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On Canonical Surfaces of General Type with $K^2 = 3\chi - 10$

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1. Introduction

Let X be a smooth minimal complex surface such that $|K_X|$ has no base points and the associated map $\phi_K: X \to \Sigma \subseteq \mathbb{P}^{p_g-1}$ is birational. In this article I will present a classification of such surfaces under the additional assumption that $K^2 = 3\chi - 10$, and also discuss in some detail the case $K^2 = 14$, $\chi = 8$, which yields an interesting example. This classification was essentially known to Castelnuovo [C], which is not surprising since the main ingredient of the argument is an analysis of curves which are "extremal" in the sense of having the largest possible genus for their degree, and this genus bound itself was proved by Castelnuovo.

If $K^2 \leqq 3\chi - 11$, then since $\chi \leqq 1 + p_g$, we have $K^2 \leqq 3p_g - 8$. Beauville has shown [B, Theorem 5.5] that in this case ϕ_K is birationally a double covering onto a ruled surface. Moreover if $K^2 = 3\chi - 10$ and $q \neq 0$, then $\chi \leqq p_g$, so that $K^2 \leqq 3p_g - 10$, also forcing ϕ_K to have degree two. Hence with our assumptions $q = 0$ and we are considering regular surfaces at the limit of Beauville's theorem: $\chi = 1 + p_g$ and $K^2 = 3p_g - 7$, with $p_g \geqq 4$.

From the point of view of Beauville's theorem one might regard these surfaces as having minimal K^2 for their p_g. Harris, in [H], has taken the opposite tack and considered varieties with maximum p_g for their degree (which in our case is K^2), which he calls "Castelnuovo varieties". He extends Castelnuovo's arguments to varieties of any dimension, and my brief treatment here for surfaces does not differ essentially from his. Although his article is basically correct,

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he makes a small error in the computations for one special case, when the varieties lie inside the cone over the Veronese surface. This has also been noticed by Ciliberto, who gives a complete treatment in [Ci], and I will refer to [Ci] in several places.

Such a mistake would ordinarily not deserve more than a small note. However it is exactly in this case, which for surfaces occurs when $p_g=7$ and $K^2=14$, that *every* such surface (inside the cone over the Veronese) has a (-2)-curve on it, so that their canonical models all have at least one A_1 singularity. Hence any first-order deformation in which the singularity smooths is obstructed; by a result of Burns and Wahl [BW] the general first order deformation smooths the singularity, so therefore the Kuranishi family for any one such surface cannot be reduced. In fact, in the last section of this paper I show that such a surface depends on 54 parameters, while the dimension of $H^1(\Theta_X)$ is 55.

As far as I know, there is only one other example of such a phenomenon in the literature [Ho]. In order to understand it properly, one needs to understand Castelnuovo's and Harris' approach in some detail, and so I felt it worthwhile to briefly review the entire picture, in the surface case.

I understand that Professor F. Catanese has succeeded in developing other examples of surfaces inside the cone over the Veronese which are obstructed.

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2. Rational Normal Scrolls

Most of the surfaces in question lie on rational normal scrolls of dimension 3, so I'll briefly review the facts about them, without any proofs. All of the statements can be found in either [GH], [ACGH], or [H], where these varieties are discussed in detail. The reader should note that I follow the conventions of Hartshorne [Ha] for projective space bundles.

Let $a_1, \ldots, a_n$ be n nonnegative integers. Let

$$\mathbb{P}=\mathbb{P}_{a_1,\ldots,a_n}=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\mathcal{O}_{\mathbb{P}^1}(a_2)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(a_n));$$

there is a natural structure map $\pi\colon \mathbb{P}\to\mathbb{P}^1$ exhibiting the n-fold $\mathbb{P}$ as a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$. The Picard group of $\mathbb{P}$ has rank 2, generated by the class H of the tautological bundle $\mathcal{O}(1)$, and the class L of $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. The linear system $|H|$ has no base points and the corresponding morphism maps $\mathbb{P}$ birationally onto an n-fold $\mathbb{S}=\mathbb{S}_{a_1,\ldots,a_n}$, which is called a *rational normal scroll;* the fibres of π are mapped onto linear spaces, so $\mathbb{S}$ is "hyper-ruled" in this sense. If $a_i\geqq 1$ for every i, then $|H|$ is very ample and $\mathbb{S}\cong\mathbb{P}$. If $a_n=0$, then $\mathbb{S}_{a_1,\ldots,a_n}$ is the cone over $\mathbb{S}_{a_1,\ldots,a_{n-1}}$. In any case the degree of $\mathbb{S}_{a_1,\ldots,a_n}$ is $\sum a_i$, and $\mathbb{S}_{a_1,\ldots,a_n}$ is an irreducible nondegenerate n-fold in a projective space of dimension $r=\sum a_i+n-1$.

This is the minimum degree for an n-fold in this space, and in fact almost all varieties of minimal degree are scrolls:

(2.1) **Theorem.** *Any irreducible nondegenerate n-fold of degree $r-n+1$ in $\mathbb{P}^r$ is either a rational normal scroll, a cone over the Veronese surface in $\mathbb{P}^5$, or a quadric hypersurface of rank $\geqq 5$.*

It will also be useful to note the following:

(2.2) **Proposition:** *A rational normal scroll is projectively normal, and $\mathbb{S} = \mathbb{S}_{a_1, \ldots, a_n}$ lies on $(\sum a_i)(\sum a_i - 1)/2$ linearly independent quadrics whose equations generate the ideal of $\mathbb{S}$.*

Note that the veronese in $\mathbb{P}^5$ (and every cone over it) has degree 4 and also lies on $4(4-1)/2 = 6$ linearly independent quadrics which cut it out.

3. Extremal Curves

The hyperplane sections of the surfaces in question turn out to have maximum genus for their degree, and the essential step in their classification is the detailed description of these curves. In this section I'll present Castenuovo's bound for the genus, and state the relevant facts concerning those curves achieving this bound. All of this material may be found in [ACGH, Chap. III, Sect. 2] and I will not present any proofs.

(3.1) **Castelnuovo's Bound.** *Let C be a smooth curve of genus g that admits a birational mapping onto a nondegenerate curve of degree d in $\mathbb{P}_r$. Let $m = [(d-1)/(r-1)]$ and $\varepsilon = (d-1) - m(r-1)$. Then $g \leqq \pi(d, r)$, where $\pi(d, r) = m(m-1)(r-1)/2 + m\varepsilon$.*

A curve C as above such that $g = \pi(d, r)$ is called an *extremal* curve. In our application we will always have $r \geqq 3$ and $d \geqq 2r+1$, so I'll present the known facts about extremal curves only in this case.

(3.2) **Theorem.** *Let C be an extremal curve, with $\phi\colon C \to D \subseteq \mathbb{P}^r$ the birational map of C onto a nondegenerate curve D of degree d in $\mathbb{P}^r$, with $g(C) = \pi(d, r)$. Assume $r \geqq 3$ and $d \geqq 2r+1$. Then:*

(3.2.1) *ϕ is an isomorphism and $D \subseteq \mathbb{P}^r$ is projectively normal. We will henceforward identify D with C.*

(3.2.2) *C lies on $(r-1)(r-2)/2$ linearly independent quadrics.*

(3.2.3) *Let S be the intersection of the quadrics containing C. Then $S \subseteq \mathbb{P}^r$ is a nondegenerate irreducible surface of degree $r-1$, and hence is either a rational normal scroll, the Veronese in $\mathbb{P}^5$, or a quadric in $\mathbb{P}^3$ (which is a scroll too!).*

Armed with this, one now simply inspects the curves on the Veronese and the scrolls to check which of them are extremal. I'll use a minor change of notation here and switch to $\mathbb{P}_{a,b}$ instead of $\mathbb{P}_{a_1, a_2}$, etc. The result is as follows.

(3.3) **Theorem.** *Assume $r \geqq 3$ and $d \geqq 2r+1$. Let $C \subseteq \mathbb{P}^r$ be an extremal curve of degree d, and define m and ε as in* (3.1). *Then C is either*

(3.3.1) *the image of a smooth plane curve of degree d/2 under the double Veronese mapping of $\mathbb{P}^2$ onto the Veronese surface in $\mathbb{P}^5$; here $r=5$ and d must be even,*

(3.3.2) *the image in $\mathbb{S}_{a,b}$ of a smooth member of the linear system*

$$|(m+1)H-(r-\varepsilon-2)L|$$

on $\mathbb{P}_{a,b}$; here $a+b=r-1$,

or

(3.3.3) *the image $\mathbb{S}_{a,b}$ of a smooth member of the linear system $|mH+L|$ on $\mathbb{P}_{a,b}$; here $a+b=r-1$ and ε must be* 0.

4. The Classification of Canonical Surfaces with $K^2=3\chi-10$

Let X be a smooth minimal complex surface such that $|K_X|$ has no base points, $\phi_K\colon X \to \Sigma \subseteq \mathbb{P}^{p_g-1}$ is birational, and $K^2=3\chi-10$. As was noticed in Sect. 1, the irregularity $q=0$, so $\chi=1+p_g$, hence $K^2=3p_g-7$, with $p_g \geqq 4$. If $p_g=4$, then $K^2=5$, and Σ is a quintic surface in $\mathbb{P}^3$ [Ho]; in what follows I'll assume that $p_g \geqq 5$.

Let P be a general hyperplane in $\mathbb{P}^{p_g-1}$, and let $C \in |K_X|$ be the corresponding curve on X, so that $D=\phi_K(C)=\Sigma \cap P$. D is a nondegenerate irreducible curve in $\mathbb{P}^{p_g-2}$ of degree $K^2=3p_g-7$. Let then $d=3p_g-7$ and $r=p_g-2$; with the notations of (3.1) $m=[(d-1)/(r-1)]=3$ since $p_g \geqq 5$, and $\varepsilon=1$, so $\pi(d,r)=\pi(3p_g-7,p_g-2)=3p_g-6$. The genus of C, by the adjunction formula for curves on X, is $1+K^2=3p_g-6$ also, so that

(4.1) *the general curve $C \in |K_X|$ is extremal, with the embedding ϕ_K.*

Note that $r \geqq 3$ and $d \geqq 2r+1$, so that the results (3.2) and (3.3) apply. In particular, we have the following, noting that the case (3.3.3) cannot occur since $\varepsilon \neq 0$:

(4.2) **Proposition.** *Any smooth curve of degree $3p_g-7$ in $\mathbb{P}^{p_g-2}$ and genus $3p_g-6$, with $p_g \geqq 5$, is either*

(4.2.1) *the image of a curve of degree 7 in $\mathbb{P}^2$ under the double Veronese mapping of $\mathbb{P}^2$ onto the Veronese surface in $\mathbb{P}^5$; here $p_g=7$,*

or

(4.2.2) *the image in $\mathbb{S}_{a,b}$ of a smooth member of the linear system $|4H-(p_g-5)L|$ on $\mathbb{P}_{a,b}$; here $a+b=p_g-3$.*

Since C is projectively normal, so is Σ. Hence the natural map from $\mathrm{Symm}^2 H^0(X,K_X)$ to $H^0(X,2K_X)$ is surjective. Its kernel is $H^0(I_\Sigma(2))$, and therefore this last space has dimension

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{Symm}^2 H^0(X,K_X) - \dim_{\mathbb{C}} H^0(X,2K_X) \\ &\quad = (p_g+1)(p_g)/2-(K^2+\chi) \quad \text{[BPV, Chap. VII, Corollary (5.6)]} \\ &\quad = (r-1)(r-2)/2. \end{aligned}$$

Therefore

(4.3) *Σ is projectively normal and lies on $(r-1)(r-2)/2$ linearly independent quadrics, where $r=p_g-2$.*

Let W be the intersection of all the quadrics containing Σ. Since W is cut out by $(r-1)(r-2)/2$ linearly independent quadrics, and these quadrics all contain Σ, then $P\cap W$ is also cut out by $(r-1)(r-2)/2$ linearly independent quadrics (P is general) containing $P\cap\Sigma=C$. Therefore, by (3.2), we have

(4.4) *If P is a general hyperplane in $\mathbb{P}^{p_g-1}$, then $P\cap W\supseteq P\cap\Sigma=C$ is exactly the rational normal scroll (or the Veronese) containing C.*
Moreover, $W\subseteq\mathbb{P}^{p_g-1}$ is a nondegenerate irreducible 3-fold of degree p_g-3.

The final statement is clear: W is nondegenerate since Σ is, and irreducible of degree p_g-3 since its hyperplane section Σ is. Since that hyperplane section is a surface, W must be a threefold.

This is the minimal possible degree for such a 3-fold. Hence by (2.1) we have

(4.5) **Corollary.** *W is either*

(4.5.1) *a rational normal scroll $\mathbb{S}_{a,b,c}$; here $a+b+c=p_g-3$,*

(4.5.2) *the cone in $\mathbb{P}^6$ over the Veronese surface in $\mathbb{P}^5$, or*

(4.5.3) *a quadric in $\mathbb{P}^4$.*

Using the description of divisors in W we can now classify Σ, and of course X. We need some notation for the cone Z over the Veronese. Z is the image in $\mathbb{P}^6$ of the $\mathbb{P}^1$-bundle $\mathbb{P}=\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2))$ over $\mathbb{P}^2$ under the map given by the sections of the tautological bundle $\mathcal{O}_{\mathbb{P}}(1)$. If $\pi\colon\mathbb{P}\to\mathbb{P}^2$ is the structure map, then $\operatorname{Pic}(\mathbb{P})$ has rank 2 and is generated by the class H of $\mathcal{O}_{\mathbb{P}}(1)$ and the class L of $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. The exceptional divisor over the vertex of the cone is in the class $H-2L$.

In [Ci], the analysis of divisors on the possible threefolds W is carried out, and I will not reproduce the calculations here. The result is the following classification:

(4.6) **Theorem.** *Fix $p_g\geqq 4$, and let X be a smooth minimal complex surface with $K^2=3\chi-10=3p_g-7$, such that ϕ_K is birational. Then $\Sigma=\phi_K(K)$ has only rational double points as singularities and X is the minimal resolution. Moreover:*

(4.6.1) *If $p_g=4$ then Σ is a quintic surface in $\mathbb{P}^3$.*

(4.6.2) *If $p_g=5$ then Σ is the complete interaction of a quadric and a quartic in $\mathbb{P}^4$.*

If $p_g\geqq 6$ then Σ is either

(4.6.3) *the image in $\mathbb{S}_{a,b,c}$ of a member of the linear system $|4H-(p_g-5)L|$ on $\mathbb{P}_{a,b,c}$; here $a+b+c=p_g-3$, or*

(4.6.4) *the image (in the cone Z over the Veronese in $\mathbb{P}^5$) of a member of the linear system $|3H+L|$ on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2))$; here $p_g=7$.*

Remark. It is exactly at the case of the cone over the Veronese that Harris' error is made. He states that Σ must be a complete intersection of Z with a hypersurface not passing through the vertex of Z, implying that $\Sigma \approx rH$ for some r. The error is made on p. 64 of his article, where certain binomial coefficients appearing in his formula for p_g seem to be shifted by one.

Note that in the final case the surface Σ is a trisection in the variety $\mathbb{P}$ which is ruled over $\mathbb{P}^2$. Using [M, Proposition 8.1] a small calculation shows that $\pi_* \mathcal{O}_\Sigma \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)$, and for a general Σ in $|3H+L|$, Corollary 10.4 of [M] shows that $p_g(\Sigma)=7$ and $K_\Sigma^2=14$ as required.

As a final result of the analysis we have

(4.7) **Corollary.** *Assume Σ lies on a smooth rational normal scroll. Then there is a fibration $f\colon X \to \mathbb{P}^1$ on X with general fiber a non-hyperelliptic curve of genus* 3.

Indeed, in this case Σ is in the class $|4H-(p_g-5)L|$ in the $\mathbb{P}^2$-bundle $\mathbb{P}_{a,b,c}$, and so Σ meets the general $\mathbb{P}^2$ fiber in a smooth quartic curve. Note that the "general" scroll is smooth, so the general X lying on a scroll will have such a fibration.

There is one "obvious" restriction on the scrolls on which X lies.

(4.8) **Lemma.** *Assume $\Sigma \subseteq \mathbb{P}_{a,b,c}$ as above in the class $|4\mathrm{H}-(p_g-5)L|$, and assume that $0 \leqq a \leqq b \leqq c$. with $a+b+c=p_g-3$. Then $4c \leqq 3p_g-7$.*

Proof. The surface scroll $\mathbb{P}_{a,b}$ inside $\mathbb{P}_{a,b,c}$ is in the class $|H-cL|$; if Σ is not to contain $\mathbb{P}_{a,b}$ as a component, then $H \cdot \Sigma \cdot \mathbb{P}_{a,b} \geqq 0$, i.e., $0 \leqq H \cdot (4H-(p_g-5)L) \cdot (H-cL) = 4H^3-(4c+p_g-5)H^2L = 3p_g-4c-7$, since $H^3=p_g-3$, $H^2L=1$, and $HL^2=L^3=0$. Q.E.D.

5. Surfaces with $p_g=7$ and $K^2=14$

In this section I'll analyze more deeply the case of $p_g=7$. By Theorem (4.6), we have just two cases: either ϕ_K maps X into the Veronese cone Z, or into a rational normal scroll. Let us first take up the case of the cone.

Let $\mathbb{P}=\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ be the resolution of the cone Z, with divisor classes H (of $\mathcal{O}_{\mathbb{P}}(1)$) and L (of $\pi^* \mathcal{O}_{\mathbb{P}^2}(1)$) as in the previous section. We have $H^3=4$, $H^2L=2$, $HL^2=1$, and $L^3=0$; also the exceptional divisor $E \cong \mathbb{P}^2$ has class $H-2L$. Recall that the surface Σ is in the class $3H+L$.

(5.1) **Proposition.**

(5.1.1) *The canonical class of $\mathbb{P}$ is $-2H-L$.*

(5.1.2) *The linear system $|3H+L|$ has no base points and the general member is a minimal smooth surface X with $p_g=7$ and $K^2=14$.*

(5.1.2) *Any such X meets E in a line l of E, and the self-intersection of l on X is -2.*

Proof. The first statement is an easy exercise in the use of the adjunction formula; one need only note that a general H is isomorphic to $\mathbb{P}^2$, and a general L is isomorphic to $\mathbb{F}_2$, the minimal resolution of the quadric cone.

Since neither $|H|$ nor $|L|$ have base points, certainly $|3H+L|$ does not; hence the general member X is smooth. Its canonical class $K_X=(K_{\mathbb{P}}+X)|_X=H|_X$, so $|K_X|$ has no base points and therefore X must be minimal; moreover $(K_X)^2 =(K_{\mathbb{P}}+X)^2 X=H^2(3H+L)=14$. Using the exact sequence $0\to\mathcal{O}_{\mathbb{P}}(-3H-L) \to\mathcal{O}_{\mathbb{P}}\to\mathcal{O}_X\to 0$ we see that $p_g(X)=h^2(\mathcal{O}_X)=h^3(\mathcal{O}_{\mathbb{P}}(-3H-L))$ since $h^2(\mathcal{O}_{\mathbb{P}}) =h^3(\mathcal{O}_{\mathbb{P}})=0$ ($\mathbb{P}$ is rational). By Serre duality $h^3(\mathcal{O}_{\mathbb{P}}(-3H-L))=h^0(\mathcal{O}_{\mathbb{P}}(H)) =h^0(\mathcal{O}_{\mathbb{P}}(1))=h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2))=7$, proving (5.1.2).

The degree of $l=X\cap E$ is measured by intersecting with L; so degree(l) $=(3H+L)(H-2L)L=3H^2L-5HL^2=1$. The self-intersection of l on X is $E^2X=(H-2L)^2(3H+L)=3H^3-11H^2L+8HL^2=12-22+8=-2$. Q.E.D.

(5.2) **Corollary.** *If ϕ_K maps X into the cone over the Veronese, then X has a (-2)-curve l, whose image under ϕ_K is at least an A_1 singularity at the vertex of the cone.*

Let us now calculate the number of moduli for these surfaces.

(5.3) **Proposition.**

(5.3.1) $\dim|3H+L|=69$,

(5.3.2) $\dim(\mathrm{Aut}(\mathbb{P}))=15$,

(5.3.3) *the number of moduli for those X's mapping into the cone over the Veronese is* 54.

Proof. Let $\pi\colon \mathbb{P}\to\mathbb{P}^2$ be the structure map for $\mathbb{P}$, exhibiting it as a $\mathbb{P}^1$-bundle. Then

$$\begin{aligned} h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3H+L)) &= h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3)\oplus\pi^*\mathcal{O}_{\mathbb{P}^2}(1)) \\ &= h^0(\mathbb{P}^2, \pi_*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}_{\mathbb{P}^2}(1)) \\ &= h^0(\mathbb{P}^2, \mathrm{Symm}^3(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2))\otimes\mathcal{O}_{\mathbb{P}^2}(1)) \\ &= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(3)\oplus\mathcal{O}_{\mathbb{P}^2}(5)\oplus\mathcal{O}_{\mathbb{P}^2}(7)) \\ &= 3+10+21+36=70, \end{aligned}$$

proving (5.3.1).

Let $\mathrm{Aut}_{\mathbb{P}^2}(\mathbb{P})$ be the automorphisms of $\mathbb{P}$ inducing the identity on $\mathbb{P}^2$; we have an exact sequence

$$1\to\mathrm{Aut}_{\mathbb{P}^2}(\mathbb{P})\to\mathrm{Aut}(\mathbb{P})\to\mathrm{Aut}(\mathbb{P}^2)\to 1,$$

so

$$\dim(\mathrm{Aut}(\mathbb{P}))=8+\dim(\mathrm{Aut}_{\mathbb{P}^2}(\mathbb{P}))=7+\dim(\mathrm{Aut}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2))),$$

since $\mathrm{Aut}_{\mathbb{P}^2}(\mathbb{P})$ is the projectivization of the automorphism group of the vector bundle. The endomorphisms of the bundle can be viewed as 2×2 matrices $\begin{pmatrix} f_0 & g_2 \\ 0 & h_0 \end{pmatrix}$, where f_0 and h_0 have degree 0 and g_2 has degree 2. These are forms in the three homogeneous variables of $\mathbb{P}^2$, so the number of degree of freedom is then 8, so that $\dim(\mathrm{Aut}(\mathbb{P}))=7+8=15$, as stated. The final statement follows by subtraction, after noting that no positive dimensional subgroup of $\mathrm{Aut}(\mathbb{P})$ can stabilize X. Q.E.D.

Let now turn to the surfaces which are mapped by ϕ_K into scrolls. There are only four scrolls in $\mathbb{P}^6$ of degree 4, namely $\mathbb{S}_{112}$, $\mathbb{S}_{022}$, $\mathbb{S}_{013}$, and $\mathbb{S}_{004}$, and X cannot map into $\mathbb{S}_{004}$ by (4.8). Recall that the surface Σ in $\mathbb{P}_{abc}$ is in the class $4H-2L$ in each case, by Lemma (4.6.3). Let us count the number of moduli for these surfaces.

(5.4) **Proposition.**

(5.4.1) $$\dim|4H-2L| = \begin{cases} 64 & on\ \mathbb{P}_{112} \\ 65 & on\ \mathbb{P}_{022}. \\ 65 & on\ \mathbb{P}_{013} \end{cases}$$

(5.4.2) $\dim(\mathrm{Aut}(\mathbb{P}_{112}))=11$,
$\dim(\mathrm{Aut}(\mathbb{P}_{022}))=13$, *and*
$\dim(\mathrm{Aut}(\mathbb{P}_{013}))=14$.

(5.4.3) *The number of moduli for those X's mapping into scrolls*
$$is \begin{cases} 53 & if\ \Sigma\subset\mathbb{P}_{112} \\ 52 & if\ \Sigma\subset\mathbb{P}_{022}. \\ 51 & if\ \Sigma\subset\mathbb{P}_{013} \end{cases}$$

Proof. I'll give the calculations for the "general" scroll $\mathbb{P}_{112}$, and leave the other two, which are identical in form, to the reader. If π: $\mathbb{P}_{112}\to\mathbb{P}^1$ is the structure map, we have

$$\begin{aligned} h^0(\mathbb{P}_{112}, \mathcal{O}_{\mathbb{P}_{112}}(4H-2L)) &= h^0(\mathbb{P}_{112}, \mathcal{O}_{\mathbb{P}_{112}}(4)\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(-2)) \\ &= h^0(\mathbb{P}^1, \pi_*[\mathcal{O}_{\mathbb{P}_{112}}(4)]\otimes\mathcal{O}_{\mathbb{P}^1}(-2)) = h^0(\mathbb{P}^1, \mathrm{Symm}^4[\mathcal{O}(1)\oplus\mathcal{O}(1)\oplus\mathcal{O}(2)]\otimes\mathcal{O}(-2)) \\ &= h^0(\mathbb{P}^1, [\mathcal{O}(4)^{\oplus 5}\oplus\mathcal{O}(5)^{\oplus 4}\oplus\mathcal{O}(6)^{\oplus 3}\oplus\mathcal{O}(7)^{\oplus 2}\oplus\mathcal{O}(8)]\otimes\mathcal{O}(-2)) \\ &= h^0(\mathbb{P}^1, \mathcal{O}(2)^{\oplus 5}\oplus\mathcal{O}(3)^{\oplus 4}\oplus\mathcal{O}(4)^{\oplus 3}\oplus\mathcal{O}(5)^{\oplus 2}\oplus\mathcal{O}(6)) \\ &= 5h^0(\mathcal{O}(2))+4h^0(\mathcal{O}(3))+3h^0(\mathcal{O}(4))+2h^0(\mathcal{O}(5))+h^0(\mathcal{O}(6)) \\ &= 5\cdot 3+4\cdot 4+3\cdot 5+2\cdot 6+7=65, \end{aligned}$$

so that $\dim|4H-2L|=64$.

As above we have

$$\dim(\mathrm{Aut}(\mathbb{P}_{112}))=\dim(\mathrm{Aut}(\mathbb{P}^1))-1+\dim(\mathrm{Aut}_{\mathbb{P}^1}(\mathcal{O}(1)\oplus\mathcal{O}(1)\oplus\mathcal{O}(2))),$$

and $\mathrm{End}_{\mathbb{P}^1}(\mathcal{O}(1)\oplus\mathcal{O}(1)\oplus\mathcal{O}(2))$ can be viewed as 3×3 matrices of the form

$$\begin{pmatrix} f_0 & g_0 & r_1 \\ h_0 & k_0 & s_1 \\ 0 & 0 & t_0 \end{pmatrix},$$

where subscript denotes the degree of the forms in the two homogeneous variables of $\mathbb{P}^1$. Hence the number of parameters here is 9, so that $\dim(\mathrm{Aut}(\mathbb{P}_{112}))=11$; this proves (5.4.2), and (5.4.3) is obtained by subtracting. Q.E.D.

(5.5) **Corollary.** *In the moduli space for surfaces with $p_g=7$, $q=0$, and $K^2=14$, the surfaces for which ϕ_K is a birational morphism into the cone over the Veronese form an irreducible component of dimension* 54.

Proof. For ϕ_K to be a birational morphism is an open condition, and we have just seen that the number of moduli for all other cases is strictly smaller than the number for those mapping into the cone. Q.E.D.

In fact, by a result of Pinkham [P], it is clear that we have two components here; one for the cone and one for the scrolls. If the surfaces in the scrolls were a degeneration of those in the cone, then the scrolls would be a degeneration of the cone; after taking a hyperplane section, we would conclude that the surface scroll $\mathbb{S}_{22}$ was degeneration of the Veronese surface. This Pinkam shows is not the case.

The interest of this result is now apparent.

(5.6) **Corollary.** *Let X be the general surface with $p_g=7$ and $K^2=14$ such that ϕ_K maps X into the cone over the Veronese. Then the Kuranishi deformation space M for X is not reduced.*

Proof. The point is that X has a (-2)-curve on it which does not smooth in any nearby surface, by (5.2). However, a theorem of Burns and Wahl [BW] states that the general vector in $H^1(\Theta_X)$ smooths every (-2)-curve on X. Therefore, since $H^1(\Theta_X)$ is the Zariski tangent space to the Kuranishi family M, we must have that $h^1(\Theta_X)$ is strictly bigger than the number of moduli of X, which is the dimension of M. Therefore M is not smooth at the point corresponding to X; since X is general, and since M is versal, M cannot be reduced, by the openess of versality. Q.E.D.

For completeness, let us calculate $h^1(\Theta_X)$ for these surfaces inside the cone over the Veronese explicitly. Since $h^0(\Theta_X)=0$ [Ma], it suffices to know $h^2(\Theta_X)$, since $\chi(\Theta_X)=2K^2-10\chi(\mathcal{O}_X)=-52$; hence $h^1(\Theta_X)=52+h^2(\Theta_X)$.

For a sheaf F, we will say its *cohomology dimensions* are $(h^0(F), h^1(F), \ldots)$ for brevity. Again let $\mathscr{E}=\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2)$, let $\mathbb{P}=\mathbb{P}(\mathscr{E})$, and let $\pi\colon \mathbb{P}\to\mathbb{P}^2$ be the natural structure map. The cohomology dimensions of $\mathcal{O}_{\mathbb{P}}$ are $(1,0,0,0)$ since $\mathbb{P}$ is rational, and those of $\mathcal{O}_{\mathbb{P}}(X)=\mathcal{O}_{\mathbb{P}}(3H+L)$ are $(70,0,0,0)$ by the analysis leading to (5.3.1). Hence the exact sequence $0\to\mathcal{O}_{\mathbb{P}}\to\mathcal{O}_{\mathbb{P}}(X)\to\mathcal{O}_X(X)\to 0$ gives that the cohomology dimensions of $\mathcal{O}_X(X)$ are $(69,0,0)$. Using the exact sequence $0\to\Theta_X\to\Theta_{\mathbb{P}}|_X\to\mathcal{O}_X(X)\to 0$, we see that $h^2(\Theta_X)=h^2(\Theta_{\mathbb{P}}|_X)$. It is this that we will compute.

By using the sequence $0\to\Omega^1_{\mathbb{P}^2}(2)\to\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}\to\mathcal{O}_{\mathbb{P}^2}(2)\to 0$, we see that

$$h^0(\Omega^1_{\mathbb{P}^2}(2))=3. \tag{5.7}$$

(the map on the right is clearly surjective on global sections.)

Next, note that $\pi^*\Theta_{\mathbb{P}^2}\otimes\mathcal{O}_{\mathbb{P}}(-X)\cong\mathcal{O}_{\mathbb{P}}(-3)\otimes\pi^*\Theta_{\mathbb{P}}{}^2(-1)$, and since by (5.1.1) $\Omega^3_{\mathbb{P}}=\mathcal{O}_{\mathbb{P}}(-2)\otimes\pi^*\mathcal{O}_{\mathbb{P}^2}(-1)$, we have using Serre duality that

$$\begin{aligned} h^3(\pi^*\Theta_{\mathbb{P}^2}\otimes\mathcal{O}_{\mathbb{P}}(-X)) &= h^0(\mathcal{O}_{\mathbb{P}}(1)\otimes\pi^*\Omega^1_{\mathbb{P}^2})=h^0(\pi_*\mathcal{O}_{\mathbb{P}}(1)\otimes\Omega^1_{\mathbb{P}^2})\\ &= h^0([\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2)]\otimes\Omega^1_{\mathbb{P}^2})=h^0(\Omega^1_{\mathbb{P}^2})+h^0(\Omega^1_{\mathbb{P}^2}(2))=3\end{aligned}$$

by (5.7). Thus

(5.8) $$h^3(\pi^* \Theta_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}}(-X)) = 3.$$

Note that the relative tangent bundle $\Theta_{\mathbb{P}/\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(-2)$ [Ha], so that $\Theta_{\mathbb{P}/\mathbb{P}^2}(-X) \cong \mathcal{O}_{\mathbb{P}}(-1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(-3)$. Using Serre duality, $h^3(\Theta_{\mathbb{P}/\mathbb{P}^2}(-X)) = h^0(\mathcal{O}_{\mathbb{P}}(-1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(2))$, which is clearly zero; hence

(5.9) $$h^3(\Theta_{\mathbb{P}/\mathbb{P}^2}(-X)) = 0.$$

The exact sequence $0 \to \Theta_{\mathbb{P}/\mathbb{P}^2}(-X) \to \Theta_{\mathbb{P}}(-X) \to \pi^* \Theta_{\mathbb{P}^2}(-X) \to 0$ then implies, using (5.8) and (5.9), that

(5.10) $$h^3(\Theta_{\mathbb{P}}(-X)) = 3.$$

Since

$$\begin{aligned} R^i \pi_*(\pi^* \mathscr{E}^*(1)) &= R^i \pi_*(\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* [\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)] \\ &= R^i \pi_* \mathcal{O}_{\mathbb{P}}(1) \otimes [\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)] \\ &= \begin{cases} \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2) & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases}, \end{aligned}$$

we have

$$h^i(\pi^* \mathscr{E}^*(1)) = h^i(\pi_* \pi^* \mathscr{E}^*(1)) = h^i(\mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)),$$

so that the cohomology dimensions of $\pi^* \mathscr{E}^*(1)$ are (8, 0, 0, 0). Using the exact sequence $0 \to \mathcal{O}_{\mathbb{P}} \to \pi^* \mathscr{E}^*(1) \to \Theta_{\mathbb{P}/\mathbb{P}^2} \to 0$, we see that the cohomology dimensions of $\Theta_{\mathbb{P}/\mathbb{P}^2}$ are (7, 0, 0, 0).

Since

$$R^i \pi_*(\pi^* \Theta_{\mathbb{P}^2}) = \begin{cases} \Theta_{\mathbb{P}^2} & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases}, \quad h^i(\pi^* \Theta_{\mathbb{P}^2}) = h^i(\pi_* \pi^* \Theta_{\mathbb{P}^2}) = h^i(\Theta_{\mathbb{P}^2})$$

which can be easily computed from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to \Theta_{\mathbb{P}^2} \to 0$, yielding that the cohomology dimensions of $\Theta_{\mathbb{P}^2}$ are (8, 0, 0). Combining the last two calculations with the exact sequence $0 \to \Theta_{\mathbb{P}/\mathbb{P}^2} \to \Theta_{\mathbb{P}} \to \pi^* \Theta_{\mathbb{P}^2} \to 0$ gives that the cohomology dimensions of $\Theta_{\mathbb{P}}$ are (15, 0, 0, 0). In particular,

(5.11) $$h^2(\Theta_{\mathbb{P}}) = h^3(\Theta_{\mathbb{P}}) = 0.$$

Finally the exact sequence $0 \to \Theta_{\mathbb{P}}(-X) \to \Theta_{\mathbb{P}} \to \Theta_{\mathbb{P}}|_X \to 0$ can be applied, using (5.10) and (5.11), to give $h^2(\Theta_{\mathbb{P}}|_X) = 3$. As remarked above, this implies that $h^2(\Theta_X) = 3$, so that we have proved the following.

(5.12) **Proposition.** *Let X be a general surface with $p_g = 7$ and $K^2 = 14$ such that ϕ_K maps X birationally into the cone over the Veronese. Then $H^1(\Theta_X)$ has dimension* 55.

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