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On Extremal Rational Elliptic Surfaces

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0. Introduction

The systematic study of elliptic surfaces was begun by Kodaira [K1]. He listed all the possible singular fibers, introduced their local (monodromy) invariants, and calculated the possible values of the elliptic modulus J. Finally, he presented the outline of a complete classification.

The simplest non-trivial example of an elliptic surface is a rational elliptic surface. Such a surface is the blow-up of $\mathbb{P}^2$ at nine points; hence, its Neron-Severi group is very explicit. It thus gives the unique opportunity to "see" an elliptic surface from two sides: on one hand, using the general (external) approach, and on the other hand the specific (internal) view in the setting of classical geometry. This "accident" of nature does provide (at least to the authors) an unending source of fascination.

In this paper we are going to restrict ourselves to elliptic surfaces with sections, so-called Jacobians (or in the terminology of Kodaira, basic elliptic surfaces). A rational elliptic surface turns out to be a Jacobian if and only if it has no multiple fibers. (In fancy terminology, the Tate-Shafarevich group vanishes).

So, unless stated otherwise, an elliptic surface will mean a Jacobian together with a distinguished section, the so-called 0-section.

A rational elliptic surface can be presented in two different ways.

The first is intimately related to the so-called Weierstrass representation, and as such natural to generalize. Namely, consider the involution $z \to -z$; it exhibits the surface as a double covering of a ruled surface branched over at least two irreducible components, one of them being a smooth rational curve of self-intersection -2 corresponding to the distinguished section. If we blow down the exceptional divisors not meeting this component, we get a normalized situation which we are going to describe in fuller detail later.

The second is much more ad-hoc (and interesting?). It concerns the surface as the blow-up of the nine basepoints of a cubic pencil.

To every (Jacobian) elliptic fibration X there is a group of sections $\Phi(X)$ with the distinguished section as zero. Up to a finite group (generically $\mathbb{Z}_2$, in

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very special cases $\mathbb{Z}_4$ or $\mathbb{Z}_6$) $\Phi(X)$ is identified with the relative automorphism group of the fibration.

Due to a formula of Shioda-Tate (to be proved later) we have the basic inequalities

$$0 \leqq rk\Phi \leqq \rho(X) - 2 \quad \text{(where } \rho \text{ is the Picard number)}$$

where the discrepancy in the upper bound is related to the degree of reducibility of the fibers.

In the rational case $\rho(X) = 10$ (always) and thus

$$0 \leqq rk\Phi \leqq 8.$$

Generically $rk\Phi = 8$, while $rk\Phi = 0$ is the most special.

Definition. An elliptic fibration X is called extremal if and only if $\rho(X) = h^{1,1}(X)$ (maximal Picard number) and $rk\Phi(X) = 0$.

In other words, the extremal fibrations are the most reducible. (For an elaboration of this concept see [P2].)

Proposition. *Let X be a rational elliptic surface. Then the following are equivalent:*

(i) *X is extremal (i.e., $rk\Phi(X) = 0$)*
(ii) *the relative automorphism group* $\mathrm{Aut}_0(X)$ *is finite*
(iii) *the number of representations as a blow-up of* $\mathbb{P}^2$ *is finite*
(iv) *the number of rational curves C with $C^2 < 0$ is finite*
(v) *the number of reduced curves C with $C^2 < 0$ is finite.*

Proof. The equivalence (i)$\Leftrightarrow$(ii) is clear and the implications (v)$\Rightarrow$(iv)$\Rightarrow$(iii) are obvious. If F is a fiber then $K_X = -F$; thus, E being exceptional is equivalent with E being a section, proving (i)$\Leftrightarrow$(iv). If C irreducible and $C^2 < 0$ then $KC + C^2 < 0$; thus, $KC + C^2 = -2$ and (iv)$\Rightarrow$(v). The fact (Proposition 6.1) that any rational elliptic surface is a blow-up of $\mathbb{P}^2$ shows (iii)$\Rightarrow$(iv). Q.E.D.

The object of this paper is to classify all extremal rational elliptic surfaces.

The complete list encompasses 16 cases (15 discrete and one continuous) and is presented in Sect. 4.

It naturally splits up in six semi-stable cases (i.e., the fibers are all semi-stable (type I_n)) and ten unstable cases.

The case of semi-stable fibrations was studied by A. Beauville [B] using a different approach. The extremal semi-stable fibrations are the semi-stable fibrations with the minimal ($=4$) number of singular semi-stable fibers. Beauville relates those six fibrations to certain elliptic modular sufaces specifying the corresponding subgroups of finite index in $SL_2(\mathbb{Z})$.

U. Hirzebruch [H] lists in a Diplom-Arbeit all elliptic fibrations with at most three singular fibers. Except for the trivial case, those correspond to rational or $K-3$ surfaces. Our list of ten unstable extremal fibrations is a sublist of the rational examples in her list.

Thus nothing is in a sense fundamentally new. The justification of our paper lies, we hope, in the new point of view and its systematic exploration.

The plan of the paper is to review the general theory of elliptic surfaces (Sects. 1 through 3), then to present the complete list (Sect. 4) and in the next section (Sect. 5) to give all the Weierstrass equations. (This incidentally proves that all the cases do occur.)

The Weierstrass models have a geometric interpretation as quartics with a distinguished point; this ties in with the notion of maximizing quartics [P1] and provides a (painless) graphical representation of all the cases.

Finally, we discuss various relationships between the different extremal fibrations, exhibited by coverings of one by another.

1. Glossary on (Jacobian) Elliptic Fibrations

Let C be a smooth curve ($\cong \mathbb{P}^1$ in our case) and let $\pi: X \to C$ be a (relatively) minimal elliptic surface over C with a distinguished section S_0.

The complete list of possible fibers has been given by Kodaira [K1] and is, of course, very well known. It encompasses two infinite families (I_n, I_n^*, $n \geqq 0$) and six exceptional cases (II, III, IV, II^*, III^*, IV^*). Associated with those are a host of various invariants, some of which we are going to present below in tabular form.

To each (Jacobian) elliptic fibration $\pi: X \to C$ there corresponds a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \bar{X} \\ & \pi\searrow \quad \swarrow\bar{\pi} & \\ & C & \end{array}$$

where α contracts all components of fibers not meeting S_0.

$\bar{\pi}: \bar{X} \to C$ is called the Weierstrass fibration associated with X. The fibers of $\bar{X}$ are all irreducible, but $\bar{X}$ is always singular (unless α is the identity); its singularities are mild. In fact, they are all rational doublepoints and X is the minimal desingularization.

$\bar{X}$ has a natural section $\bar{S}_0(=\alpha(S_0))$ and the natural involutions $z \to -z$ on both X and $\bar{X}$ commute and respect the fibrations, giving a diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & R \\ \Big\downarrow{\scriptstyle\alpha} & \searrow \; C \; \swarrow & \Big\downarrow{\scriptstyle\alpha'} \\ & \nearrow \quad \nwarrow & \\ \bar{X} & \xrightarrow[\bar{i}]{} & \bar{R} \end{array}$$

where R, $\bar{R}$ are ruled surfaces and $\bar{R}$ is the minimal model of R (not affecting $i(S_0)$). The involutions i, $\bar{i}$ exhibit X, $\bar{X}$ as double coverings of R, $\bar{R}$ branched along four-sections B, $\bar{B}(\bar{B}=\alpha'(B))$. We can write $B=i(S_0)+T$, and $\bar{B}=\bar{i}(\bar{S}_0)+\bar{T}$; T and $\bar{T}$ are obviously trisections and disjoint from $i(S_0)$, $\bar{i}(\bar{S}_0)$. T is smooth while $\bar{T}$ in general has simple singularities (corresponding to the rational doublepoints of $\bar{X}$).

If we localize the picture to a point $c=c(0)$ on C with local parameter t, we can present $\bar{X}$ in Weierstrass normal form

$$y^2=x^3+A(t)\,x+B(t),$$

where $A(t)$, $B(t)$ are regular functions at c. The three roots of the cubic in x make up the component $\bar{T}$ and $\bar{i}(\bar{S}_0)$ corresponds to $x=\infty$. Let $\Delta(t)=4A^3+27B^2$ be the discriminant. The fiber over $c(t_0)$ is singular iff $\Delta(t_0)=0$.

We can now define the function $J(t)=4A^3/\Delta$. If $\Delta(t_0)\neq 0$, $J(t_0)$ simply gives the modulus of the elliptic fiber over $c(t_0)$.

Define $m(J(t_0))$ (or simply $m(t_0)$) to be the order of vanishing of $J(t)-J(t_0)$ at t_0.

It is clear that the type of singular fiber on X can be read off the singularity of $\bar{T}$ (together with its position relative to the fiber of $\bar{R}\to C$). The Weierstrass model allows us to read off the same information from the orders of vanishing of the functions A, B and Δ at the various singular base values. Let us denote those vanishing orders with a, b and δ, respectively.

To each fiber F on X we can associate the rational doublepoint X_n on $\alpha(F)$ or equivalently the singular point x_n on $\bar{T}$. In fact, the former has as a resolution the union of the components of F not meeting S_0. Thus, they can be considered as sublattices of the Neron-Severi group of X and as such they have rank $(=r(F))$ and discriminant $(=d(F))$. [If $r(F)=0$ then $d(F)=1$ by convention.] Finally, if $e(F)$ denotes the euler number of the fiber F as a reduced divisor, we can set up the following table.

Table 1.1

		a	b	δ	J	$m(J)$	e	r	d	RDP
I_0		0	0	0	$\neq 0, 1, \infty$	–	0	0	1	–
I_0		$a\geq 1$	0	0	0	$3a$	0	0	1	–
I_0		0	$b\geq 1$	0	1	$2b$	0	0	1	–
I_n	$n\geq 1$	0	0	n	∞	n	n	$n-1$	n	A_{n-1}
I_0^*		2	3	6	$\neq 0, 1, \infty$	–	6	4	4	D_4
I_0^*		$a\geq 3$	3	6	0	$3a-6$	6	4	4	D_4
I_0^*		2	$b\geq 4$	6	1	$2b-6$	6	4	4	D_4
I_n^*	$n\geq 1$	2	3	$n+6$	∞	n	$n+6$	$n+4$	4	D_{n+4}
II		$a\geq 1$	1	2	0	$3a-2$	2	0	1	–
III		1	$b\geq 2$	3	1	$2b-3$	3	1	2	A_1
IV		$a\geq 2$	2	4	0	$3a-4$	4	2	3	A_2
IV^*		$a\geq 3$	4	8	0	$3a-8$	8	6	3	E_6
III^*		3	$b\geq 5$	9	1	$2b-9$	9	7	2	E_7
II^*		$a\geq 4$	5	10	0	$3a-10$	10	8	1	E_8

We think of this table as describing the function whose input is a, b and δ and whose output is the type of fiber and the values of J, $m(J)$, e, r and d. (Note that e, r and d depend only on the type of fiber and not on the values of a, b or δ.)

Note that we have $a \leqq 3$ or $b \leqq 5$; otherwise the singularity of $\bar{X}$ is not a rational doublepoint and consequently not allowed.

Several useful observations can be made at this point.

Corollary 1.2. *With the above notations $\delta = e$.*

Corollary 1.3. *In all cases $0 \leqq e - r \leqq 2$. Moreover*

a) $e - r = 0 \Leftrightarrow$ *the fiber F is smooth, i.e. of type I_0.*
b) $e - r = 1 \Leftrightarrow$ *the fiber F is semistable, i.e. of type I_n, $n \geqq 1$.*

2. Formulae

The local invariants of the previous section are related by various global constraints.

First we observe that the nonsingular fibers of $\pi: X \to C$ form a locally trivial fiber bundle, hence

$$\sum_{(F \text{ singular})} e(F) = e(X) \qquad (= 12\chi(\mathcal{O}_X)) \tag{2.1}$$

(the last equality follows from Noether's formula and the vanishing of K_X^2). If S is any section of π we have $S^2 = -\chi(\mathcal{O}_X)$. (This follows from the formula for the canonical divisor (see [K2]).

Let N denote the Neron-Severi group of X. Define

$$\Sigma = \{\sigma \in N \mid \sigma \cdot F = 1, \sigma^2 = -\chi(\mathcal{O}_X)\}.$$

The set Σ consists of "numerical sections"; it contains as a subset Φ the set of irreducible sections.

Let U be the sublattice of N generated by S_0 and the class of a fiber F. Since U is unimodular, we get a splitting $N = U \oplus U^\perp$; let $p: N \to U^\perp$ be the projection.

Lemma 2.2. *The map p restricts to a bijection $p: \Sigma \to U^\perp$.*

Proof. For $\sigma \in \Sigma$ note that

$$\begin{aligned} p(\sigma) &= \sigma + [(\sigma \cdot F)(S_0^2) - (\sigma \cdot S_0)]\, F - (\sigma \cdot F)\, S_0 \\ &= \sigma + [(S_0^2) - (\sigma \cdot S_0)]\, F - S_0. \end{aligned}$$

For $\tau \in U^\perp$ define

$$p'(\tau) = \tau - \tfrac{1}{2}(\tau^2)\, F + S_0.$$

(Note that $\tau \cdot F = 0 \Rightarrow \tau K = 0$, hence (τ^2) is even.) We leave to the reader to check that $p|_\Sigma$ and p' are indeed inverse bijections. Q.E.D.

The lattice $U^\perp$ is of course an additive subgroup of N; the above lemma allows us to transport this group structure to Σ.

Formally, $\sigma_1 \oplus \sigma_2 = \sigma_1 + \sigma_2 - S_0 + lF$ (where l is chosen so as to make the sum a numerical section). The genesis of this formula is clear when we restrict to a (generic) fiber. Then

$$\sigma_1 \oplus \sigma_2 - S_0 \equiv (\sigma_1 - S_0) + (\sigma_2 - S_0)$$

which is simply the definition of the group law on an elliptic curve. Geometrically the group law on Σ is the addition induced fiber by fiber.

Observe that Φ is not a subgroup of Σ. (The latter may often be torsion-free, while the former frequently has torsion.)

We have, however, a projection $q: \Sigma \to \Phi$. Indeed, any numerical section σ may be decomposed as

$$\sigma = \sigma_0 + r,$$

where σ_0 is an irreducible section and $rF = 0$ (i.e. r consists of components of fibers) ([M-P]). Then $q(\sigma) = \sigma_0$.

Let R be the sublattice of N generated by the components of fibers not meeting S_0. R is a negative definite sublattice with a natural decomposition as $R \cong \oplus R(F)$, where F runs over the reducible fibers of π. (Note that $N(\bar{X}) \cong N(X)/R$.)

Clearly $R \subseteq U^{\perp}$ so that $p': R \to \Sigma$ is defined.

Lemma 2.3. $0 \to p'(R) \to \Sigma \xrightarrow{q} \Phi \to 0$ *is exact.*

Proof. Clearly $p'(R) \subseteq \operatorname{Ker} q$. Conversely, assuming that $q(\sigma) = S_0$, then $\sigma = S_0 + \Sigma n_i E_i + lF$ where $E_i \in U^{\perp}$. As $\sigma^2 = S_0^2$ we must have $l = -\frac{1}{2}(\Sigma n_i E_i)^2$. Thus, $\sigma = p'(\Sigma n_i E_i)$. Q.E.D.

We can now conclude some useful corollaries.

Corollary 2.4. *The rank ρ of N (the Picard number) satisfies*

$$\rho = 2 + rkR + rk\Phi = 2 + \sum_F r(F) + rk\Phi.$$

This formula is due to Shioda and Tate (see [S], [T]). If $rk\Phi = 0$ we have a refinement.

Corollary 2.5. *If N is torsion-free (which incidentally is always the case) and Φ is finite then*

$$\operatorname{disc}(R) = \prod_F d(F) = |\Phi|^2 \operatorname{disc}(N).$$

Proof. Since U is unimodular, $\operatorname{disc}(N) = \operatorname{disc}(U^{\perp}) = \operatorname{disc}(\Sigma)$. The result now follows from the exact sequence of Lemma 2.3. Q.E.D.

The Weierstrass equation for the surface $\bar{X}$ which we discussed locally in the previous section has the following well-known global version.

Let L^{-1} be the normal bundle of the section S_0 in X. Then $\bar{X}$ is isomorphic to the closed subscheme of $\mathbb{P} = \mathbb{P}(L^{-2} \oplus L^{-3} \oplus \mathcal{O}_C)$ defined by

$$y^2 z = x^3 + Axz^2 + Bz^3,$$

where $A \in H^0(C, L^4)$, $B \in H^0(C, L^6)$ and $[x, y, z]$ is the global coordinate system on $\mathbb{P}$ relative to $(L^{-2}, L^{-3}, \mathcal{O}_C)$.

The discriminant $\Delta \in H^0(C, L^{12})$ vanishes at singular fibers of π. Since $\deg L = -(S_0^2) = \chi(\mathcal{O}_X)$ we recover (2.1) from Corollary 1.2.

From now on, let us restrict to the case of $C \cong \mathbb{P}^1$, and we assume that the fibration $\pi: X \to \mathbb{P}^1$ is not trivial, i.e. not a product surface.

Then $L \cong \mathcal{O}_{\mathbb{P}^1}(M)$ for some $M \geq 1$ amd A, B and Δ are forms of degree $4M$, $6M$ and $12M$, respectively. Moreover, $H^1(\mathcal{O}_X) = 0$, $p_g = \dim H^2(\mathcal{O}_X) = M - 1$, and the canonical class $K_X = (M-2)F$. If $M = 1$, X is a rational surface. If $M = 2$, X is a $K-3$ surface.

Let us focus on the rational case $M = 1$. Recall that X is extremal if and only if Φ is finite.

Corollary 2.6. *Assume X is rational and extremal. Then*

a) $\sum_F (e(F) - r(F)) = 4$

b) $\prod_F d(F)$ *is a perfect square*

c) $|\Phi(X)| = \sqrt{\Pi d(F)}$.

Proof. Since X is rational, $\Sigma e(F) = 12$. Also, since X is extremal, $\Sigma r(F) = \rho - 2 = 8$ as $\rho = 10 - K^2$ for rational surfaces. This proves a). Now for rational surfaces, disc $N = 1$; hence; b) and c) follow from Corollary 2.5. Q.E.D.

3. The J-map

There is yet another ingredient to be exploited and that is the modulus function $J: C \to \mathbb{P}^1$ $(\cong \bar{\mathcal{M}}_1)$, defined by $J(c) = J(\pi^{-1}(c))$. The local data of J are given by the values of J and the concomitant multiplicities. The global datum is its degree. Those are obviously related via the Hurwitz formula.

Let us adopt the following notation: Let i_n denote the number of singular fibers of π of type I_n. Similarly we define i_n^*, ii, iii, iv, iv^*, iii^*, ii^*. By $i_0(j)$, $i_0^*(j)$ we denote fibers of type I_0, I_0^* with modulus $J = j$.

Lemma 3.1. $\deg J = \sum_{n \geq 1} n(i_n + i_n^*)$.

Proof. Count $J^{-1}(\infty)$ with appropriate multiplicities (according to Table 1.1). Q.E.D.

Formula 3.2. If degree $J \neq 0$, *then* $2g(C) - 2 = -2 \sum_{n \geq 1} n(i_n + i_n^*) + \sum_F (m(F) - 1)$, *where $g(C)$ is the genus of C and $m(F)$ is the multiplicity of J at the fiber F.*

Proof. This is simply a reformulation of Hurwitz formula. Q.E.D.

Note for every $j \in \mathbb{P}^1$: $\sum_{J(F) = j} m(F) = \deg J$. Therefore, by ignoring all fibers with $J \neq 0, 1, \infty$, we have the following inequality:

Firmula 3.2'. $2g(C) - 2 \geq \deg J - |J^{-1}(0)| - |J^{-1}(1)| - |J^{-1}(\infty)|$ if degree $J \neq 0$.

We now want to estimate the orders of the reduced fibers $|J^{-1}(j)|$.

Lemma 3.3. $|J^{-1}(0)| \leqq \frac{1}{3}\deg J + \frac{2}{3}(ii + iv^*) + \frac{1}{3}(iv + ii^*)$ *and*

$$|J^{-1}(1)| \leqq \tfrac{1}{2}\deg J + \tfrac{1}{2}(iii + iii^*) \qquad \textit{if degree } J \neq 0.$$

Proof. Using Table 1.1 we obtain the estimates

$$\deg J \geqq 3(i_0(0) + i_0^*(0)) + ii + iv^* + 2(iv + ii^*)$$
$$\deg J \geqq 2(i_0(1) + i_0^*(1)) + iii + iii^*.$$

As

$$|J^{-1}(0)| = i_0(0) + i_0^*(0) + ii + ii^* + iv + iv^*$$
$$|J^{-1}(1)| = i_0(1) + i_0^*(1) + iii + iii^*$$

we are done. Q.E.D.

Observing that $|J^{-1}(\infty)| = \sum_{n \geqq 1} (i_n + i_n^*)$ we can conclude

Proposition 3.4. *For an elliptic fibration* $\pi: X \to C$ *with degree* $J \neq 0$ *we have the estimate*

$$\deg J \leqq 6 \sum_{n \geqq 1} (i_n + i_n^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) + 12g(C) - 12.$$

4. The List of Rational Extremal Fibrations

We are now going to classify the configurations of singular fibers which can occur on a rational extremal fibration. It turns out that the only restrictions are those imposed by Corollary 2.6 and Proposition 3.4.

Theorem 4.1. *Assume* $\pi: X \to \mathbb{P}^1$ *is a rational extremal elliptic fibration, with* $\Phi(X)$ *as a group of sections.*

Then the set of singular fibers of π, *together with the order of* Φ *and the degree of the modulus function* $J: \mathbb{P}^1 \to \mathbb{P}^1$ *must be in the following table:*

Singular fibers	Degree of J	Order of Φ	Notation
II, II^*	0	1	X_{22}
III, III^*	0	2	X_{33}
IV, IV^*	0	3	X_{44}
I_0^*, I_0^*	0	4	$X_{11}(j), j \in \mathbb{C}$
$II^* I_1 I_1$	2	1	X_{211}
$III^* I_2 I_1$	3	2	X_{321}
$IV^* I_3 I_1$	4	3	X_{431}
$I_4^* I_1 I_1$	6	2	X_{411}
$I_1^* I_4 I_1$	6	4	X_{141}
$I_2^* I_2 I_2$	6	4	X_{222}
$I_9 I_1 I_1 I_1$	12	3	X_{9111}
$I_8 I_2 I_1 I_1$	12	4	X_{8211}
$I_5 I_5 I_1 I_1$	12	5	X_{5511}
$I_6 I_3 I_2 I_1$	12	6	X_{6321}
$I_4 I_4 I_2 I_2$	12	8	X_{4422}
$I_3 I_3 I_3 I_3$	12	9	X_{3333}

Proof. Observe that Corollaries 1.3 and 2.6(a) shows that there are at least two and at most four singular fibers.

Case 1. There are two singular fibers.
Now both fibers have to be unstable (as $e(F)-r(F)=2$). Since $\Sigma e(F)=12$ we have only the four possibilities in the table plus the possible configurations $\{I_2^*, IV\}$, $\{I_3^*, III\}$ and $\{I_4^*, II\}$. The first two violate Corollary 2.6(b) and the third leads to $\deg J<0$ by Proposition 3.4.

Case 2. There are three singular fibers.
Now we see that exactly two fibers have to be semistable. By using 2.1 and 2.6(b) we are left with the cases listed here plus the following seven others:

$$\{IV, I_2, I_6\}, \quad \{III, I_1, I_8\}, \quad \{III, I_3, I_6\}, \quad \{II, I_1, I_9\},$$
$$\{II, I_2, I_8\}, \quad \{II, I_5, I_5\}, \quad \{I_0^*, I_3, I_3\}.$$

All of those cases are ruled out by Proposition 3.4 together with Lemma 3.1.

Case 3. There are four singular fibers.
Now all fibers have to be semistable. Thus, $\deg J=\Sigma e(F)=12$ and Proposition 3.4 offers no restriction.

Given four positive integers $n_1, \ldots, n_4$ such that $\Sigma n_i=12$ and Πn_i is a square $(=|\Phi|^2)$ gives the final six possibilities. Q.E.D.

By exhibiting the Weierstrass equations we will see that all the cases actually do occur.

5. Weierstrass Equations for Rational Extremal Fibrations

In this section we construct all the rational extremal fibrations by exhibiting their Weierstrass equations

$$y^2 z = x^3 + Axz^2 + Bz^3,$$

where A and B are binary forms of degrees 4 and 6, respectively. The forms A and B are unique up to the action of $GL(2, \mathbb{C})$. (Note that the center acts non-trivially; for a scalar λ we have $\lambda(A, B)=(\lambda^4 A, \lambda^6 B)$.)

In Table 5.1 we list the four fibrations with two singular fibers.

Table 5.1

Surface	A	B	Δ	J	$\pi^{-1}(0)$	$\pi^{-1}(\infty)$	Φ
					$u=0$	$v=0$	$[x, y, z]$
X_{22}	0	uv^5	$27u^2v^{10}$	0	II	II^*	$[0, 1, 0]$
X_{33}	uv^3	0	$4u^3v^9$	1	III	III^*	$[0, 1, 0]$ $[0, 0, 1]$
X_{44}	0	u^2v^4	$27u^4v^8$	0	IV	IV^*	$[0, 1, 0]$ $[0, uv^2, 1]$ $[0, -uv^2, 1]$
$X_{11}(j)$	ru^2v^2	su^3v^3	$(4r^3+27s^2)u^6v^6$	$\frac{4r^3}{4r^3+27s^2}$	I_0^*	I_0^*	$[0, 1, 0]$ $[x_i\, uv, 0, 1]$

In the $X_{11}(j)$ case $r, s \in \mathbb{C}$ with $4r^3+27s^2 \neq 0$ where $\{x_i\}$ are the three roots of $x^3+rx+s=0$; $\Phi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. These equations can be easily derived from [M] and the surfaces are unique.

The six cases with three singular fibers require more work and are listed below.

Table 5.2

Surface	A, B, Δ, J	Ramification of J and singular fibers				Sections of Φ $[x, y, z]$
		$[u, v]$	J	$m(j)$	fiber type	
X_{211}	$A=-3u^4$	$[0, 1]$	0	2	II^*	$[0, 1, 0]$
	$B=2u^5v$	$[1, 0]$	1	2	smooth	
	$\Delta=-108u^{10}(u^2-v^2)$	$[1, 1]$	∞	1	I_1	
	$J=u^2/(u^2-v^2)$	$[-1, 1]$	∞	1	I_1	
X_{321}	$A=-uv^3$	$[0, 1]$	0	3	smooth	$[0, 1, 0]$
	$B=v^5(u-v)$	$[1, 0]$	1	1	III^*	$[v^2, 0, 1]$
	$\Delta=-v^9(u-3v)^2(4u-3v)$	$[1, 1]$	1	2	smooth	
	$J=4u^3/(u-3v)^2(4u-3v)$	$[3, 1]$	∞	2	I_2	
		$[3, 4]$	∞	1	I_1	
X_{431}	$A=v^3(24u-27v)$	$[9, 8]$	0	3	smooth	$[0, 1, 0]$
	$B=v^4(16u^2-72uv+54v^2)$	$[1, 0]$	0	1	IV^*	$[3v^2, \pm 4uv^2, 1]$
	$\Delta=256\cdot 27u^3v^8(u-v)$	$[9\pm 3\sqrt{3}, 4]$	1	2	smooth	
	$J=\dfrac{v(24u-27v)^3}{64\cdot 27u^3(u-v)}$	$[0, 1]$	∞	3	I_3	
		$[1, 1]$	∞	1	I_1	
X_{411}	$A=-3v^2(u^2-3v^2)$	$[\pm\sqrt{3}, 1]$	0	3	smooth	$[0, 1, 0]$
	$B=uv^3(2u^2-9v^2)$	$[0, 1]$	1	2	smooth	$[uv, 0, 1]$
	$\Delta=-27^2v^{10}(u^2-4v^2)$	$[3, \pm\sqrt{2}]$	1	2	smooth	
	$J=\dfrac{4(u^2-3v^2)^3}{27v^4(u^2-4v^2)}$	$[1, 0]$	∞	4	I_4^*	
		$[\pm 2, 1]$	∞	1	I_1	
X_{141}	$A=-3(u^2-3v^2)(u-2v)^2$	$[\pm\sqrt{3}, 1]$	0	3	smooth	$[0, 1, 0]$
	$B=u(2u^2-9v^2)(u-2v)^3$	$[0, 1]$	1	2	smooth	$[u^2-2uv, 0, 1]$
	$\Delta=-27^2v^4(u-2v)^6(u^2-4v^2)$	$[3, \pm\sqrt{2}]$	1	2	smooth	$[(u-3v)(u-2v),$
	$J=\dfrac{4(u^2-3v^2)^3}{27v^4(u^2-4v^2)}$	$[2, 1]$	∞	1	I_1^*	$\pm 3\sqrt{3}v(u-2v)^2, 1]$
		$[-2, 1]$	∞	1	I_1	
		$[1, 0]$	∞	4	I_4	
X_{222}	$A=-3uv(u-v)^2$	$[0, 1]$	0	3	smooth	$[0, 1, 0]$
	$B=(u-v)^3(u^3+v^3)$	$[1, 0]$	0	3	smooth	$[(v-u)(u+v), 0, 1]$
	$\Delta=27(u^3-v^3)^2(u-v)^6$	$[-1, 1]$	1	2	smooth	$[(v-u)(wu+w^2v), 0, 1]$
	$J=-4u^3v^3/(u^3-v^3)^2$	$[-w, 1]$	1	2	smooth	$[(v-u)(w^2u+wv), 0, 1]$
		$[-w^2, 1]$	1	2	smooth	
	Here $w^3=1$,	$[1, 1]$	∞	2	I_2^*	
	$w \neq 1$	$[w, 1]$	∞	2	I_2	
		$[w^2, 1]$	∞	2	I_2	

Proof. The simplest case is X_{211}; here J is of degree 2 and we can choose coordinates $[u, v]$ so that J is ramified at $u=0$ (with the fiber II^*) and at $v=0$ (with a smooth fiber I_0 having $J=1$). In this case, by Table 1.1, A is a multiple of u^4 and B a multiple of $u^5 v$. By scaling the coordinates with diagonal matrices, we may achieve $A=-3u^4$ and $B=2u^5 v$. Hence, X_{211} is unique.

In the case of X_{321}, J is of degree 3 and is totally ramified over $J=0$ (with a smooth fiber) and is simply ramified over $J=1$ (with a smooth fiber) and over $J=\infty$ (with the I_2 singular fiber). If we choose coordinates so that the $J=0$ fiber is at $u=0$, the III^* fiber is over $v=0$ and the smooth $J=1$ fiber is over $u=v$, then we must have A being a multiple of uv^3 and B a multiple of $v^5(u-v)$. We can use the scalar action to assume that $A=-uv^3$ and $B=cv^5(u-v)$ for some constant c. In order that J be ramified over ∞, c must be ± 1. If $c=-1$, use the scalar i to change c to $+1$, leaving A unaffected. Hence, $A=-uv^3$, $B=v^5(u-v)$; moreover, X_{321} is unique.

For the surface X_{431}, let us choose coordinates $[u, v]$ so that the IV^* fiber is at $v=0$, the I_3 fiber is at $u=0$, and the I_1 fiber is at $u=v$. Then by scaling we may assume $A=3v^3(a_1 u+a_0 v)$ and $B=2v^4(u^2+b_1 uv+b_0 v^2)$; we must have $4A^3+27B^2=4\cdot 27\cdot v^8 u^3(u-v)$. This forces $a_1^3+2b_1=-1$, $3a_1^2 a_0+b_1^2+2b_0=0$, $3a_1 a_0^2+2b_1 b_0=0$, and $a_0^3+b_0^2=0$. The last two are equivalent to $a_0=-4b_1^2/9a_1^2$ and $b_0=-8b_1^3/27a_1^3$. The second one then gives $b_1=-9a_1^3/16$ and the first then implies $a_1^3=8$. By using λ such that $\lambda^6=1$ as a scaling factor, we may arrange λ^4 to be any cube root of unity, so we may assume $a_1=2$, forcing $b_1=-9/2$, $b_0=27/8$ and $a_0=-9/4$. Finally, by scaling with $\lambda=\sqrt{2}$ to clear the denominators, we get $A=24uv^3-27v^4$ and $B=16u^2 v^4-72uv^5+54v^6$. Here $\Delta=4^4\cdot 27\cdot u^3 v^8(u-v)$ and X_{431} is also unique.

The surfaces X_{411} and X_{141} are related; they have the same J-function. This map $J\colon \mathbb{P}^1\to\mathbb{P}^1$ is of degree 6 and is ramified only over $J=0$, 1, and ∞. Its multiplicities over 0 are (3, 3) (two smooth fibers), over 1 are (2, 2, 2) (three smooth fibers), and over ∞ (4, 1, 1) (one I_4^* fiber and two I_1 fibers). If we choose coordinates $[u, v]$ on the base curve so that the $J=0$ fibers are at $u=0$ and $u=v$ and the I_4^* fiber is at $v=0$, then the J function must be of the form $J=u^3(u-v)^3/v^4(au^2+buv+vc^2)$ for suitable constants a, b and c. These are determined by the condition that the $J=1$ locus is a perfect square. An easy calculation shows that in fact we may write

$$J=256u^3(u-v)^3/v^4(12u^2-12uv-v^2).$$

Hence, this J-map is unique.

A more elegant construction is to realize this J-map as factoring into a degree 2-map followed by a degree 3-map. If s is an affine coordinate on the intermediate $\mathbb{P}^1$, the degree 3-map can be written as $J=4s^3/27(s-1)$. It is totally ramified at $s=0$ where $J=0$ and is simply ramified at $s=\infty$ where $J=\infty$; $J=\infty$ also at $s=1$. The constants are adjusted so that it is also ramified over J at $s=3/2$; $J=1$ also at $s=-3$. The original degree 6-map is obtained by composing this with a double cover, branched at $s=-3$ and at $s=\infty$. The equation for this is $(u/v)^2=(s+3)$ or $s=(u^2-3v^2)/v^2$. This makes the degree 6 J-function $J=4(u^2-3v^2)^3/27v^4(u^2-4v^2)$.

Table 5.3

Surface: X_{9111} $|\Phi|=3$

$A=-3u(u^3+24v^3)$ $B=2(u^6+36u^3v^3+216v^6)$

$\Delta=2^8\cdot 3^6\cdot v^9(u^3+27v^3)$

Singular fibers:

$[u,v]$	$[1,0]$	$[-3,1]$	$[-3w,1]$	$[-3w^2,1]$
fibre type	I_9	I_1	I_1	I_1

where $w^3=1$, $w\neq 1$.

Sections:

[x,	y,	z]
[0,	1,	0]
[u^2,	$\pm 12\sqrt{3}v^3$,	1]

Surface: X_{8211} $|\Phi|=4$

$A=-3(u^4+4u^2v^2+v^4)$ $B=2u^6+12u^4v^2+15u^2v^4-2v^6$

$\Delta=-3^6u^2v^8(u^2+4v^2)$

Singular fibers:

$[u,v]$	$[1,0]$	$[0,1]$	$[2,\pm i]$
fibre type	I_8	I_2	I_1

Sections:

[x,	y,	z]
[0,	1,	0]
[u^2+2v^2,	0,	1]
[u^2-v^2,	$\pm 3\sqrt{3}uv^2$,	1]

Surface: X_{5511} $|\Phi|=5$

$A=-3(u^4-12u^3v+14u^2v^2+12uv^3+v^4)$

$B=2(u^6-18u^5v+75u^4v^2+75u^2v^4+18uv^5+v^6)$

$\Delta=-2^8\cdot 3^6\cdot u^5v^5(u^2-11uv-v^2)$

Singular fibers:

$[u,v]$	$[0,1]$	$[1,0]$	$[11\pm 5\sqrt{5},2]$
fibre type	I_5	I_5	I_1

Sections:

[x,	y,	z]
[0,	1,	0]
[$u^2+6uv+v^2$,	$\pm 12\sqrt{3}u^2v$,	1]
[$u^2-6uv+v^2$,	$\pm 12\sqrt{3}uv^2$,	1]

Surface: X_{6321} $|\Phi|=6$

$A=-3(u^4+4u^3v-2uv^3+v^4)$

$B=2u^6+12u^5v+12u^4v^2-14u^3v^3+3u^2v^4-6uv^5+2v^6$

$\Delta=-3^6u^3v^6(4u^3+12u^2v-15uv^2+4v^3)$

$=-3^6u^3v^6(2u-v)^2(u+4v)$

Singular fibers:

$[u,v]$	$[1,0]$	$[0,1]$	$[1,2]$	$[-4,1]$
fibre type	I_6	I_3	I_2	I_1

Sections:

[x,	y,	z]
[0,	1,	0]
[$u^2+2uv-2v^2$,	0,	1]
[$u^2+2uv+v^2$,	$\pm 3\sqrt{3}uv^2$,	1]
[$u^2-4uv+v^2$,	$\pm 3\sqrt{3}uv(v-2u)$,	1]

Table 5.3 continued

Surface: X_{4422} $|\Phi|=8$

$A=-3u^4+3u^2v^2-3v^4$ $B=2u^6-3u^4v^2-3u^2v^4+2v^6$

$\Delta=-3^6u^4v^4(u+v)^2(u-v)^2$

Singular fibers:

$[u, v]$	$[0,1]$	$[1,0]$	$[1,1]$	$[1,-1]$
fibre type	I_4	I_4	I_2	I_2

Sections:

[x,	y,	z]
[0,	1,	0]
[u^2+v^2,	0,	1]
[u^2-2v^2,	0,	1]
[$-2u^2+v^2$,	0,	1]
[$u^2+3uv+v^2$,	$\pm 3\sqrt{3}uv(u+v)$,	1]
[$u^2-3uv+v^2$,	$\pm 3\sqrt{3}uv(u-v)$,	1]

Surface: X_{3333} $|\Phi|=9$

$A=-3u^4+24uv^3$ $B=2u^6+40u^3v^3-16v^6$

$\Delta=2^8\cdot 3^3v^3(u^3+v^3)^3$

Singular fibers:

$[u, v]$	$[1,0]$	$[-1,1]$	$[-w,1]$	$[-w^2,1]$
fibre type	I_3	I_3	I_3	I_3

where $w^3=1$, $w\neq 1$.

Sections:

[x,	y,	z]
[0,	1,	0]
[$-3u^2$,	$\pm 4i(u^3+v^3)$,	1]
[$(u-2v)^2$,	$\pm 4\sqrt{3}v(u^2-uv+v^2)$,	1]
[$(u-2wv)^2$,	$\pm 4\sqrt{3}v(wu^2-w^2uv+v^2)$,	1]
[$(u-2w^2v)^2$,	$\pm 4\sqrt{3}v(w^2u^2-wuv+v^2)$,	1]

where $w^3=1$, $w\neq 1$.

In the case of X_{411} we can choose $A=-3v^2(u^2-3v^2)$. Then $B=uv^3(2u^2-9v^2)$, which puts the I_4^* fiber at $v=0$. For the surface X_{141} we may put $A=-3(u^2-3v^2)(u-2v)^2$ and $B=u(2u^2-9v^2)(u-2v)^3$, which puts the I_1^* fiber at $u=2v$. Both surfaces are unique.

Finally, the J-map for the X_{222} case is also of degree 6 and ramified only over $J=0, 1$ and ∞. Over 0 the multiplicities are (3, 3) (two smooth fibers); over 1 they are (2, 2, 2) (three smooth fibers); and over ∞ they are (2, 2, 2) (one I_2^* fiber and two I_2 fibers). If we choose coordinates so that the $J=0$ fibers are at $u=0$ and $v=0$ and the I_2^* fiber is at $u=v$, then we must have $A=auv(u-v)^2$ and $B=(u-v)^3C$, where C is a cubic form and a is a constant. Since the discriminant must be a square, a calculation shows that this forces $C=u^3+v^3$ if $a=-3$. Hence, $A=-3uv(u-v)^2$, $B=(u-v)^3(u^3+v^3)$ and $\Delta=27(u^3-v^3)^2(u-v)^6$. Moreover, X_{222} is unique.

This J-map can also be obtained by factoring it into a degree-3 map followed by a degree-2 map. If s is an affine coordinate on the intermediate $\mathbb{P}^1$

then $J=-4(s^2-s)$. This double cover is branched over $J=1$ and $J=\infty$ (where $s=1/2$ and $s=\infty$, respectively); over $J=0$, $s=0$ and $s=1$. The triple cover is a $\mathbb{Z}/3$ cover, defined by $(u/v)^3=s/s-1$ branched over $s=0$ and $s=1$.

This completes the construction of the rational extremal fibrations with three singular fibers.

Finally, we will simply present the Weierstrass equations for the six extremal elliptic fibrations with four singular fibers in Table 5.3. These were recently studied by Beauville ([B]) in a different context. In every case the surface exists and is unique.

We have proven the following.

Theorem 5.4. *For every possible configuration of singular fibers given in Theorem 4.1, there is a unique rational extremal elliptic surface with that configuration of singular fibers, except for the surfaces $X_{11}(j)$. These surfaces each have two singular fibers of type I_0^*, with constant J-map $(=j)$, and fixing j, there is a unique such surface.*

6. Cubic Pencils and Plane Quartics

We are now going to set our list of examples into a geometrical setting. The most direct way of presenting a rational elliptic surface is through pencils of cubics curves in the projective plane.

A cubic pencil is a one-dimensional linear system of cubics, having no fixed components. Thus, a cubic pencil has only isolated basepoints, and by Bezout's theorem those are always nine in number. They are not necessarily all distinct on $\mathbb{P}^2$ but may be infinitely near.

By blowing up the basepoints of a cubic pencil we obtain a rational elliptic surface. The exceptional divisors of first kind correspond exactly to the sections. This is as we have remarked before a direct consequence of the representation of the canonical divisor.

The well-known converse is also true; we will omit its proof.

Proposition 6.1. *Every Jacobian rational elliptic surface is the blow up of the basepoints of a cubic pencil.*

Remark. Even if we drop the assumption of a Jacobian fibration, a rational elliptic surface is still a blow up of $\mathbb{P}^2$ at nine points, though not necessarily the basepoints of a cubic pencil.

There is, as has been explained above, another geometric interpretation of an elliptic fibration. As this is less well-known than the cubic pencil approach, we will go into more detail.

Letting $z\to -z$ we obtain a double covering onto $\mathbb{F}_2$ branched at the minimal section and a trisection T. The singularities of T are all simple, and there is a natural $1-1$ correspondence between those and the Kodaira fibers, as explained by the list below (cf. Table 1.1). (The notation for curve singularities in the 2^{nd} column is that used in [B-P-V].)

Table 6.2

Type of fiber	Singularity of T	Intersection T with F
I_1	–	tangency
II	–	flexed
III	a_1 (node)	tangency to one branch
IV	a_2 (cusp)	cuspidal tangency
I_n $n \geqq 2$	a_{n-1}	transversally
I_n^* $n \geqq 0$	d_{n+4}	transversally
II^*	e_8	transversally
III^*	e_7	transversally
IV^*	e_6	transversally

Observation 6.3. *The fibration is irreducible iff T is nonsingular.*

Proof. Immediate from Table 6.2. Q.E.D.

Lemma 6.4. *If X is a rational elliptic fibration with at least one reducible fiber, then X is birationally a double cover of $\mathbb{P}^2$ branched along a quartic and the elliptic fibration is induced from the pencil whose basepoint P is the image of the section.* (That basepoint is referred to as the distinguished point of the quartic.)

Proof. We simply observe that if a point is blown up on $\mathbb{F}_2$ away from the minimal section, then an elementary transformation performed at the point makes the minimal section exceptional and allows a descent onto $\mathbb{P}^2$. The fibers of $\mathbb{F}_2$ are then mapped onto the pencil whose basepoint is the image of the minimal section. Q.E.D.

Remark. Note that if we add the subindices of the simple singularities of T, we obtain the so-called index of T (cf. [P1] $\sigma(T)$. We get the estimate $\sigma(T) \leqq 8$ with equality iff the fibration is extremal.

Furthermore, if the elementary transformation is done at P on T, two cases occur: either the index of the resolved singularity drops by one or its drops by two. In the first case the index of the quartic is simply $\sigma(T)-1$, and if T corresponds to a rational extremal fibration, the ensuing quartic is maximizing ([P1]).

The following lemma gives the exact picture.

Lemma 6.5. a) *If P is a triple point on T, then the exceptional divisor $\dot{E}_p$ becomes a line component of the quartic supporting the following singularities according to the scheme below.*

singularity of P	*singularities on E_p*
d_n	$d_{n-2}+a_1$
e_6	a_5
e_7	d_6
e_8	e_7

b) *If P is a double point on T, then the exceptional divisor E_p becomes the tangent to the quartic at its distinguished point.*

singularity of P		*singularities on E_p*
a_1	*(transversal)*	–
a_1	*(III)*	*(E_p flexed at distinguished point)*
a_n	$(n \geqq 2)$	a_{n-2}
a_2	*(IV)*	*(E_p hyperflexed at distinguished point).*

We will omit the straightforward proof of the above.

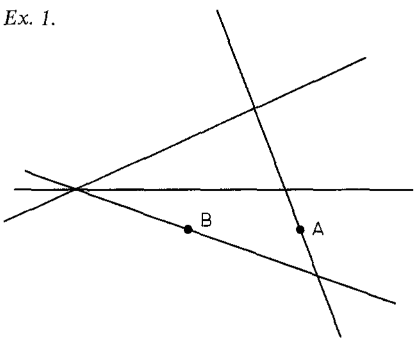
We can now state

Proposition 6.6. *Every extremal rational elliptic fibration, with the exception of X_{9111}, X_{5511} and X_{3333}, has a representation as a maximizing quartic with a distinguished point.*

The point is that there are only six maximizing quartics (cf. [P1], where unfortunately two obvious examples are missing). This gives a comparatively painless way of exhibiting the various extremal fibrations. The proof of the proposition is simply given by exhibiting the maximizing quartics, which we do in the following table.

Table 6.7 (the six maximizing quartics together with their distinguished points)

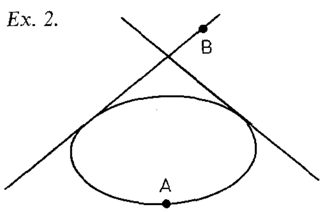
Ex. 1.



A) $X_{11}(j)$ note that A together with the other three points determine the (I_0^*) modulus j.

B) X_{222} (I_2^*)

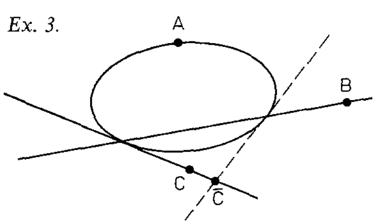
Ex. 2.



A) X_{4422} (I_2)

B) X_{141} (I_1^*)

Ex. 3.



A) X_{222} (I_2)

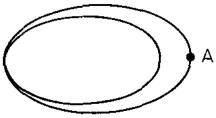
B) X_{411} (I_4^*)

C) X_{321} (III^*)

$\bar{C}$) X_{33} (III^*)

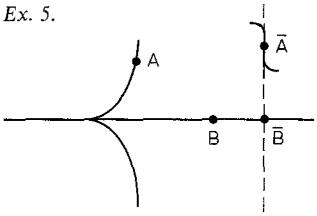
Table 6.7 continued

Ex. 4.



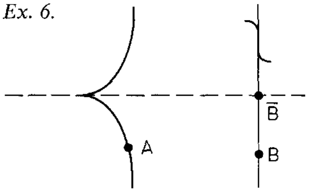
A) X_{8211} (I_2)

Ex. 5.



A) X_{321} (I_2)
Ā) X_{33} (III)
B) X_{211} (II^*)
B̄) X_{22} (II^*)

Ex. 6.

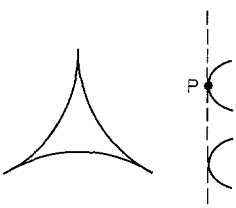


A) X_{6321} (I_2)
B) X_{431} (IV^*)
B̄) X_{44} (IV^*)

The three remaining cases have representations as quartics of index 6, with special distinguished points.

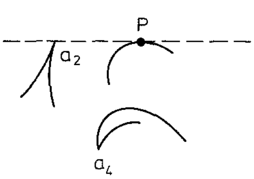
Table 6.8

Ex. 1. The Steiner quartic $(3a_2)$ together with one of its bitangency points P.



This represents X_{3333}.

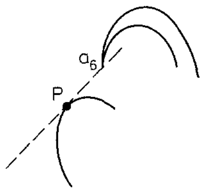
Ex. 2. The first degenerate of the Steiner quartic (a_2, a_4) together with the point P whose tangent passes through the cusp.



This represents X_{5511}.

Table 6.8 continued

Ex. 3. The second degenerate of the Steiner quartic (a_6) together with the point whose tangent passes through the singularity.



This represents X_{9111}.

Remark. The Steiner quartic together with its degenerate forms have nice geometric ways of construction:

Consider linear systems of conics with three basepoints (not necessarily distinct); those define Cremona transformations from $\mathbb{P}^2$ to $\mathbb{P}^2$. To each such system we have the notion of an inscribed conic. In the case of three distinct basepoints, the conic is tangent to the sides of the corresponding triangle. If two basepoints coalesce, they define a line, and the conic is required to be tangent to that one as well. Finally if all three points coincide, they also determine a line (all the conics are flexed to each other at one point, and the line is the common tangent) which the conic is expected to touch.

The image of the inscribed conic turns out to be a Steiner quartic with its degenerate forms.

7. Extremal Rational Elliptic Surfaces as Pull-Backs of Each Other

In this section we will analyze the following question: Given $X \xrightarrow{\pi} \mathbb{P}^1$ an extremal rational elliptic surface, and given $f: \mathbb{P}^1 \to \mathbb{P}^1$, when is the pullback surface $Y = X \times_{\mathbb{P}^1} \mathbb{P}^1 \to \mathbb{P}^1$ also an extremal rational elliptic surface? Note that, in general, if the pullback Y is extremal, then X must be, since $\Phi(X)$ injects into $\Phi(Y)$; hence there is no loss of generality in assuming X is extremal to begin with.

The first step in the analysis is to carefully understand the relationships between the singular fibers of X and those of the pullback Y. If the map f is etale at $p \in \mathbb{P}^1$, then of course the fiber of Y over p is identical to the fiber of X over $f(p)$. Hence we need only study the situation when f is ramified at p, to some order N. In this case we can choose local analytic coordinates z at p and w at $f(p)$ so that the map f is simply given by the base change $w = z^N$ of order N. Using the notation of Table 1.1, if the fiber of X over $f(p)$ has invariants a, b, and δ, then the invariants of the fiber of Y over p will be $a' = Na$, $b' = Nb$, and $\delta' = N\delta$, respectively. If $a' \geqq 4$ and $b' \geqq 6$, Y has a non-rational singularity which can be partially resolved by replacing a' by $a' - 4$, b' by $b' - 6$, and δ' by $\delta' - 12$. Repeated application of this partial resolution eventually gives $a' \leqq 3$ or $b' \leqq 5$, and at this point the fiber of Y over p can be calculated using Table 1.1. We present the results in the following table.

Table 7.1. Base change of elliptic fibers, to order N

Fiber of X over $f(p)$	Fiber of Y over P		
I_0	I_0		
I_M	I_{NM}		
I_M^*	I_{NM} if N even;	I_{NM}^* if N odd	
II	I_0 if $N\equiv 0 \pmod 6$);	II if $N\equiv 1 \pmod 6$)	
	IV if $N\equiv 2 \pmod 6$);	I_0^* if $N\equiv 3 \pmod 6$)	
	IV^* if $N=4 \pmod 6$);	II^* if $N\equiv 5 \pmod 6$)	
III	I_0 if $N\equiv 0 \pmod 4$);	III if $N\equiv 1 \pmod 4$)	
	I_0^* if $N\equiv 2 \pmod 4$);	III^* if $N\equiv 3 \pmod 4$)	
IV	I_0 if $N\equiv 0 \pmod 3$);	IV if $N\equiv 1 \pmod 3$);	IV^* if $N\equiv 2 \pmod 3$)
IV^*	I_0 if $N\equiv 0 \pmod 3$);	IV^* if $N\equiv 1 \pmod 3$);	IV if $N\equiv 2 \pmod 3$)
III^*	I_0 if $N\equiv 0 \pmod 4$);	III^* if $N\equiv 1 \pmod 4$)	
	I_0^* if $N\equiv 2 \pmod 4$);	III if $N\equiv 3 \pmod 4$)	
II^*	I_0 if $N\equiv 0 \pmod 6$);	II^* if $N\equiv 1 \pmod 6$)	
	IV^* if $N\equiv 2 \pmod 6$);	I_0^* if $N\equiv 3 \pmod 6$)	
	IV if $N\equiv 4 \pmod 6$);	II if $N\equiv 5 \pmod 6$)	

Also note that if J_X is the J-map for X, and J_Y is that for Y, then

$$\text{degree } J_Y = (\text{degree } f)(\text{degree } J_X). \tag{7.2}$$

We can now state the results when degree $J_X \neq 0$.

Theorem 7.3. *Assume degree $J_X \neq 0$, and that Y is an extremal rational elliptic surface also. Then X, Y, and the pullback map f are on the following list:*

X	Y	f
X_{431}	X_{6321}	degree 3, totally ramified over the IV^* fiber, and simply ramified over the I_3 and I_1 fibers.
X_{431}	X_{9111}	degree 3, totally ramified over the IV^* fiber and over the I_3 fiber.
X_{431}	X_{3333}	degree 3, totally ramified over the IV^* fiber and over the I_1 fiber.
X_{411}	X_{8211}	degree 2, ramified over the I_4^* fiber and one of the I_1 fibers.
X_{141}	X_{8211}	degree 2, ramified over the I_1^* fiber and the I_4 fiber.
X_{141}	X_{4422}	degree 2, ramified over the I_1^* fiber and the I_1 fiber.
X_{222}	X_{4422}	degree 2, ramified over the I_2^* fiber and one of the I_2 fibers.

Proof. We will take up the possibilities for the surface X one at a time.

Case 1. $X = X_{211}$.
Here degree $J_X = 2$, so degree $f = 2$, 3, or 6 (since degree $J_Y = 2$, 3, 4, 6, or 12, and degree $f \geqq 2$). If degree $f = 2$, then by (7.2), degree $J_Y = 4$, so $Y = X_{431}$. The

singular fiber I_3 of Y must be obtained by base change from one of the two I_1 fibers of X; this is not possible, since degree $f=2$, and a base change of order 3 is necessary to produce an I_3 fiber from an I_1, by Table 7.1. Hence degree $f\neq 2$. If degree $f=3$, then Y has an I^*_J with $M\geqq 1$, which cannot be produced by base change from the fibers of X. Finally assume degree $f=6$. Then degree $J_U=12$ and all 4 singular fibers of Y are of type I_J. Hence by Table 7.1, f must be totally ramified over the II^* fiber of X, and the corresponding fiber of Y is smooth. The 4 singular fibers of Y are obtained from the two I_1 fibers of X; hence over the two points of $\mathbb{P}^1$ giving the two I_1 fibers of X there are exactly 4 pre-images via the degree 6 map f. There are two cases to consider. If over one point there is only one pre-image, then f is totally ramified over that point, and hence by Hurwitz's formula f has no other ramification points; therefore there are 6 pre-images above the other point, and 7 pre-images altogether, not 4. Finally assume that over both points t_1 and t_2 there are two pre-images. Then the ramification of f over t_i must be to orders r_i and $6-r_i$; hence by Hurwitz's formula we have

$$-2\geqq -2(6)+5+(r_1-1)+(5-r_1)+(r_2-1)+(5-r_2),$$

or $-2\geqq 1$, a contradiction.

Hence $X\neq X_{211}$.

Case 2. $X=X_{321}$.
Here degree $f=2$ or 4. However, if degree $f=2$, then degree $J_Y=6$, and Y has an I^*_M fiber, which cannot be produced from the fibers of X. Hence degree $f=4$, and as above f must be totally ramified over the III^* fiber, producing a smooth fiber of Y. Again we must have 4 pre-images over the two points t_1 and t_2 with the I_1 and I_2 fibers, and again there must be two over each. Hence by Hurwitz, using the same notation as in Case 1,

$$-2\geqq -2(4)+3+(r_1-1)+(3-r_1)+(r_2-1)+(3-r_2)$$

or $-2\geqq -1$. Therefore $X\neq X_{321}$.

Case 3. $X=X_{431}$.
Here degree $f=3$, and f must be totally ramified over the IV^* fiber. If f is also totally ramified over the I_3 fiber, then there is no other ramification and Y is X_{9111}. If f is also totally ramified over the I_1 fiber, then Y is X_{3333}. Otherwise there are two preimages of f over both the I_1 and I_3 fibers, and in each case f is simply ramified at one pre-image and etale at the other; this gives $Y=X_{6321}$.

Case 4. $X=X_{411}$.
Here degree $f=2$, and f must be ramified over the I_4^* fiber (producing an I_8 fiber in Y) and over one other point. That point must be under one of the I_1 fibers, and $Y=X_{8211}$.

Case 5. $X=X_{141}$.
Again degree $f=2$, and f is ramified over the I_1^* fiber, and over one of the other two singular fibers. If it is ramified over the I_4 fiber, $Y=X_{8211}$; if over the I_1 fiber, $Y=X_{4422}$.

Case 6. $X = X_{222}$.
Again degree $f=2$, and f is ramified over the I_2^* fiber, and over one of the two I_2 fibers, producing $Y = X_{4422}$. Q.E.D.

The situation where degree $J_X = 0$ is at once simpler and more complicated. It is simpler because degree $J_Y = 0$ in this case, hence Y has only two singular fibers, etc. It is much more complicated because the degree of the pullback map f is now unrestricted, and the reader can check that there are an infinite number of possibilities for f. We will be satisfied to analyze only the case where the two singular fibers of X are at 0 and ∞ and f is an N^{th} power map from $\mathbb{P}^1 \to \mathbb{P}^1$, i.e., a map of degree N totally ramified over 0 and ∞, (and hence unramified elsewhere). An elementary analysis using Table 7.1 gives the following.

Theorem 7.4. *Assume degree $J_X = 0$ and f is an N^{th} power map, totally ramified over the two singular fibers of X. Then Y is either an extremal rational elliptic surface with degree $J_Y = 0$, or Y is a product surface $E \times \mathbb{P}^1$, where the J-invariant of E is equal to the J-invariant of a general fiber of X. Moreover, given X and N, Y is determined by the following table.*

Table 7.5

X	N	Y
$X_{11}(j)$	$N \equiv 0 \pmod 2$	$E \times \mathbb{P}^1$, $J(E) = j$
	$N = 1 \pmod 2$	$X_{11}(j)$
X_{22}	$N \equiv 0 \pmod 6$	$E \times \mathbb{P}^1$, $J(E) = 0$
	$N \equiv 1, 5 \pmod 6$	X_{22}
	$N \equiv 2, 4 \pmod 6$	X_{44}
	$N \equiv 3 \pmod 6$	$X_{11}(0)$
X_{33}	$N = 0 \pmod 4$	$E \times \mathbb{P}^1$, $J(E) = 1$
	$N = 1, 3 \pmod 4$	X_{33}
	$N = 2 \pmod 4$	$X_{11}(1)$
X_{44}	$N = 0 \pmod 3$	$E \times \mathbb{P}^1$, $J(E) = 1$
	$N = 1, 2 \pmod 3$	X_{44}

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