} \end{cases}, \quad \textit{and}$$

$$\mathbb{W}_{R_k}(\Gamma_m) = \begin{cases} -1 & \textit{if } k = 5m-1 \\ -\frac{1}{2} & \textit{if } k = 5m+7 \textit{ or } 5m+9 \\ +\frac{1}{2} & \textit{if } k = 5m+2 \textit{ or } 5m+4 \\ 0 & \textit{otherwise} \end{cases}.$$

Now the Wahl map on the elements $K_{r,\alpha}$ generating the kernel of Φ can be found:

(8.5) COROLLARY. *Fix any* α.

(8.5.1) $\mathbb{W}_{P_i}(K_{1,\alpha}) = 0$ *for every* i,

$$\mathbb{W}_{Q_j}(K_{1,\alpha}) = \begin{cases} -2 & \textit{if } j = 5\alpha \\ -1 & \textit{if } j = 5\alpha + 5 \\ +\frac{1}{2} & \textit{if } j = 5\alpha+1, 5\alpha+3, 5\alpha+9, \textit{ or } 5\alpha+11, \\ -\frac{1}{2} & \textit{if } j = 5\alpha+2, 5\alpha+4, 5\alpha+8, \textit{ or } 5\alpha+10 \\ 0 & \textit{if } j = 5\alpha+6 \textit{ or } 5\alpha+7 \end{cases},$$

and

$$\mathbb{W}_{R_k}(K_{1,\alpha}) = \begin{cases} -\frac{3}{2} & \textit{if } k = \alpha \\ -1 & \textit{if } k = \alpha + 4 \\ +\frac{1}{2} & \textit{if } k = \alpha + 3,\ \alpha + 7,\ \textit{or } \alpha + 9 \\ -\frac{1}{2} & \textit{if } k = \alpha + 2,\ \alpha + 5,\ \alpha + 8,\ \textit{or } \alpha + 10 \\ 0 & \textit{if } k = \alpha + 1,\ \alpha + 6,\ \textit{or } \alpha + 11 \end{cases}.$$

(8.5.2) $$\mathbb{W}_{P_i}(K_{2,\alpha}) = \begin{cases} -1 & \textit{if } i = \alpha - 1 \\ 0 & \textit{if } i \neq \alpha - 1 \end{cases}.$$

(8.5.3) $\mathbb{W}_{P_i}(K_{3,\alpha}) = 0$ *for every* i,

$$\mathbb{W}_{Q_j}(K_{3,\alpha}) = \begin{cases} +1 & \textit{if } k = 5\alpha + 1 \\ -1 & \textit{if } k = 5\alpha \textit{ or } 5\alpha + 2 \\ +\frac{1}{2} & \textit{if } k = 5\alpha + 3 \textit{ or } 5\alpha + 11, \\ -\frac{1}{2} & \textit{if } k = 5\alpha + 4 \textit{ or } 5\alpha + 10 \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and}$$

$$\mathbb{W}_{R_k}(K_{3,\alpha}) = \begin{cases} -\frac{3}{2} & \textit{if } k = \alpha \\ +1 & \textit{if } k = \alpha + 5 \\ -1 & \textit{if } k = \alpha + 2 \textit{ or } \alpha + 10 \\ +\frac{1}{2} & \textit{if } k = \alpha + 7 \\ -\frac{1}{2} & \textit{if } k = \alpha + 3 \textit{ or } \alpha + 8 \\ 0 & \textit{otherwise} \end{cases}.$$

(8.5.4) $\mathbb{W}_{P_i}(K_{4,\alpha}) = 0$ *for every* i,

$$\mathbb{W}_{Q_j}(K_{4,\alpha}) = \begin{cases} -\frac{3}{2} & \textit{if } j = 5\alpha + 4 \\ -\frac{1}{2} & \textit{if } j = 5\alpha + 11, \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and}$$

$$\mathbb{W}_{R_k}(K_{4,\alpha}) = \begin{cases} -\frac{3}{2} & \textit{if } k = \alpha + 8 \\ -1 & \textit{if } k = \alpha + 7 \\ -\frac{1}{2} & \textit{if } k = \alpha \\ 0 & \textit{otherwise} \end{cases}.$$

We are now in a position to make the analysis. We must show that the vectors in $\mathbb{C}^{36}$ whose coordinates are the $\mathbb{W}_{P_i}$, $\mathbb{W}_{Q_j}$, and $\mathbb{W}_{R_k}$ values of the elements $\sigma_\alpha \wedge \sigma_{\alpha+6}$, and the $K_{r,\alpha}$'s, span $\mathbb{C}^{36}$. Since $\mathbb{W}_{P_i}(K_{2,\alpha})$ span the P_i coordinates, and the $\mathbb{W}_{P_i}$ coordinates of all other elements are zero, it suffices to show that the $\mathbb{W}_{Q_j}$ and $\mathbb{W}_{R_k}$ coordinates of the elements $\sigma_\alpha \wedge \sigma_{\alpha+6}$ and the $K_{r,\alpha}$'s, $r \neq 2$, span $\mathbb{C}^{24}$. In fact, using the elements $\sigma_\alpha \wedge \sigma_{\alpha+6}$, it suffices to show that the $\mathbb{W}_{Q_j} + \mathbb{W}_{Q_{j+6}}$

($j = 0, \ldots, 5$) and the $\mathbb{W}_{R_k}$ ($k = 0, \ldots, 11$) coordinates of the elements $K_{1,\alpha}$, $K_{3,\alpha}$, and $K_{4,\alpha}$ span $\mathbb{C}^{18}$. After multiplying by 2 to eliminate denominators, these 36 vectors in $\mathbb{C}^{18}$ are the following:

	[Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}]
$K_{1,0}$:	[−4	+1	−2	+2	−2	−1	−3	0	−1	+1	−2	−1	0	+1	−1	+1	−1	0]
$K_{1,1}$:	[+1	−2	+2	−2	−1	−4	0	−3	0	−1	+1	−2	−1	0	+1	−1	+1	−1]
$K_{1,2}$:	[−2	+2	−2	−1	−4	+1	−1	0	−3	0	−1	+1	−2	−1	0	+1	−1	+1]
$K_{1,3}$:	[+2	−2	−1	−4	+1	−2	+1	−1	0	−3	0	−1	+1	−2	−1	0	+1	−1]
$K_{1,4}$:	[−2	−1	−4	+1	−2	+2	−1	+1	−1	0	−3	0	−1	+1	−2	−1	0	+1]
$K_{1,5}$:	[−1	−4	+1	−2	+2	−2	+1	−1	+1	−1	0	−3	0	−1	+1	−2	−1	0]
$K_{1,6}$:	[−4	+1	−2	+2	−2	−1	0	+1	−1	+1	−1	0	−3	0	−1	+1	−2	−1]
$K_{1,7}$:	[+1	−2	+2	−2	−1	−4	−1	0	+1	−1	+1	−1	0	−3	0	−1	+1	−2]
$K_{1,8}$:	[−2	+2	−2	−1	−4	+1	−2	−1	0	+1	−1	+1	−1	0	−3	0	−1	+1]
$K_{1,9}$:	[+2	−2	−1	−4	+1	−2	+1	−2	−1	0	+1	−1	+1	−1	0	−3	0	−1]
$K_{1,10}$:	[−2	−1	−4	+1	−1	+2	−1	+1	−2	−1	0	+1	−1	+1	−1	0	−3	0]
$K_{1,11}$:	[−1	−4	+1	−1	+2	−2	0	−1	+1	−2	−1	0	+1	−1	+1	−1	0	−3]
$K_{3,0}$:	[−2	+2	−2	+1	−2	+1	−3	0	−2	−1	0	+2	0	+1	−1	0	−2	0]
$K_{3,1}$:	[+2	−2	+1	−2	+1	−2	0	−3	0	−2	−1	0	+2	0	+1	−1	0	−2]
$K_{3,2}$:	[−2	+1	−2	+1	−2	+2	−2	0	−3	0	−2	−1	0	+2	0	+1	−1	0]
$K_{3,3}$:	[+1	−2	+1	−2	+2	−2	0	−2	0	−3	0	−2	−1	0	+2	0	+1	−1]
$K_{3,4}$:	[−2	+1	−2	+2	−2	+1	−1	0	−2	0	−3	0	−2	−1	0	+2	0	+1]
$K_{3,5}$:	[+1	−2	+2	−2	+1	−2	+1	−1	0	−2	0	−3	0	−2	−1	0	+2	0]
$K_{3,6}$:	[−2	+2	−2	+1	−2	+1	0	+1	−1	0	−2	0	−3	0	−2	−1	0	+2]
$K_{3,7}$:	[+2	−2	+1	−2	+1	−2	+2	0	+1	−1	0	−2	0	−3	0	−2	−1	0]
$K_{3,8}$:	[−2	+1	−2	+1	−2	+2	0	+2	0	+1	−1	0	−2	0	−3	0	−2	−1]
$K_{3,9}$:	[+1	−2	+1	−2	+2	−2	−1	0	+2	0	+1	−1	0	−2	0	−3	0	−2]
$K_{3,10}$:	[−2	+1	−2	+2	−2	+1	−2	−1	0	+2	0	+1	−1	0	−2	0	−3	0]
$K_{3,11}$:	[+1	−2	+2	−2	+1	−2	0	−2	−1	0	+2	0	+1	−1	0	−2	0	−3]
$K_{4,0}$:	[0	0	0	0	−3	−1	−1	0	0	0	0	0	0	−2	−3	0	0	0]
$K_{4,1}$:	[0	0	0	−3	−1	0	0	−1	0	0	0	0	0	0	−2	−3	0	0]
$K_{4,2}$:	[0	0	−3	−1	0	0	0	0	−1	0	0	0	0	0	0	−2	−3	0]
$K_{4,3}$:	[0	−3	−1	0	0	0	0	0	0	−1	0	0	0	0	0	0	−2	−3]
$K_{4,4}$:	[−3	−1	0	0	0	0	−3	0	0	0	−1	0	0	0	0	0	0	−2]
$K_{4,5}$:	[−1	0	0	0	0	−3	−2	−3	0	0	0	−1	0	0	0	0	0	0]
$K_{4,6}$:	[0	0	0	0	−3	−1	0	−2	−3	0	0	0	−1	0	0	0	0	0]
$K_{4,7}$:	[0	0	0	−3	−1	0	0	0	−2	−3	0	0	0	−1	0	0	0	0]
$K_{4,8}$:	[0	0	−3	−1	0	0	0	0	0	−2	−3	0	0	0	−1	0	0	0]
$K_{4,9}$:	[0	−3	−1	0	0	0	0	0	0	0	−2	−3	0	0	0	−1	0	0]
$K_{4,10}$:	[−3	−1	0	0	0	0	0	0	0	0	0	−2	−3	0	0	0	−1	0]
$K_{4,11}$:	[−1	0	0	0	0	−3	0	0	0	0	0	0	−2	−3	0	0	0	−1]

It may be checked by any reasonably intelligent machine, using a reasonably persistent typist, that this matrix in fact does have full rank 18. We used MACSYMA. Therefore:

(8.6) THEOREM. *The Wahl map for the graph curve $X_{12,5}$ is surjective.*

Since the surjectivity of the Wahl map is a Zariski open condition in moduli, we have the following "generic" statement, using (7.1), (7.2), and (8.6):

(8.7) THEOREM. *Assume that $g \geqslant 12$ or $g = 10$. Then the Wahl map for the general curve of genus g is surjective.*

This is our main theorem. Note that it is the best possible result in this vein; as mentioned in the introduction, it is known that for all stable curves of genus $\leqslant 9$ or genus 11, the Wahl map fails to be surjective.

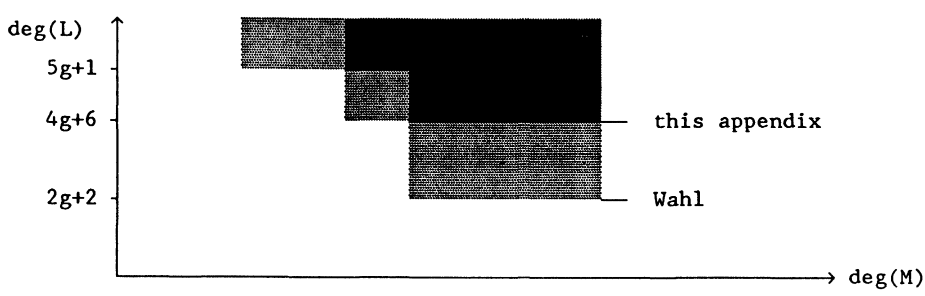
Appendix: Surjectivity of the map $\mathbb{W}_L$ for L of large degree. In the text, it is mentioned that the map

$$\mathbb{W}_L \colon \Lambda^2 H^0(G, L) \to H^0(C, KL^2)$$

is surjective whenever the degree d of L is large enough with respect to the genus g of C. More generally, Wahl [W] has shown that the map $\mathbb{W}_{L,M}$ is surjective if $\deg(L) \geqslant 5g + 1$ and $\deg(M) \geqslant 2g + 2$, implying of course that $\mathbb{W}_L$ is surjective when $d \geqslant 5g + 1$. In this appendix, we give an elementary proof that $\mathbb{W}_L$ is surjective whenever $d \geqslant 4g + 6$ (and, more generally, that $\mathbb{W}_{L,M}$ is surjective if both $\deg(L)$ and $\deg(M) \geqslant 4g + 6$). The set of pairs of degrees for which $\mathbb{W}_{L,M}$ is known to be surjective is thus the shaded area in the diagram below; it seems likely that the actual region is somewhat larger, though we have no guess how large.

To prove our statement, we use the characterization given in the text, that the Wahl map $\mathbb{W}_L$ associated to a complete linear series $V = H^0(C, L)$ will be surjective whenever

$$H^1(C \times C, N(-2\Delta)) = 0,$$



where π_1, π_2: $C \times C \to C$ are the projection maps and $N = \pi_1^*L \otimes \pi_2^*L$. To check this condition, observe first that a curve embedded by a complete linear series $|M|$ of degree $2g + 1$ or greater is quadratically normal, so that the multiplication map

$$H^0(C, M) \otimes H^0(C, M) \to H^0(C, M^2)$$

is surjective. It follows, since $H^1(C \times C, \pi_1^*M \times \pi_2^*M) = 0$, that

$$H^1(C \times C, \pi_1^*M \otimes \pi_2^*M \otimes \mathcal{O}(-\Delta)) = 0.$$

From this we see that if M is any line bundle of degree $2g + 3$ or more, the restriction map from the line bundle $\pi_1^*M \otimes \pi_2^*M \otimes \mathcal{O}(-\Delta)$ on $C \times C$ to any pair of fibers of $C \times C$ over C is surjective on global sections. Since $\pi_1^*M \otimes \pi_2^*M \otimes \mathcal{O}(-\Delta)$ is very ample on each fiber, it follows that it is very ample (and has vanishing first cohomology) on $C \times C$.

Now suppose that L has degree at least $4g + 6$; as before, set $N = \pi_1^*L \otimes \pi_2^*L$. If the degree of L is even, we can write the line bundle $N(-2\Delta)$ as a square

$$N(-2\Delta) = P^2,$$

where

$$P = \pi_1^*M \otimes \pi_2^*M \otimes \mathcal{O}(-\Delta)$$

for some line bundle M on C of degree at least $2g + 3$. By the above, the linear system $|P|$ will contain smooth irreducible curves D; consider the sequence

$$0 \to \mathcal{O}_{C\times C}(P) \to \mathcal{O}_{C\times C}(2P) \to \mathcal{O}_D(2P) \to 0.$$

By degree considerations the line bundle $\mathcal{O}_D(2P)$ is nonspecial; since we already know that $H^1(C \times C, \mathcal{O}_{C\times C}(P)) = 0$, it follows that $H^1(C \times C, N(-2\Delta)) = 0$.

Essentially the same argument applies in case the degree of L is odd, and to show that the map $\mathbb{W}_{L,M}$ is surjective whenever the degrees of both L and M are greater than or equal to $4g + 6$.

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