Irrational Nef Rays at the Boundary of the Mori Cone for Very General Blowups of the Plane

Ciro Ciliberto, Rick Miranda, & Joaquim Roé

ABSTRACT. In this paper, we develop a technique for discovering (non-effective) irrational rays at the boundary of the Mori cone for linear systems on a general blowup of the plane and give examples of such irrational rays.

Introduction

Let X_s be the blowup of the complex projective plane at s very general points. We will usually assume that $s \geq 10$. Let $\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_s)$, $d > 0$, be the linear system on X_s corresponding to plane curves of degree d having multiplicities at least $m_1, \ldots, m_s$ at the given points (we will use exponential notation for repeated multiplicities). If we denote by H the class of the pullback to X_s of a general line in the plane and by E_i the exceptional divisor over the ith point that is blown up, then the linear system $\mathcal{L}$ corresponds to the global sections of the line bundle $\mathcal{O}_{X_s}(dH - \sum_i m_i E_i)$. The (projective) dimension of the linear system is $\dim(H^0(X_s, \mathcal{O}_{X_s}(dH - \sum_i m_i E_i))) - 1$. The system is said to be *effective* if the dimension is nonnegative, that is, there are effective divisors in the linear system. If the dimension is -1, then we will say that the system is *noneffective* or *empty*.

We note that the Picard group of X_s is the free Abelian group of rank $s + 1$ generated by the classes of H and $E_1, \ldots, E_s$. We define $N = N^1(X_s)$ to be $\mathrm{Pic}(X_s) \otimes_{\mathbb{Z}} \mathbb{R}$; this is a real vector space of dimension $s + 1$.

A *ray* in N is the set of all nonnegative real multiples of a nonzero vector in N. Many of the concepts applicable to elements of the Picard group $\mathrm{Pic}(X_s)$ may be extended to N. A ray is called *effective* if there is a (necessarily integral) vector in the ray that represents an effective divisor class in $\mathrm{Pic}(X_s)$. The ray is called *rational* if it contains a nonzero integral vector (i.e., an element of the Picard group). The *degree* $\deg(R)$ of a ray R, the coefficient of H, is not well defined, but its sign is; hence we may speak of a ray of positive/zero/negative degree. Every effective ray R must have a nonnegative degree. Similarly, if R and R' are two rays, the intersection $R \cdot R'$ is not well defined, but its sign is. In particular, this applies to the self-intersection of a ray. Any rational ray with $\deg(R) > 0$ and $R^2 > 0$ is effective (by the Riemann–Roch theorem).

The *effective cone* is the cone generated by effective rays, that is, all finite linear combinations of effective divisors with positive real coefficients. In general,

the effective cone is not closed; its closure is called the *Mori cone*. The dual of the Mori cone is the cone of *nef* divisors or rays; these are divisors/rays that nonnegatively intersect all effective divisors. This nef cone is closed.

In [C+13] a *good ray* was defined as a noneffective rational ray with nonnegative degree (intersection with H) and zero self-intersection. Such a ray is extremal for the Mori and nef cones and is nef (see [C+13, Lemma 3.8]).

In that paper, we also defined a *wonderful* ray as a nef ray that has self-intersection zero and is irrational. Up to this point, no wonderful ray (with all coordinates nonzero) for any $s > 10$ has appeared in the literature. Some of the motivation for discovering wonderful rays is as follows.

Define the *De Fernex ray F_s* as the ray generated by $\sqrt{s-1}H - \sum_{i=1}^{s} E_i$. A ray R is said to be *De Fernex positive, negative, or orthogonal* according to $R \cdot F_s$ being positive, negative, or null. The *strong Δ-conjecture* (see [C+13, Conjecture 3.10]) is that if $s \geq 11$, and R is a rational De Fernex nonpositive ray of self-intersection zero, then R is not effective and therefore is a good ray. (See [Fer]; there is a refinement for the case $s = 10$.) Note that $R \cdot F_s \leq 0$ implies $R \cdot K_s > 0$, where $K_s = -3H + \sum_i E_i$ is the canonical divisor on X_s.

The strong Δ-conjecture would imply that an irrational De Fernex nonpositive ray R with self-intersection zero is a wonderful ray, since any such ray would then be a limit of good rays which are nef.

This remark implies, in particular, that the *Nagata ray $\sqrt{s}H - \sum_i E_i$* would be wonderful (if s is not a square), and this would prove the Nagata conjecture (see [Nag59]). Of course, the Nagata conjecture is very important and has been shown for decades that it is difficult to prove. However, in view of the strong Δ-conjecture, from a conceptual Mori cone viewpoint, proving that the Nagata ray is wonderful is not philosophically more important than proving that any other De Fernex negative irrational ray with self-intersection zero is wonderful.

In the unpublished work [Fer], De Fernex showed that for $s = 10$, all rays R of self-intersection zero with $R \cdot K_{10} = 0$ are nef. Since such rays can be irrational, they provide examples of wonderful rays according to the definition of [C+13], and they are the only wonderful rays ever produced in the literature so far. Note that these are De Fernex orthogonal and hence not the most relevant for our motivation. After the first version of this paper appeared on math.arxiv, J. C. Ottem kindly communicated to us that he knew about the existence of such rays in some particular cases, using completely different methods from ours presented here. For instance, for 12 points, he could construct wonderful rays orthogonal to K_{12} (so De Fernex positive) on the blowup of the dual Hesse configuration, exploiting the fact [Tot10; RU15] that it is the minimal desingularization of a quotient of a well-understood Abelian surface. As far as we know, general constructions for arbitrary s have not been known. In this paper, we fill this gap by proving the following:

THEOREM 1. *For all $s \geq 10$, wonderful rays R exist. For all $s \geq 13$, wonderful rays R with $R \cdot K_s > 0$ exist. For $s = 14$, for all $s \geq 13$ such that $s - 4$ is a square, and for all $s \geq 18$ such that $s - 2$ is a square, De Fernex negative wonderful rays exist.*

Our proof is explicit in that for each s, we exhibit an irrational ray with self-intersection zero, all of whose coordinates are positive, and prove that it is wonderful by proving that it is a limit of explicit good rays. For certain values of s, as stated in Theorem 1, these wonderful rays intersect the canonical divisor positively and the De Fernex ray negatively (note that all good rays known so far were isolated, and in fact the only previously known cases of nonisolated nef rays of self-intersection zero are the unpublished examples due to De Fernex and Ottem, so this is also the first example of accumulation of self-intersection zero classes on the boundary of the Mori cone with $R \cdot K_s > 0$).

Specifically, we start by exploiting an infinite sequence of Cremona transformations, which we then apply to a carefully chosen good ray, and show that the limit ray exists and is irrational. Since Cremona transformations preserve the "goodness" of the rays, the limit is therefore a limit of good rays and is therefore wonderful. The rays obtained in this way are orthogonal to K_s.

To produce wonderful rays that meet K_s positively and are De Fernex-negative, we exploit a degeneration technique (described in Section 1), which allows us to coherently reduce K_s-positive and De Fernex-negative systems to the systems generating the sequence of good rays found in the first step above. The limit of these systems then provides the examples that prove Theorem 1.

The construction that we present and develop is a general technique for iteratively generating good rays, based on a judicious use of degeneration procedures and Cremona transformations. This technique can be applied more extensively to produce additional wonderful rays, some of which have additional properties; we are developing this. In particular, using this same strategy, we can get many other irrational points on the boundary of the Mori cone, though we do not present them in this paper. Our approach to the results of this paper (namely using Cremona transformations to get wonderful rays orthogonal to the canonical system and then applying the *uncollision technology* that we present in Section 1) actually can be shown to produce all the wonderful rays of the unpublished De Fernex paper [Fer] that are orthogonal to the canonical system. However, we will not dwell on this here.

The linear systems we consider will have points with at most three distinct multiplicities, that is, they will be of the form $\mathcal{L}_d(a^{s_a}, b^{s_b}, c^{s_c})$. This is only a technical device to simplify computations; clearly, many other examples can be found with more different multiplicities.

There is a close relationship between our results here and the possible existence of irrational Seshadri constants. Namely, there are many rays of self-intersection zero (like the Nagata ray), which, if proven to be wonderful, would imply that some Seshadri constant at a blowup of $\mathbb{P}^2$ is irrational. The wonderful rays presented here are not among these; however, it is possible that our techniques can be used to show the existence of such rays. (See [HR08], [C+13], [D+16], and [HH18] for more detail.)

In the first two sections, we present the technical tools used to obtain wonderful rays. In the last three, we prove separately the three existence claims of our main theorem.

1. Collision of r^2 Points

A key part of the construction involves a degeneration of a linear system $\mathcal{L}_d(m_1, \ldots, m_s)$ where r^2 of the s points, of equal multiplicity m, come together; following Évain [Eva99], who treated particular cases of the situation in the 1990s, we call this a *collision*. We will index the points so that the first r^2 multiplicities are all equal to m. The analysis of this situation for $r = 2$ was developed in [CM05]; here we need to allow any integer $r \geq 2$, but the same technique, combined with Nagata's [Nag59] result on square numbers of points, suffices for our purposes.

We consider the trivial family $\mathcal{X} = X_{s-r^2} \times \Delta$ over a disc Δ and blow up a general point in the central fiber over $0 \in \Delta$ to obtain the threefold $\mathcal{X}'$. This produces a degeneration of X_{s-r^2} to a union of two surfaces, a plane (the exceptional divisor for the blowup) and the proper transform F of the original X_{s-r^2} fiber, which is now isomorphic to X_{s-r^2+1}. These two surfaces intersect transversely along a smooth rational curve R, which is a line in the plane and a (-1)-curve in F.

We now choose r^2 general points on the plane, extend these r^2 general points to the general fiber using r^2 sections of the projection of $\mathcal{X}'$ to Δ, and blow up those r^2 sections to ruled surfaces $\mathcal{E}_1, \ldots, \mathcal{E}_{r^2}$. This then produces a threefold $\mathcal{Y}$, which is a degeneration of X_s, to a union of a surface $P \cong X_{r^2}$ and $F \cong X_{s-r^2+1}$, intersecting transversely along the double curve R. This smooth rational curve R is the pullback of a general line in the surface P and remains a (-1)-curve in the surface F.

We have the line bundle corresponding to $\mathcal{L}_d(m_{r^2+1}, \ldots, m_s)$ on X_{s-r^2} and can extend it trivially to $\mathcal{X}$. If we pull that back to the first blowup $\mathcal{X}'$, then we see that this restricts to the bundle corresponding to $\mathcal{L}_d(0, m_{r^2+1}, \ldots, m_s)$ on the surface F and to the trivial bundle on the plane. We then pull that back to the second blowup $\mathcal{Y}$ and tensor by $\mathcal{O}_{\mathcal{Y}}(-tP - m\sum_{i=1}^{r^2} \mathcal{E}_i)$ with a nonnegative integer t (called the *twisting parameter*). This produces a line bundle $\mathcal{M}$ on $\mathcal{Y}$, which restricts to the general fiber in a bundle whose associated linear system is the original system $\mathcal{L}_d(m_1, \ldots, m_s)$.

The restriction of $\mathcal{M}$ to P is a bundle with associated linear system $\mathcal{L}_t(m^{r^2})$, and the restriction of $\mathcal{M}$ to F is a bundle with associated linear system $\mathcal{L}_d(t, m_{r^2+1}, \ldots, m_s)$.

At this point, we choose the twisting parameter t to be the minimum t such that the linear system $\mathcal{L}_t(m^{r^2})$ is effective. By Nagata's theorem, if $r > 3$, then $t > rm$, and it is not hard to see that if $r \leq 3$, then $t = rm$. The principle of semicontinuity guarantees that the dimension of the general linear system $\mathcal{L}_d(m_1, \ldots, m_s)$ is at most equal to the dimension of the linear system on the reducible surface $P + F$. The linear system on the reducible surface corresponds to the fiber product of the space of sections on P with the space of sections on F, fibered over the restriction to the space of sections on R.

Let us investigate this in more detail in the cases $r = 2, 3$; we do not need a precise description if $r > 3$. We note that since we are taking the minimum

twist parameter t to make the system on P effective, the restriction map from the space of sections on P to the space of sections on the double curve R (which is $H^0(\mathcal{O}_R(t))$) is injective, onto a subspace $W \subset H^0(R, \mathcal{O}_R(t))$ of codimension c. Hence the fiber product is isomorphic to the subspace of $H_F = H^0(F, \mathcal{O}_F(dH - tR - \sum_{i=r^2+1}^s m_i E_i)$, which restricts to elements of W. By the generality of the points chosen on P the subspace W is transversal to the full restriction (see [CM98, Section 3]), and so the desired subspace also has codimension c in H_F.

If $r = 2$, then the linear system on P is $\mathcal{L}_{2m}(m^4)$, whose elements are sums of m conics in the system $\mathcal{L}_2(1^4)$. This system restricts to R in a linear system on R, which is not complete (being of degree $2m$ and dimension m); it has codimension m. This imposes the additional matching condition on the linear system on $F = X_{s-3}$: it must restrict to divisors on R, which are members of the linear system coming from P. We see then that the fiber product has codimension m in the system on F; these are the additional "matching" conditions to be imposed on the system on F (over and above the point of multiplicity $t = 2m$).

If $r = 3$, then the linear system on P is $\mathcal{L}_{3m}(m^9)$, which has a unique effective element, the unique cubic through the 9 general points, taken with multiplicity m. This cubic meets the double curve R in three (general) points and on the other component X_{s-8} imposes three points of certain multiplicity on the double curve. We do not need to explicitly determine the multiplicity of these additional points; for our purposes, it suffices to observe that the restricted system on R has degree $3m$ and dimension 0, which means that the restriction subspace W has codimension $3m$. Hence the number of additional "matching" conditions imposed on $F = X_{s-8}$ is at least $3m$ (over and above the point of multiplicity $t = 3m$).

In particular, whenever the linear system on F (with the matching conditions) is empty, the collision shows that the original system was empty. It will be useful for us to proceed in reverse: from a system on the plane with $s - r^2 + 1$ points, whose dimension is known, to another system of the same degree with s points obtained replacing the first point of multiplicity rm by r^2 points of multiplicity m. We call this an *uncollision step*.

We summarize this in the following:

LEMMA 2. *Fix $r \geq 2$, $s \geq r^2 + 1$, and multiplicities $m, m_{r^2+1}, \ldots, m_s$.*

(a) *If $r = 2$ and $\dim \mathcal{L}_d(2m, m_5, \ldots, m_s) < m$, then $\mathcal{L}_d(m^4, m_5, \ldots, m_s)$ is empty.*

(b) *If $r = 3$ and $\dim \mathcal{L}_d(3m, m_{10}, \ldots, m_s) < 3m$, then $\mathcal{L}_d(m^9, m_{10}, \ldots, m_s)$ is empty.*

(c) *If $r \geq 4$ and $\mathcal{L}_d(rm, m_{r^2+1}, \ldots, m_s)$ is empty, then the uncollided system $\mathcal{L}_d(m^{r^2}, m_{r^2+1}, \ldots, m_s)$ is also empty.*

Three comments are in order. First, the above reductions are sharp for $r = 2$ but not for $r = 3$ or $r \geq 4$: there are additional matching conditions for a curve in

the degenerate surface $P + F$ to be a limit of a curve on the general fiber. In particular, as mentioned above, by Nagata's theorem, if $r \geq 4$, then the minimum t such that the linear system $\mathcal{L}_t(m^{r^2})$ is effective satisfies $t > rm$, so it would in fact suffice that $\mathcal{L}_d(rm + 1, m_{r^2+1}, \ldots, m_s)$ is empty to conclude that $\mathcal{L}_d(m^{r^2}, m_{r^2+1}, \ldots, m_s)$ is also empty.

Second, note that for all r, the systems before and after the collision have the same degree and self-intersection. In particular, if one is of self-intersection zero, so is the other; this will be important in our application.

Third, the process of considering an "uncollision" behaves well with taking limits of rays. Given a linear system $\mathcal{L}$, and an index i denoting one of the multiplicities, we may define the *uncollision* $\mathrm{Uncoll}_r(\mathcal{L}, i)$ as the system replacing the ith multiplicity m_i by r^2 points of multiplicity m_i/r. For instance,

$$\mathrm{Uncoll}_r(\mathcal{L}_d(rm, m_{r^2+1}, \ldots, m_s), 1) = \mathcal{L}_d(m^{r^2}, m_{r^2+1}, \ldots, m_s).$$

This makes sense at the level of linear systems if m_i is divisible by r, but also makes sense as elements of N, and additionally makes sense for rays in N. In particular, if $\mathcal{L}_k$ is a sequence of linear systems giving rise to rays $[\mathcal{L}_k] \in N$, then

$$\lim_{k \to \infty} [\mathrm{Uncoll}_r(\mathcal{L}_k, i)] = \left[\mathrm{Uncoll}_r \left(\lim_{k \to \infty} (\mathcal{L}_k), i \right) \right] \tag{1}$$

as rays in N. We finally note that the uncollision process is given by rational parameters and so preserves rationality and irrationality of rays.

2. Useful Cremona Transformations

In this section, we explore some Cremona maps that act on linear systems of the form $\mathcal{L}_d(a^{s_a}, b^{s_b}, c^{s_c})$.

Consider the Cremona–Kantor (CK) group $\mathcal{G}_s$ generated by quadratic transformations based at s general points $p_1, \ldots, p_s$ of the plane and by permutations of these points (see [Du 36; Alb02], [Dol12, Chapter 7]). The group $\mathcal{G}_s$ acts on the set of linear systems of the type $\mathcal{L}_d(m_1, \ldots, m_s)$. All systems in the same (CK)-orbit (or (CK)-equivalent) have the same expected, virtual, and true dimension. A linear system $\mathcal{L}_d(m_1, \ldots, m_s)$ is *Cremona reduced* if it has minimal degree in its (CK)-orbit, and this is the case if and only if the degree is greater than or equal to the sum of the three largest multiplicities (see [Coo31, p. 402–402, Thms. 8 and 10]).

An element $\phi \in \mathcal{G}_s$, seen as a linear automorphism of $\mathrm{Pic}(X_s) = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_s$, can be specified by giving its *characteristic matrix,* that is, the matrix with respect to the standard basis $(H, E_1, \ldots, E_s)$. The *homaloidal net* of ϕ (i.e., the pullback of the net of lines by ϕ) is $\mathcal{L}_d(m_1, \ldots, m_s)$, where $(d, -m_1, \ldots, -m_s)$ is the first column of the characteristic matrix of ϕ.

EXAMPLE 3. There exist four homaloidal types with homogeneous multiplicities (see [Hud27, IV,13], [Alb02, 2.5.5], [Dol12, 7.2.2]). The simplest one corresponds to the *(standard) quadratic map* on three points with characteristic matrix

$$Q = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

Any permutation of the three rightmost columns gives rise to a *distinct* quadratic Cremona map; the given matrix is the most symmetric, and it has the property of preserving the order of repeated multiplicities; for example, the pullback by ϕ_Q of a system $\mathcal{L}_d(m, m, n)$ is $\mathcal{L}_{d'}(m', m', n')$.

Of course, the quadratic Cremona map can be applied on any subset of three points among the set of points $\{p_1, \ldots, p_s\}$ that we blow up, for any $s \geq 3$; the characteristic matrix of the corresponding element in $\mathcal{G}_s$ is obtained from Q by adding suitably many rows and columns of the identity matrix. Similarly, any Cremona map defined for X_s can be applied to $X_{s'}$ with $s' > s$ by selecting a suitable set of s points among the s', and the matrix is obtained by adding rows and columns of the identity.

The other three homogeneous homaloidal types are attributed to Sturm, Geiser, and Bertini; as in the quadratic case, for each of them, there is a unique $\phi \in \mathcal{G}_s$ with that type and of order 2. Their characteristic matrices are

$$S = \begin{pmatrix} 5 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} 8 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -2 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -2 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix},$$

and

$$B = \begin{pmatrix}
17 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
-6 & -3 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
-6 & -2 & -3 & -2 & -2 & -2 & -2 & -2 & -2 \\
-6 & -2 & -2 & -3 & -2 & -2 & -2 & -2 & -2 \\
-6 & -2 & -2 & -2 & -3 & -2 & -2 & -2 & -2 \\
-6 & -2 & -2 & -2 & -2 & -3 & -2 & -2 & -2 \\
-6 & -2 & -2 & -2 & -2 & -2 & -3 & -2 & -2 \\
-6 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & -2 \\
-6 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -3
\end{pmatrix},$$

respectively.

EXAMPLE 4. In addition to the homogeneous Cremona maps of the previous example, we will use as building blocks two families of *quasi-homogeneous* involutions. The first is the *de Jonquières map* on $2n + 1$ points with characteristic matrix

$$J_n = \begin{pmatrix}
1+n & n & 1 & 1 & \cdots & 1 \\
-n & 1-n & -1 & -1 & \cdots & -1 \\
-1 & -1 & -1 & 0 & \cdots & 0 \\
-1 & -1 & 0 & -1 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
-1 & -1 & 0 & \cdots & 0 & -1
\end{pmatrix}$$

(this matrix is not explicitly given in [Alb02, 2.6.10, 3.4.3] or [Dol12, 7.2.3], but it is easy to recover it from the fact that the de Jonquières Cremona map is the composition of n quadratic Cremona transformations based at points $\{p_1, p_{2i}, p_{2i+1}\}$ for $i = 1, \ldots, n$ and permuting some points).

The second family has an even number of base points and was described by Montesano [Mon18] (see also [Hud27, IV.13]). The characteristic matrix defining the map with $2n + 2$ base points is

$$C_n = \begin{pmatrix}
1+n^2 & -n+n^2 & n & n & \cdots & n \\
n-n^2 & 2n-n^2 & 1-n & 1-n & \cdots & 1-n \\
-n & 1-n & 0 & -1 & \cdots & -1 \\
-n & 1-n & -1 & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & -1 \\
-n & 1-n & -1 & \cdots & -1 & 0
\end{pmatrix}.$$

The reader may check that this is the result of applying a de Jonquières map based at points $p_2, p_3, \ldots, p_{2n+2}$ followed by another de Jonquières map based at $p_1, p_3, \ldots, p_{2n+2}$ (where $p_3, \ldots, p_{2n+2}$ are the simple points of each de Jonquières homaloidal net).

EXAMPLE 5. In our constructions, we will use some particular famiies of Cremona maps of infinite order, built composing maps of the previous kinds. The maps

in the first family involve $s = 2n + 7$ points; they are obtained by composing a quintic Sturm map based on points $p_{2n+2}, \ldots, p_{2n+7}$ followed by a de Jonquières map based on (disjoint) points $p_1, \ldots, p_{2n+1}$. Extending the matrices J_n and S with the suitable number of rows and columns and multiplying, we obtain the corresponding characteristic matrix

$$JS_n = \begin{pmatrix}
5+5n & n & 1 & 1 & \cdots & 1 & 2+2n & 2+2n & 2+2n & 2+2n & 2+2n & 2+2n \\
-5n & 1-n & -1 & -1 & \cdots & -1 & -2n & -2n & -2n & -2n & -2n & -2n \\
-5 & -1 & -1 & 0 & \cdots & 0 & -2 & -2 & -2 & -2 & -2 & -2 \\
-5 & -1 & 0 & -1 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-5 & -1 & 0 & \cdots & 0 & -1 & -2 & -2 & -2 & -2 & -2 & -2 \\
-2 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-2 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 \\
-2 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & 0 & -1 & -1 & -1 \\
-2 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & -1 & 0 & -1 & -1 \\
-2 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\
-2 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}.$$

In particular, the homaloidal systems of these maps have the form $\mathcal{L}_{5+5n}(5n, 5^{2n}, 2^6)$. Note that if we transform a linear system $\mathcal{L}$ whose parameters have the "shape" $\mathcal{L}_d(a, b^{2n}, c^6)$, then the result is a linear system with that same shape.

An additional useful family, involving $s = 2n + 8$ points for $n \geq 1$, can be obtained by composing an octic Geiser map based on points $p_2, \ldots, p_8$ followed by a quasi-homogeneous map with characteristic matrix C_{n+3} based on all points $p_1, \ldots, p_{2n+8}$. Extending the matrix S with the suitable number of rows and columns and multiplying, we obtain the corresponding characteristic matrix CG_n, which we omit for brevity, noting only that its homaloidal system is $\mathcal{L}_{8n^2+27n+17}(8n^2 + 19n + 6, (8n+6)^7, (8n+3)^{2n})$.

Cremona maps with characteristic matrix JS_n and CG_n will be further extensively used. For simplicity, we fix the notations ϕ_n (or simply ϕ if n is understood) for a map with characteristic matrix JS_n and ψ_n (or simply ψ if n is understood) for a map with characteristic matrix CG_n.

3. Wonderful Rays in $K_s^\perp$

Because the intersection of the Mori cone with any hyperplane of the form $d = $ constant is compact, any infinite set of rays on the Mori cone has some accumulation ray. This simple observation is sufficient to provide many interesting rays on the boundary of the Mori cone. Indeed, if $s \geq 9$, then every divisor class not multiple of K_s has an infinite orbit under the action of the Cremona group, and if R is a ray of self-intersection zero on the boundary of the Mori cone, then every ray in its orbit also has self-intersection zero and lies on the boundary of the Mori cone. As a consequence, there exist accumulation rays of self-intersection zero on the boundary of the Mori cone for every $s \geq 10$, an important fact which

has not received much attention before. A careful choice of Cremona maps allows us to obtain explicit *irrational* rays with such properties:

Proposition 6. *Let n be an integer, and consider $\alpha_n = \sqrt{n(n-1)}$ and $\beta_n = \sqrt{49n^2 - 28}$. The rays generated by*

$$W_{\text{odd}} = (5n^2 + 4n)H - n(3n + 2\alpha_n)E_1$$

$$- (3n + 2\alpha_n) \sum_{i=2}^{1+2n} E_i - n(2 + n - \alpha_n) \sum_{i=2n+2}^{2n+7} E_i$$

on X_{7+2n} if $n \geq 2$ and by

$$W_{\text{even}} = 14n(8n^2 + 27n + 16)H - 7n(n+2)(9n + \beta_n + 6)E_1$$

$$- n(21n^2 - 3n\beta_n + 126n - 2\beta_n + 84) \sum_{i=2}^{8} E_i$$

$$- 7n(9n + \beta_n + 6) \sum_{i=9}^{2n+8} E_i$$

on X_{8+2n} if $n \geq 1$ are wonderful.

Corollary 7. *For every $s \geq 10$, there exist wonderful rays on X_s.*

Remark 8. For $s = 10$, De Fernex [Fer] proved that every ray of self-intersection zero in $K_{10}^{\perp}$ is nef. It can be proved that every such ray is a limit of rays in the orbit of the nef class $\mathcal{L}_3(1^9, 0)$ under the action of the Cremona–Kantor group. Also, for other even $s > 9$, our methods do provide many other wonderful rays, some of which are simpler than W_{even} on X_s; we choose this divisor because it will be useful later on, to construct De Fernex negative wonderful rays.

Proof. Let us first tackle the odd case. We are assuming that $n \geq 2$, so α_n is irrational; moreover, the self-intersection of the given system is readily computed to be zero. We only have to show that it is nef or, equivalently, that the ray it spans is a limit of nef rays.

Let ϕ be the Cremona map with characteristic matrix JS_n given in Example 5. The image under ϕ of a linear system of the form $\mathcal{L}_d(a, b^{2n}, c^6)$ is again of that form and is computed multiplying the matrix JS_n by $(d, -a, (-b)^{2n}, (-c)^6)$; the result has the parameters $(d', -a', (-b')^{2n}, (-c')^6)$, where

$$\begin{pmatrix} d' \\ a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} 5n+5 & -n & -2n & -12n-12 \\ 5n & 1-n & -2n & -12n \\ 5 & -1 & -1 & -12 \\ 2 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix}.$$

So $\mathcal{L}_d(a, b^{2n}, c^6)$ is mapped to $\mathcal{L}_{d'}(a', (b')^{2n}, (c')^6)$. This matrix diagonalizes with eigenvalues 1 and $2n \pm \alpha_n - 1$; thus applied iteration of ϕ to a general

(d, a, b, c) converges to the eigenspace of the dominant eigenvalue $2n + \alpha_n - 1$, and this eigenspace is spanned by

$$(5n^2 + 4n, 3n^2 + 2n\alpha_n, 3n + 2\alpha_n, n(2 + n - \alpha_n)).$$

Since the vector $(1, 0, 0, 0)$ corresponding to the ample class $\mathcal{L}_1(0^{2n+7})$ does not belong to the span of the eigenspaces of eigenvalues 1 and $2n - \alpha_n - 1$, the iterated application of ϕ to $\mathcal{L}_1(0^{2n+7})$ (which gives the homaloidal systems of the powers of ϕ, obviously, nef) converges to the claimed ray.

The even case is treated similarly; it suffices to show a sequence of nef classes converging to the given ray, and these nef classes will be the homaloidal classes of the powers of a suitable Cremona map. In this case, we use the Cremona map ψ with characteristic matrix CG_n of Example 5. The image under ψ of a linear system of the form $\mathcal{L}_d(a, b^7, c^{2n})$ is $\mathcal{L}_{d'}(a', (b')^7, (c')^{2n})$, where in this case

$$\begin{pmatrix} d' \\ a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} 8n^2 + 27n + 17 & -n^2 - 5n - 6 & -21n^2 - 70n - 42 & -2n^2 - 6n \\ 8n^2 + 19n + 6 & -n^2 - 4n - 3 & -21n^2 - 49n - 14 & -2n^2 - 4n \\ 8n + 6 & -n - 2 & -21n - 15 & -2n \\ 8n + 3 & -n - 2 & -21n - 7 & -2n + 1 \end{pmatrix}$$
$$\times \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix}.$$

This matrix diagonalizes with dominant eigenvalue $(n\beta_n + 7n^2 - 2)/2$ and associated eigenspace spanned by

$$(14n(8n^2 + 27n + 16), 7n(n + 2)(9n + \beta_n + 6),$$
$$n(21n^2 - 3n\beta_n + 126n - 2\beta_n + 84), 7n(9n + \beta_n + 6)). \qquad \square$$

The reader may check that the wonderful classes of Proposition 6 are orthogonal to the canonical divisor. In fact, this will be the case for every wonderful ray constructed by iterating Cremona maps, because, on one hand, a converging sequence of such classes necessarily has increasing degrees, and, on the other hand, Cremona maps preserve the canonical class and the intersection product. This implies that $\lim(K_s \cdot (\phi^k(\mathcal{L}))/\deg(\phi^k(\mathcal{L}))) = 0$ or, in other words, the limit ray is orthogonal to the canonical class.

4. Wonderful Rays in K_s^+

By the Cone theorem the shape of the Mori cone on the half-space K_s^- of classes that negatively intersect the canonical divisor is governed by the rays generated by (-1)-curves. On the orthogonal $K_s^\perp$ hyperplane, this is no longer quite the case, as shown by the existence of wonderful rays, but these wonderful rays are very particular, as they are also limits of (-1)-rays (indeed, if C is a (-1)-curve whose class does not belong to a certain linear space, then the class of $\phi^k(C)$ converges to the wonderful ray of Proposition 6 as well). So it will be much more

compelling evidence in favor of the strong Δ-conjecture to show wonderful rays on K_s^+.

To construct a sequence of good rays converging to a wonderful ray in K_s^+, we will use again the Cremona maps ϕ and ψ introduced in Example 5 and the uncollision described in Section 1. We are guided by the commutativity of taking limits and uncollisions, as noted in (1).

REMARK 9. If W is an $\mathbb{R}$-divisor class with $W \cdot K_s = 0$ and W' is obtained from W by uncolliding a point of multiplicity $rm > 0$ to $r^2 \geq 4$ points of multiplicity m, then $W' \cdot K_{s+r^2-1} > 0$. Indeed, writing $W = dL - \sum m_i E_i$, we have $W \cdot K_s = \sum m_i - 3d$ and

$$W' \cdot K_{s+r^2-1} = \sum m_i - rm + r^2 m - 3d = W \cdot K_s + (r^2 - r)m > W \cdot K_s.$$

It would be convenient if we could simply uncollide the wonderful rays found in Section 3 and prove that those rays were also wonderful; by the above remark they would lie in $K_{s+r^2-1}^+$. However, the collision/uncollision analysis and construction are only available for actual linear systems and not for irrational rays.

Hence we must finesse this by uncolliding each (integral) linear system in the sequence and show that the limit of these uncollided systems is wonderful. Indeed, since the constructions of the uncollision are relatively simple linear transformations of the parameters, it is elementary that the limit ray of the uncollided systems will be the formal uncollision of the wonderful ray found earlier, as noted in (1).

We can codify this approach with the following:

LEMMA 10. *Suppose that $\{\mathcal{L}_k\}$ is a sequence of linear system rays such that $\mathrm{Uncoll}_r(\mathcal{L}_k, 1)$ is good for all large k and that the limit ray $R = \lim_{k \to \infty} \mathcal{L}_k$ is irrational. Then $W = \mathrm{Uncoll}_r(R, 1)$ is a wonderful ray.*

Proof. Using (1), we see that the ray W is the limit of the sequence of eventually good rays $\mathrm{Uncoll}_r(\mathcal{L}_k, 1)$, and hence W is nef. It is also irrational, since R is. $\square$

As our first application, we consider the matrix

$$A_n = \begin{pmatrix} 5n+5 & -n & -2n & -12n-12 \\ 5n & 1-n & -2n & -12n \\ 5 & -1 & -1 & -12 \\ 2 & 0 & 0 & -5 \end{pmatrix}$$

associated with the Cremona map ϕ introduced in Example 5 and define the integer numbers $d_{n,k}, a_{n,k}, b_{n,k}, c_{n,k}$ by

$$\begin{pmatrix} d_{n,k} \\ a_{n,k} \\ b_{n,k} \\ c_{n,k} \end{pmatrix} = A_n^k \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

PROPOSITION 11. *For every $n \geq 0$ and every $k \geq 0$, the linear system*

$$\mathcal{P}_{n,k} = \mathcal{L}_{d_{n,k}}(a_{n,k}, b_{n,k}^{2n}, c_{n,k}^6)$$

is a pencil of rational curves having self-intersection zero. For every $n \geq 2$ and every $k \geq 1$, the linear system

$$\mathcal{G}_{n,k} = \mathcal{L}_{2d_{n,k}}(a_{n,k}^4, 2b_{n,k}^{2n}, 2c_{n,k}^6),$$

whose self-intersection is again zero, is empty, and all its multiples $m\mathcal{G}_{n,k}$ are empty for $m \geq 1$.

Proof. The first claim is obvious, because $\mathcal{P}_{n,k} = \phi^k(\mathcal{P}_{n,0})$ is a Cremona transform of the pencil $\mathcal{P}_{n,0} = \mathcal{L}_1(1, 0^{2n+6})$ of lines through the first point.

To prove that $m\mathcal{G}_{n,k}$ is empty, we collide its four points of multiplicity $ma_{n,k}$ and apply Lemma 2(a). It suffices therefore to show that $\dim \mathcal{L}_{2md_n,k}(2ma_{n,k}, 2mb_{n,k}^{2n}, 2mc_{n,k}^6) < ma_{n,k}$. This system is exactly $2m\mathcal{P}_{n,k}$, which is composed with the pencil $\mathcal{P}_{n,k}$ and has dimension $2m$. Hence it suffices to prove that $a_{n,k} > 2$ (for $n \geq 2$ and $k \geq 1$).

The vectors $v = (1, -1, 0, -2)$ and $w = (0, -1, n, 0)$ satisfy $vA_n = v$ and $wA_n = w$, so the quantities $d_{n,k} - a_{n,k} - 2c_{n,k}$ and $nb_{n,k} - a_{n,k}$ are independent of k, and looking at the case $k = 0$, we see that they equal 0 and -1, respectively. So to see that $a_{n,k} = 1 + nb_{n,k} > 2$, it will suffice to see that $b_{n,k} > 0$ for $k > 1$ (because $n \geq 2$). On the other hand, $\mathcal{P}_{n,k}$ is a pencil, so in particular it is nef, and therefore $d_{n,k}, a_{n,k}, b_{n,k}, c_{n,k} \geq 0$; thus it only remains to prove that $b_{n,k} \neq 0$ for $k \geq 1$.

Again using that $a_{n,k-1} = 1 + nb_{n,k-1}$ and $d_{n,k-1} = a_{n,k-1} + 2c_{n,k-1} = 1 + nb_{n,k-1} + 2c_{n,k-1}$, we have

$$\begin{pmatrix} b_{n,k} \\ c_{n,k} \end{pmatrix} = \begin{pmatrix} 5 & -1 & -1 & -12 \\ 2 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} d_{n,k-1} \\ a_{n,k-1} \\ b_{n,k-1} \\ c_{n,k-1} \end{pmatrix} = \begin{pmatrix} 4n-1 & -2 \\ 2n & -1 \end{pmatrix} \begin{pmatrix} b_{n,k-1} \\ c_{n,k-1} \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

This immediately shows that $b_{n,k} - c_{n,k} = (2n-1)b_{n,k-1} - c_{n,k-1} + 2$, which allows us to prove by induction that $b_{n,k} > c_{n,k} \geq 0$. For $k = 1$, a direct computation gives $b_{n,1} = 4 > c_{n,1} = 2$. For $k > 1$, we then have

$$b_{n,k} - c_{n,k} = (2n-1)b_{n,k-1} - c_{n,k-1} + 2 > (2n-2)b_{n,k-1} + 2 > 0,$$

as needed. $\qquad\square$

We now apply Lemma 10 with $r = 2$ and $\mathcal{L}_k = \mathcal{P}_{n,k}$; it follows that the rays generated by $\mathcal{G}_{n,k} = \mathrm{Uncoll}_2(2\mathcal{P}_{n,k}, 1)$ and $\mathrm{Uncoll}_2(\mathcal{P}_{n,k}, 1)$ coincide, and the above shows that $[\mathcal{G}_{n,k}]$ is good for all $n \geq 2$ and $k \geq 1$. Since $\lim_{k \to \infty} \mathcal{P}_{n,k}$ corresponds to the eigenvector for the dominant eigenvalue for the matrix A_n, we have that this limit is W_{odd}. We conclude that the ray generated by the limit of the $\mathcal{G}_{n,k}$ is wonderful, and this limit is equal to $\mathrm{Uncoll}_2(W_{odd}, 1)$, which we denote by W_{even}^+:

Corollary 12. *Each ray spanned by the class $\mathcal{G}_{n,k}$ is good for all $n \geq 2$ and $k \geq 1$. Therefore their limit is wonderful; it is the ray spanned by*

$$W_{even}^{+} = (10n^2 + 8n)H - n(3n + 2\alpha_n) \sum_{i=1}^{4} E_i$$

$$- (6n + 4\alpha_n) \sum_{i=5}^{4+2n} E_i - 2n(2 + n - \alpha_n) \sum_{i=5+2n}^{10+2n} E_i,$$

where $\alpha_n = \sqrt{n(n-1)}$.

Computing the intersection with the De Fernex ray, we see that it is negative for $n = 2$; this gives us the following:

Corollary 13. *For every even $s \geq 14$, there exist wonderful rays R on X_s with $R \cdot K_s > 0$. For $s = 14$, there is a wonderful ray that is De Fernex negative.*

Now to take care of the odd cases, we consider the matrix

$$B_n = \begin{pmatrix} 8n^2 + 27n + 17 & -n^2 - 5n - 6 & -21n^2 - 70n - 42 & -2n^2 - 6n \\ 8n^2 + 19n + 6 & -n^2 - 4n - 3 & -21n^2 - 49n - 14 & -2n^2 - 4n \\ 8n + 6 & -n - 2 & -21n - 15 & -2n \\ 8n + 3 & -n - 2 & -21n - 7 & -2n + 1 \end{pmatrix}$$

associated with the Cremona map ψ introduced in Example 5 and define the integer numbers $d'_{n,k}, a'_{n,k}, b'_{n,k}, c'_{n,k}$ by

$$\begin{pmatrix} d'_{n,k} \\ a'_{n,k} \\ b'_{n,k} \\ c'_{n,k} \end{pmatrix} = B_n^k \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Proposition 14. *For every $n \geq 1$ and every $k \geq 0$, the linear system*

$$\mathcal{P}'_{n,k} = \mathcal{L}_{d'_{n,k}}(a'_{n,k}, (b'_{n,k})^7, (c'_{n,k})^{2n})$$

is a pencil of rational curves of self-intersection zero. For every $n \geq 1$ and every $k \geq 1$, the linear system

$$\mathcal{G}'_{n,k} = \mathcal{L}_{2d'_{n,k}}((a'_{n,k})^4, (2b'_{n,k})^7, (2c'_{n,k})^{2n})$$

has zero self-intersection and is empty, and all its multiples $m\mathcal{G}'_{n,k}$ are empty for $m \geq 1$.

Proof. The proof follows the lines of the previous proposition. The statements about $\mathcal{P}'_{n,k}$ are obvious.

To prove that $m\mathcal{G}'_{n,k}$ is empty, we collide its four points of multiplicity $ma'_{n,k}$, and using Lemma 2(a) again, we need to prove that $a'_{n,k} > 2$ for $n \geq 1$ and $k \geq 1$.

The vectors $v = (3, 0, -7, -3n - 2)$ and $w = (0, 1, 0, -n - 2)$ satisfy $vB_n = v$ and $wB_n = w$, so the quantities $3d'_{n,k} - 7b'_{n,k} - (3n+2)c'_{n,k}$ and $a'_{n,k} - (n+2)c'_{n,k}$

are independent of k, and looking at the case $k = 0$, we see that they equal 3 and 1, respectively. So to see that $a'_{n,k} = 1 + (n+2)c'_{n,k} > 2$, it will suffice to see that $c'_{n,k} > 0$ for $k > 1$. On the other hand, $\mathcal{P}'_{n,k}$ is nef, and therefore $d'_{n,k}$, $a'_{n,k}$, $b'_{n,k}$, $c'_{n,k} \geq 0$; thus it only remains to prove that $c'_{n,k} \neq 0$ for $k \geq 1$.

Again using that $d'_{n,k-1} = (7/3)b'_{n,k-1} + (3n+2)c'_{n,k-1} + 3$ and $a'_{n,k} = 1 + (n+2)c'_{n,k}$, we have

$$\begin{pmatrix} b'_{n,k} \\ c'_{n,k} \end{pmatrix} = \begin{pmatrix} -\frac{7}{3}n - 1 & 7n^2 + \frac{16}{3}n \\ -\frac{7}{3}n & 7n^2 + \frac{7}{3}n - 1 \end{pmatrix} \begin{pmatrix} b'_{n,k-1} \\ c'_{n,k-1} \end{pmatrix} + \begin{pmatrix} 7n+4 \\ 7n+1 \end{pmatrix}.$$

We will use this expression to show that $3c'_{n,k} > b'_{n,k} \geq 0$, which will finish the proof. For $k = 1$, we have $b'_{n,1} = 7n+4$ and $c'_{n,1} = 7n+1$ satisfying the inequality (because $n \geq 1$). For $k > 1$, we can argue by induction on k. Indeed, it follows from the latter matrix equality that

$$3c'_{n,k} - b'_{n,k} > \left(14n^2 + \frac{5}{3}n - 3\right)c'_{n,k-1} - \left(\frac{14}{3}n - 1\right)b'_{n,k-1}$$

$$> \left(\frac{14}{3}n - 1\right)(3c'_{n,k-1} - b'_{n,k-1}) > 0,$$

and we are done. $\qquad\square$

Again Lemma 10 applies, with $r = 2$ and $\mathcal{L}_k = \mathcal{P}'_{n,k}$; we again have that the rays generated by $\mathcal{G}'_{n,k}$ and $\mathrm{Uncoll}_2(\mathcal{P}'_{n,k}, 1)$ coincide. The above shows that $[\mathcal{G}'_{n,k}]$ is good for all $n \geq 1$ and $k \geq 1$, so that since $\lim_{k\to\infty} \mathcal{P}'_{n,k} = W_{\mathrm{even}}$, we have that the limit of $\mathcal{G}'_{n,k}$ generates a wonderful ray, and this limit is equal to $\mathrm{Uncoll}_2(W_{\mathrm{odd}}, 1)$, which we denote by W^+_{odd}:

COROLLARY 15. *Each ray spanned by the class* $\mathcal{G}'_{n,k}$ *is good for all* $n \geq 1$ *and* $k \geq 1$. *Therefore their limit is wonderful; it is the ray spanned by*

$$W^+_{\mathrm{odd}} = 28n(8n^2 + 27n + 16)H - 7n(n+2)(9n + \beta_n + 6)\sum_{i=1}^{4} E_i$$

$$- 2n(21n^2 - 3n\beta_n + 126n - 2\beta_n + 84)\sum_{i=5}^{11} E_i$$

$$- 14n(9n + \beta_n + 6)\sum_{12}^{11+2n} E_i,$$

where $\beta_n = \sqrt{49n^2 - 28}$.

A computation shows that it is De Fernex negative for $n = 1$. Hence we have the following:

COROLLARY 16. *For every odd* $s \geq 13$, *there exist wonderful rays* R *on* X_s *with* $R \cdot K_s > 0$. *For* $s = 13$, *there is a wonderful ray that is De Fernex negative.*

5. De Fernex Negative Wonderful Rays

In the previous section, we used uncollision of a point to four points on a wonderful ray to obtain a wonderful ray on K_s^+; in the initial cases of each sequence ($n = 2$ in Corollary 12 or $n = 1$ in Corollary 15), the resulting ray is De Fernex negative, but for larger n, the multiplicities obtained become too inhomogeneous, and the rays become De Fernex positive. This can be remedied by using an uncollision to a variable number of points to obtain infinite sequences of De Fernex negative wonderful rays at the price of covering only some particular values of s.

PROPOSITION 17. *For every $n \geq 1$ and every $k \geq 1$, the linear systems*

$$\mathcal{G}''_{n,k} = \mathcal{L}_{(n+1)d'_{n,k}}((a'_{n,k})^{(n+1)^2}, ((n+1)b'_{n,k})^7, ((n+1)c'_{n,k})^{2n}),$$

$$\mathcal{G}'''_{n,k} = \mathcal{L}_{(n+2)d'_{n,k}}((a'_{n,k})^{(n+2)^2}, ((n+2)b'_{n,k})^7, ((n+2)c'_{n,k})^{2n})$$

have zero self-intersection and are empty, and all their multiples are empty.

Proof. The claim for $\mathcal{G}''_{1,k} = \mathcal{G}'_{1,k}$ has already been proved.

For $m\mathcal{G}'''_{1,k}$ and $m\mathcal{G}''_{2,k}$, we collide the 9 points of multiplicity $ma'_{n,k}$ to a point of multiplicity $3ma'_{n,k}$ with $\alpha = 3ma'_{n,k}$ matching conditions, as explained in Section 1; the resulting system is $3m\mathcal{P}'_{n,k}$ with α additional conditions. Since we already proved that $a'_{n,k} > 2$, we get $\alpha = 3ma'_{n,k} > 6m > \dim(3m\mathcal{P}'_{n,k})$, we can apply Lemma 2(b), and we conclude that the system is empty.

For all other cases, write $r = n + 1$ (in the case of $\mathcal{G}''$) or $r = n + 2$ (in the case of $\mathcal{G}'''$) and collide the r^2 points of multiplicity $ma'_{n,k}$ to a point of multiplicity $t > rma'_{n,k}$, because $r > 3$. The resulting linear system is the subsystem of $rm\mathcal{P}'_{n,k}$ formed by the curves with a point of multiplicity t at the first point. However, curves in the rational pencil $\mathcal{P}'_{n,k}$ have multiplicity exactly $a'_{n,k}$ at the first point, so curves in $rm\mathcal{P}'_{n,k}$, being sums of rm curves in $\mathcal{P}'_{n,k}$, have multiplicity at most $rma_{n,k} < t$; we conclude that the collided system is empty, and hence by Lemma 2(c) we have the result. □

Using the notation introduced, we see that $\mathcal{G}''_{n,k} = \mathrm{Uncoll}_{n+1}(\mathcal{P}'_{n,k}, 1)$ and $\mathcal{G}'''_{n,k} = \mathrm{Uncoll}_{n+2}(\mathcal{P}'_{n,k}, 1)$ as rays. The above shows that these rays are good. The limit of the $\mathcal{P}'_{n,k}$ systems is the irrational ray W_{even}. Hence using Lemma 10, we conclude that the two formal uncollisions of the limit of the W_{even} rays are wonderful. We denote these two uncollisions by W_{sq4}^+ and W_{sq2}^+, and so we have the following:

COROLLARY 18. *The rays spanned by the classes $\mathcal{G}''_{n,k}$ and $\mathcal{G}'''_{n,k}$ are good for every $n \geq 1$, $k \geq 1$. Therefore their limits as $k \to \infty$ are wonderful; these are the rays*

spanned by

$$W_{sq4}^+ = 14n(8n^2 + 27n + 16)L - 7n\frac{n+2}{n+1}(9n + \beta_n + 6)\sum_{i=1}^{(n+1)^2} E_i$$

$$- n(21n^2 - 3n\beta_n + 126n - 2\beta_n + 84)\sum_{i=(n+1)^2+1}^{(n+1)^2+7} E_i$$

$$- 7n(9n + \beta_n + 6)\sum_{(n+1)^2+8}^{(n+2)^2+4} E_i$$

and

$$W_{sq2}^+ = 14n(8n^2 + 27n + 16)L - 7n(9n + \beta_n + 6)\sum_{i=1}^{(n+2)^2} E_i$$

$$- n(21n^2 - 3n\beta_n + 126n - 2\beta_n + 84)\sum_{i=(n+2)^2+1}^{(n+2)^2+7} E_i$$

$$- 7n(9n + \beta_n + 6)\sum_{(n+2)^2+8}^{(n+3)^2+2} E_i,$$

respectively, where $\beta_n = \sqrt{49n^2 - 28}$.

Note that the first $(n + 2)^2$ and the last $2n$ points have the same multiplicity in W_{sq2}^+, so this class is a permutation of

$$W_{sq2}^+ = 14n(8n^2 + 27n + 16)L - n(21n^2 - 3n\beta_n + 126n - 2\beta_n + 84)\sum_{i=1}^{7} E_i$$

$$- 7n(9n + \beta_n + 6)\sum_{8}^{(n+3)^2+2} E_i.$$

We were led to these examples since in the wonderful ray W_{even}, the first multiplicity is exactly equal to $n + 2$ times the multiplicity of the last $2n$ points. This means that uncolliding that first point to a collection of $(n + 2)^2$ points will yield a wonderful ray with only two distinct multiplicities, which is more uniform. A closer examination reveals that these rays intersect the De Fernex ray negatively:

PROPOSITION 19. *The classes* W_{sq4}^+ *and* W_{sq2}^+ *are De Fernex negative.*

Proof. Since the classes W_{sq4}^+ and W_{sq2}^+ and the De Fernex rays are given explicitly in terms of n, this is essentially a calculus exercise. We indicate explicitly

how to obtain the inequality in the case of W^+_{sq2}, leaving the other class to the interested reader.

The computation of the intersection product as a function of n is straightforward and gives

$$W^+_{sq2} \cdot F_{(n+3)^2+2} = -63n^4 - 567n^3 - 1386n^2 - 756n$$
$$+ 14(8n^3 + 27n^2 + 16n)\sqrt{n^2 + 6n + 10}$$
$$- 7(n^2 + 3n + 2)\sqrt{49n^4 - 28n^2}.$$

We check by hand that this is negative for $n = 1, 2, 3, 4$. On the other hand, it is clear that $\sqrt{n^2 + 6n + 10} < n + 3 + 1/2n$ and $\sqrt{49n^4 - 28n^2} > 7n^2 - 3$ for all $n \geq 1$. Therefore

$$W^+_{sq2} \cdot F_{(n+3)^2+2} < -63n^4 - 567n^3 - 1386n^2 - 756n$$
$$+ 14(8n^3 + 27n^2 + 16n)(n + 3 + 1/2n)$$
$$- 7(n^2 + 3n + 2)(7n^2 - 3)$$
$$= 7(-7n^2 + 24n + 22),$$

which is negative for $n \geq 5$. □

These two sets of examples give the following:

COROLLARY 20. *For every $s \geq 13$ such that $s - 4$ is a square and every $s \geq 18$ such that $s - 2$ is a square, there exist De Fernex negative wonderful rays.*

This is the final ingredient in the proof of Theorem 1, which follows from Corollaries 7, 13, 16, and 20.

References

[Alb02] M. Alberich-Caramiñana, *Geometry of the plane cremona maps,* Lecture Notes in Math., 1769, Springer-Verlag, Berlin, 2002, xvi+257 pp.

[C+13]   C. Ciliberto, B. Harbourne, R. Miranda, and J. Roé, *Variations on Nagata's conjecture,* Clay Math. Proc. 18 (2013), 185–203.

[CM98]   C. Ciliberto and R. Miranda, *Degenerations of planar linear systems,* J. Reine Angew. Math. 501 (1998), 191–220.

[CM05]   ———, *Matching conditions for degenerating plane curves and applications,* Projective varieties with unexpected properties, Proceedings of the Siena conference (C. Ciliberto, A. V. Geramita, B. Harbourne, R. M. Miró-Roig, K. Ranestad, eds.), pp. 177–198, W. de Gruyter, 2005.

[Coo31]   J. L. Coolidge, *A treatise on algebraic plane curves,* Oxford University Press, 1931.

[Fer]   T. de Fernex, *On the Mori cone of blow-ups of the plane,* arXiv:1001.5243.

[Dol12] I. V. Dolgachev, *Classical algebraic geometry. A modern view,* Cambridge University Press, 2012, xii, 639 p.

[Du 36] P. Du Val, *On the Kantor group of a set of points in a plane,* Proc. Lond. Math. Soc. 42 (1936), 18–51.

[D+16]   M. Dumnicki, A. Küronya, C. McLean, and T. Szemberg, *Rationality of Seshadri constants and the Segre–Harbourne–Gimigliano–Hirschowitz conjecture,* Adv. Math. 303 (2016), 1162–1170.

[Eva99] L. Évain, *La fonction de Hilbert de la réunion de 4h gros points génériques de $\mathbb{P}^2$ de même multiplicité,* J. Algebraic Geom. 8 (1999), no. 4, 787–796.

[HH18]   K. Hanumanthu and B. Harbourne, *Single point Seshadri constants on rational surfaces,* J. Algebra 499 (2018), 37–42.

[HR08]   B. Harbourne and J. Roé, *Discrete behavior of Seshadri constants on surfaces,* J. Pure Appl. Algebra 212 (2008), no. 3, 616–627.

[Hud27] H. P. Hudson, *Cremona transformations in plane and space,* Cambridge University Press, 1927. xx+454 pp.

[Mon18] D. Montesano, *Su le corrispondenze birazionali piane emisimmetriche,* Rend. della R. Acc. delle Scienze Fisiche e Mat. di Napoli, (3) 24 (1918), 31–78.

[Nag59] M. Nagata, *On the fourteenth problem of Hilbert,* Amer. J. Math. 81 (1959), 766–772.

[RU15]   X. Roulleau and G. Urzúa, *Chern slopes of simply connected complex surfaces of general type are dense in* [2, 3]*,* Ann. of Math. (2) 182 (2015), no. 1, 287–306.

[Tot10] B. Totaro, *The cone conjecture for Calabi–Yau pairs in dimension 2,* Duke Math. J. 154 (2010), no. 2, 241–263.

C. Ciliberto   R. Miranda
Dipartimento di Matematica   Department of Mathematics
Università di Roma Tor Vergata   Colorado State University
Via O. Raimondo   Fort Collins (CO), 80523
00173 Roma   USA
Italia

cilibert@axp.mat.uniroma2.it   rick.miranda@colostate.edu

J. Roé
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona)
Catalunya

jroe@mat.uab.cat