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NONCLASSICAL GODEAUX SURFACES IN CHARACTERISTIC FIVE

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ABSTRACT. A classical Godeaux surface is a smooth minimal projective surface X, with $K_X^2 = 1$, $p_a = p_g = 0$ and $\mathrm{Pic}^\tau(X) = \mathbf{Z}/5\mathbf{Z}$. A *nonclassical* Godeaux surface is a smooth minimal projective surface X with $K_X^2 = 1$, $p_a = 0$, $p_g = 1$ and $\mathrm{Pic}^\tau(X) = \mu_5$ or α_5; such surfaces should exist in characteristic 5. It is the purpose of this note to construct nonclassical Godeaux surfaces in characteristic 5, with $\mathrm{Pic}^\tau(X) = \mu_5$. The method is to exhibit a smooth quintic surface on which $\mathbf{Z}/5\mathbf{Z}$ acts, so that the quotient is smooth; this quotient is the desired surface.

Let k be an algebraically closed field of characteristic 5. Choose coordinates $[x, y, z, w]$ for $\mathbf{P}_k^3$; this allows us to identify $\mathrm{Aut}_k(\mathbf{P}^3)$ with $\mathrm{PGL}(3, k)$, which I will do. Let σ be the automorphism represented by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This element has order 5 and generates a cyclic group $\langle\sigma\rangle$ isomorphic to $\mathbf{Z}/5\mathbf{Z}$ on $\mathbf{P}^3$. It is not difficult to see that every action of $\mathbf{Z}/5\mathbf{Z}$ on $\mathbf{P}^3$ which has only isolated fixed points is conjugate to the action of $\langle\sigma\rangle$. The only fixed point of σ is $[1, 0, 0, 0]$.

For any linear form l in x, y, z and w let $N(l) = \prod_{i=0}^4 \sigma^i(l)$ denote the norm of l under the action of σ; $N(l)$ is a quintic form which is invariant under σ. Let V be the subspace of the space of quintic forms which is generated by norms of linear functions. Let ϕ: $\mathbf{P}^3 \dashrightarrow \mathbf{P}(V)$ be the (a priori rational) map defined by the subspace V. I claim that, in fact, ϕ is defined everywhere and is étale away from the fixed point $[1, 0, 0, 0]$. The argument proceeds in several steps.

Let $C = \{\alpha \in \mathrm{PGL}(3) | \alpha\sigma = \sigma\alpha\}$ be the centralizer of σ. Then

$$C = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} \middle| \, a \neq 0 \right\},$$

and the action of C on $\mathbf{P}^3$ decomposes $\mathbf{P}^3$ into 4 orbits A_0, A_1, A_2 and A_3, where $A_0 = \{[1, 0, 0, 0]\}$ is the fixed point, $A_1 = \{[x, 1, 0, 0]\}$, $A_2 = \{[x, y, 1, 0]\}$ and

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$A_3 = \{[x, y, z, 1]\}$. Note that C acts on forms in x, y, z and w in the standard way.

LEMMA 1. (1.1) *If f is a quintic form in x, y, z and w which is invariant under σ, and $\alpha \in C$, then $\alpha \cdot f$ is an invariant quintic.*

(1.2) *Let $p \in \mathbf{P}^3$, and assume that there exists an invariant quintic form f such that $f(p) \neq 0$. Then for any q in the orbit of p under C, there exists an invariant quintic form g such that $g(q) \neq 0$.*

(1.3) *Let $(p_1, p_2) \in \mathbf{P}^3 \times \mathbf{P}^3$ and assume that there exists an invariant quintic form f such that $f(p_1) = 0$ and $f(p_2) \neq 0$. Then for any (q_1, q_2) in the orbit of (p_1, p_2) under C (using the diagonal action), there exists an invariant quintic form g such that $g(q_1) = 0$ and $g(q_2) \neq 0$.*

(1.4) *Let $p \in \mathbf{P}^3$ and assume that the space V of norms of linear functions separates tangent vectors at p. Then V separates tangent vectors at any q in the same orbit of p under C.*

PROOF. The first statement is trivial; the rest follow from it, using the general principle that if an invariant f has a certain property at p, and if $\alpha \in C$, then the invariant $\alpha^{-1} \cdot f$ has that property at $q = \alpha \cdot p$. Q.E.D.

LEMMA 2. *V has no base locus.*

PROOF. By (1.2), we need only check that one point in each orbit of C is not a base point for V. It is easily seen that $N(x)$ is not zero at $[1, 0, 0, 0] \in A_0$, $N(y)$ is not zero at $[0, 1, 0, 0] \in A_1$, $N(z)$ is not zero at $[0, 0, 1, 0]$, and $N(w) = w^5$ is not zero at $[0, 0, 0, 1]$. Q.E.D.

LEMMA 3. *V separates the orbits of $\langle\sigma\rangle$.*

PROOF. Assume that p and q are in different orbits of $\langle\sigma\rangle$. By (1.3), we may assume that $p = [1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$ or $[0, 0, 0, 1]$, and (by symmetry) that the dimension of the C-orbit of q is no greater than that of the C-orbit of p. This eliminates $p = [1, 0, 0, 0]$ from consideration and leaves three cases.

Case 1. $p = [0, 0, 0, 1]$, $q = [x_0, y_0, z_0 \cdot w_0]$. In this case write $\sigma^i \cdot q = [x_i, y_i, z_i, w_i]$ for $0 \leq i \leq 4$. Since q is not in the orbit of p under $\langle\sigma\rangle$, $(x_i, y_i, z_i) \neq (0, 0, 0)$ for any i. Hence there is a linear form $l(x, y, z)$ such that $l(x_i, y_i, z_i) \neq 0$ for $0 \leq i \leq 4$. Then $f = N(l)$ is in V, $f(q) \neq 0$ and $f(p) = 0$.

Case 2. $p = [0, 0, 1, 0]$, $q = [x_0, y_0, z_0, 0]$. Using the same technique as in Case 1, and writing $\sigma^i \cdot q = [x_i, y_i, z_i, 0]$, one can choose a linear form $l(x, y)$ such that if $f = N(l)$, then $f(q) \neq 0$ and $f(p) = 0$.

Case 3. $p = [0, 1, 0, 0]$, $q = [x_0, y_0, 0, 0]$. In this case $f = N(x)$ vanishes at p and $f(q) \neq 0$. Q.E.D.

LEMMA 4. *V separates tangent vectors at all $p \neq [1, 0, 0, 0]$ in $\mathbf{P}^3$.*

PROOF. By (1.4), we need only check this at one point in each orbit A_1, A_2 and A_3 of C. Again there are 3 cases to consider.

Case 1. $p = [0, 0, 0, 1]$. In this case a computation shows that $N(ax + by + cz)$ has, at p, the linear term

$$(ax + by + cz)[c(b + 2c)(a + 3b + 3c)(4a + b + 4c)]$$

which is general, for general a, b and c.

Case 2. $p = [0,0,1,0]$. Here the linear term of $N(ax + by + cw)$ at p is

$$(ax + by + cw)[b(a + 2b)(3a + 3b)(a + 4b)]$$

which is general, for general a, b and c.

Case 3. $p = [0,1,0,0]$. Here the linear term of $N(ax + bz + cw)$ at p is $(ax + bz + cw)[4a^4]$ and is therefore general. Q.E.D.

The previous lemmas provide exactly what is needed to verify the claim and the proof of the following is complete.

PROPOSITION 5. *The map* $\phi\colon \mathbf{P}^3 \to \mathbf{P}(V)$ *is a regular map which is étale away from the fixed point* $[1,0,0,0]$ *and separates the orbits of* $\langle\sigma\rangle$.

COROLLARY 6. *The image* Z *of* $\mathbf{P}^3$ *under the map* ϕ *is the quotient of* $\mathbf{P}^3$ *under the action of* $\langle\sigma\rangle$, *and is smooth except possibly at* $\phi([1,0,0,0])$.

COROLLARY 7. *There exists a smooth quintic surface* Y *in* $\mathbf{P}^3$ *which is invariant under* $\langle\sigma\rangle$.

PROOF. By Bertini's theorem, there exists a smooth hyperplane section X of Z in $\mathbf{P}(V)$ since Z has only one isolated singularity. The quintic surface is the pull-back of X to $\mathbf{P}^3$. Q.E.D.

Of course, the surface X is the quotient of Y by $\langle\sigma\rangle$ and is the alleged nonclassical Godeaux surface. It remains to compute the invariants of the general such X. Let $\pi\colon Y \to X$ be the quotient map. By [**1**, Theorem 2.1],

$$\hat{\mathbf{Z}}_5 = \mu_5 \cong \ker(\pi\colon \operatorname{Pic} X \to \operatorname{Pic} Y),$$

where $\hat{G}$ denotes the Cartier dual of a finite group scheme G. This is precisely $\operatorname{Pic}^\tau(X)$, since Y is simply connected and $H^0(Y, \Omega_Y^1) = 0$ (see [**1**, Example 2.3]). Therefore, $H^1(X, \mathcal{O}_X)$ has dimension one. Since π is étale, $c_2(X) = \frac{1}{5}c_2(Y) = 11$, and $K_Y = \pi^* K_X$, so that $K_X^2 = \frac{1}{5}K_Y^2 = 1$. By Noether's formula $\chi(\mathcal{O}_X) = 1$, so $p_g(X) = 1$ and $p_a(X) = 0$.

Classical Godeaux surfaces have been constructed in all characteristics by W. Lang [**2**]. I wish to thank him for suggesting this problem. I am also indebted to M. Artin for a helpful conversation.

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