

EMPTINESS OF HOMOGENEOUS LINEAR SYSTEMS WITH TEN GENERAL BASE POINTS

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ABSTRACT. In this paper we give a new proof of the fact that for all pairs of positive integers (d, m) with $d/m < 117/37$, the linear system of plane curves of degree d with ten general base points of multiplicity m is empty.

INTRODUCTION

We will denote by $\mathcal{L}_d(m_1^{s_1}, \dots, m_n^{s_n})$ the linear system of plane curves of degree d having multiplicities at least m_i at s_i fixed points, $i = 1, \dots, n$. The points in question may be proper or infinitely near, but often we will assume them to be general. In the *homogeneous case*, he *expected dimension* of the linear system $\mathcal{L}_d(m^n)$ is

$$e(\mathcal{L}_d(m^n)) = \max\left\{-1, \frac{d(d+3)}{2} - \frac{nm(m+1)}{2}\right\}.$$

Nagata's conjecture for ten general points states that if $\frac{d}{m} < \sqrt{10} \approx 3.1622$ then $\mathcal{L}_d(m^{10})$ is empty. Harbourne and Roé [7] proved that if $\frac{d}{m} < 177/56 \approx 3.071$ then $\mathcal{L}_d(m^{10})$ is empty. Then Dumnicki [5] (see also [1]), combining various techniques, among which methods developed by Ciliberto–Miranda [2] and Harbourne–Roé, found a better bound $313/99 \approx 3.161616$. The aim of this paper is to develop a general degeneration technique for analysing the emptiness of $\mathcal{L}_d(m^n)$ for general points, and we demonstrate it here in the case $n = 10$. This technique is based on the *blow-up and twist method* introduced in this setting by Ciliberto and Miranda in [2]. Using this, and precisely exploiting a suitable degeneration of the plane blown up at ten general points into a union of nine surfaces, we prove that $\mathcal{L}_d(m^{10})$ is empty if $\frac{d}{m} < \frac{117}{37} \approx 3.162162$. Using the same degeneration Ciliberto and Miranda recently proved in [4] the non-speciality of $\mathcal{L}_d(m^{10})$ for $\frac{d}{m} \geq \frac{174}{55}$ and, as remarked in that article, one obtains as a consequence the emptiness of $\mathcal{L}_d(m^{10})$ for $\frac{d}{m} < \frac{550}{174} \approx 3.1609$. Our emptiness result implies that the 10–point Seshadri constant of the plane is at least $117/370$ (see [7]). Recently T. Eckl [6] also obtained the same bound. Using the methods developed in [4] he constructs a more complicated degeneration of the plane into 17 surfaces to find the bound $370/117$ for asymptotic non–speciality of $\mathcal{L}_d(m^{10})$. As proved in [4] this is equivalent to saying that the Seshadri constant has to be at least $117/370$, which is the same conclusion we obtain here with considerably less effort.

The present paper has to be considered as a continuation of [4], which the interested reader is encouraged to consult for details on which we do not dwell here. From [4] we will take the general setting and most of the notation. Indeed, the degeneration we use here has been introduced in [4], §9. It is a family parametrized by a disk whose general member X_t is a plane blown up at ten general points, whereas the central fibre X_0 is a local normal crossings union of nine surfaces. This construction is briefly reviewed in §1.

A *limit line bundle* on X_0 is the datum of a line bundle on the normalization of each component, verifying *matching conditions*, i.e. the line bundles have to agree on the double curves of X_0 . In order to analyse the emptiness of $\mathcal{L}_d(m^{10})$ in the asserted range, we use the concept of *central effectivity* introduced in [4], §10.1. A line bundle \mathcal{L}_0 on X_0 is *centrally effective* if a general section of \mathcal{L}_0 does not vanish identically on any irreducible component of X_0 . In

particular, if \mathcal{L}_0 is centrally effective then its restriction to each component of X_0 is effective. If $\mathcal{L}_d(m^{10})$ is not empty, then there is a line bundle \mathcal{L} on the total space X of the family with a non-zero section s vanishing on a surface whose restriction to the general fiber X_t is a curve in $\mathcal{L}_d(m^{10})$. Then there is a limit curve in the central fiber X_0 as well, hence there is a limit line bundle \mathcal{L}_0 associated to that curve. The bundle \mathcal{L}_0 , which is the restriction to X_0 of \mathcal{L} twisted by multiples of the components of X_0 where s vanishes, is centrally effective. In conclusion, if $\mathcal{L}_d(m^{10}) \neq \emptyset$ then there is a limit line bundle which is centrally effective. Conversely if for fixed d and m no limit line bundle \mathcal{L}_0 is centrally effective, e.g. if its restriction to some component of X_0 is not effective, then we conclude that $\mathcal{L}_d(m^{10}) = \emptyset$.

In this article we will exploit this argument. We will describe in §3 limit line bundles \mathcal{L}_0 of the line bundle $\mathcal{L}_d(m^{10})$. We will see that, in order to apply the central effectivity argument, we can restrict our attention to some *extremal* limit line bundles, and verify central effectivity properties only for them. In §3 we will prove that $\mathcal{L}_d(m^{10})$ with general base points is empty if $\frac{d}{m} < \frac{117}{37}$, by showing that none of the extremal limit line bundles verifies the required central effective properties.

1. THE DEGENERATION

Consider $X \rightarrow \Delta$ the family obtained by blowing up a point in the central fiber of the trivial family over a disc $\Delta \times \mathbb{P}^2 \rightarrow \Delta$. The general fibre X_t for $t \neq 0$ is a \mathbb{P}^2 , and the central fibre X_0 is the union of two surfaces $V \cup Z$, where $V \cong \mathbb{P}^2$, $Z \cong \mathbb{F}_1$, and V and Z meet along a rational curve E which is the (-1) -curve on Z and a line on V (see Figure 1 in [4]).

Choose four general points on V and six general points on Z . Consider these as limits of ten general points in the general fibre X_t and blow them up in the family X (we abuse notation and denote by X also the new family). This creates ten exceptional surfaces whose intersection with each fiber X_t is a (-1) -curve, the exceptional curve for the blow-up of that point. The general fibre X_t of the new family is a plane blown up at ten general points. The central fibre X_0 is the union of V_1 a plane blown up at four general points, and Z_1 a plane blown up at seven general points (see Figure 2 in [4]). This is the first degeneration in [4], §3.

We will briefly recall the notion of a 2-throw as described in [4], §4.2. Consider a degeneration of surfaces containing two components V and Z , transversely meeting along a double curve R . Let E be a (-1) -curve on V intersecting R transversely twice. Blow it up in the total space. This creates a ruled surface $T \cong \mathbb{F}_1$ meeting V along E ; the double curve $V \cap T$ is the negative section of T . The surface Z is blown up twice, with two exceptional divisors G_1 and G_2 . Now blow up E again, creating a double surface $S \cong \mathbb{F}_0$ in the central fibre meeting V along E and T along the negative section. The blow-up affects Z , by creating two more exceptional divisors F_1 and F_2 which are (-1) curves, while G_1 and G_2 become (-2) -curves. Blowing S down by the other ruling contracts E on the surface V ; R becomes a nodal curve, and T changes into a plane \mathbb{P}^2 (see Figure 3 in [4]). In this process Z becomes non-normal, since we glue F_1 and F_2 . However, in order to analyse divisors and line bundles on the resulting surface we will always refer to its normalization Z .

On Z we introduced two pairs of infinitely near points p_i, q_i , corresponding to the (-1) -cycles $F_i + G_i$ and F_i , $i = 1, 2$. Given a linear system \mathcal{L} on Z , denote by \mathcal{L} also its pull-back on the blow-up and consider the linear system $\mathcal{L}(-a(F_i + G_i) - bF_i)$. We will say that this system is obtained by imposing to \mathcal{L} a *point of type* $[a, b]$ at p_i, q_i .

The above discussion is general; we now apply it to the degeneration $V_1 \cup Z_1$ described above. Perform the sequence of 2-throws along the following (-1) -curves:

- (1) The cubic $\mathcal{L}_3(2, 1^6)$ on Z_1 . This creates the second degeneration in [4], §6 (see Figure 5 there). Note that V_1 becomes a 8-fold blow up of the plane: it started as a 4-fold blow up and it acquires two more pairs of infinitely near (-1) -curves.

- (2) Six disjoint curves, i.e. two conics $C_1 = \mathcal{L}_2(1^4, [1, 0], [0, 0])$, $C_2 = \mathcal{L}_2(1^4, [0, 0], [1, 0])$ and four quartics $Q_j = \mathcal{L}_4(2^3, 1, [1, 1]^2)$ on V_1 (the multiplicity one proper point is located at the i -th point of the four we blew up on V). Throwing the conics creates the third degeneration in [4], §7 (see Figure 5 there), and further throwing the quartics creates the fourth degeneration in [4], §9 (see Figure 7 there).

By executing all these 2–throws we introduce seven new surfaces T , U_i , $i = 1, 2$ (denoted by T_4 , $U_{i,4}$, $i = 1, 2$ in [4]) and Y_j , $j = 1, \dots, 4$. They are all projective planes, except T , which is however a plane at the second degeneration level. Moreover, we have the proper transforms V and Z of V_1 and Z_1 (denoted V_4 and Z_4 in [4]). Throwing the two conics C_i both Z_1 and the plane corresponding to T undergo four blow–ups, two of them infinitely near. By throwing the four quartics Q_j , V_1 becomes more complicated with 16 additional blow ups, in eight pairs of infinitely near points.

2. THE LIMIT LINE BUNDLES

Next we describe the limit line bundles of $\mathcal{L}_d(m^{10})$. Their restrictions to the components of the central fibre will in general be of the form

$$\mathcal{L}_Z = \mathcal{L}_{d_Z}(\mu, q^6, [x_i, x'_i]_{i=1,2}), \quad \mathcal{L}_V = \mathcal{L}_{d_V}(\nu^4, [y, y']^2, [z_i, z'_i]_{i=1,\dots,4}^2)$$

$$\mathcal{L}_T = \mathcal{L}_{d_T}([x_i, x'_i]_{i=1,2}), \quad \mathcal{L}_{U_i} = \mathcal{L}_{s_i}, i = 1, 2, \quad \mathcal{L}_{Y_i} = \mathcal{L}_{t_i}, i = 1, \dots, 4$$

where the parameters $d_Z, \mu, q, x_i, x'_i, \dots$ etc. are integers. Note that in \mathcal{L}_Z and \mathcal{L}_V the points are no longer in general position, since they have to respect constraints dictated by the 2–throws.

The matching conditions involving the U_i 's and the Y_i 's, imply $s_i = x_i - x'_i$, $i = 1, 2$, and $t_i = z_i - z'_i$, $i = 1, \dots, 4$. Next we have to impose the remaining matching conditions and also the conditions that this is a limit line bundle of $\mathcal{L}_d(m^{10})$, i.e. conditions telling us that the total degree of the limit bundle is d and the multiplicity at the original blown up points is m . This would give us the form of all possible limits line bundles of $\mathcal{L}_d(m^{10})$, that we need in order to apply the central effectivity argument. However we can simplify our task, by making the following remark.

Let us go back to the 2–throw construction. Let \mathcal{L} be an effective line bundle on the total space of the original degeneration such that $\mathcal{L} \cdot E = -\sigma < 0$. Assume $\sigma = 2h$ is even (this will be no restriction in our setting). Create the two exceptional surfaces S and T and still denote by \mathcal{L} the pull–back of the line bundle on the new total space. In order to make it centrally effective we have to twist it to $\mathcal{L}(-uT - (u+v)S)$, and central effectivity requires $u \geq h$, $u \geq v \geq 0$ and $u+v \geq 2h$ (see [3], §2). The main remark is that in our setting we may assume $u+v = 2h$ by replacing (u, v) with (u', v') where $u' = \min\{u, 2h\}$, $v' = 2h - u'$. Indeed, $u+v > 2h$ means subtracting E more than $2h$ times from \mathcal{L}_V , and creating points of type $[u, v]$ rather than $[u', v']$ for \mathcal{L}_Z . In both cases, this imposes more conditions on the two systems. This is clear for \mathcal{L}_V . As for \mathcal{L}_Z , this follows from $u(F_i + G_i) + vF_i \geq u'(F_i + G_i) + v'F_i$, $i = 1, 2$. Therefore if one is able to prove that either one of the two systems on V and Z is empty, the central effectivity argument will certainly apply to the original twist $\mathcal{L}(-uT - (u+v)S)$. Note that $u+v = 2h$ is equivalent to require that $\mathcal{L}(-uT - (u+v)S) \cdot E = 0$. Essentially the same argument shows that we can also assume that $(u, v) = (h, h)$.

The above discussion shows that, in particular, we may assume $x_i = x'_i$, $i = 1, 2$, $y = y'$, and $z_i = z'_i$, $i = 1, \dots, 4$, with the further conditions that the restrictions to the the 2–thrown curves have degree 0. We call *extremal* the bundles verifying these conditions. If, for given d and m , for all extremal limit line bundles either \mathcal{L}_Z or \mathcal{L}_V are empty, then there is no centrally effective limit line bundle and therefore $\mathcal{L}_d(m^{10})$ is empty for general points.

For an extremal bundle, matching between V and T says that $d_T = 2x_1 = 2x_2$. So we set $x_1 = x_2 = x$. The multiplicity conditions for the general points on V then read

$$m = \nu + 4x + 2z_i + 4 \sum_{j \neq i} z_j, \quad i = 1, \dots, 4$$

yielding $z_1 = \dots = z_4$, which we denote by z . Thus we have eight parameters $d_V, d_Z, \nu, \mu, q, x, y, z$ subject to the following seven linear equations

$$3d_Z - 2\mu - 6q = 2d_V - 4\nu - y = 4d_V - 7\nu - 4y = 0$$

$$m = \nu + 4x + 14z = q + 2x + 16z + 2y, \quad d = d_Z + 6y + 48z + 6x, \quad d_V - 4y = \mu - 4x.$$

The first three come from the zero restriction conditions to the 2-thrown curves, the next two from the multiplicity m conditions on V and Z , the next one from the degree d condition, the last from the matching between V and Z .

Set $\alpha = d - 3m$ and $\ell = 19m - 6d$. By solving the above linear system, we find

$$d_Z = 10\alpha - 6a, \quad \mu = 6\alpha - 3a, \quad q = 3\alpha - 2a, \quad x = 5m - \frac{3}{2}d - a$$

$$d_V = 9a - 18\ell, \quad \nu = 4a - 8\ell, \quad y = 2a - 4\ell, \quad z = \frac{\ell}{2}.$$

The solutions, as natural, depend on a parameter $a \in \mathbb{Z}$ (which is the one introduced in the first degeneration in [4]). They are integers since we may assume d and m to be even.

In conclusion we proved:

Proposition 2.1. *In the above degeneration, the extremal limit line bundles \mathcal{L} of $\mathcal{L}_d(m^{10})$ with general base points restrict to the components of the central fibre X_0 as follows*

$$\mathcal{L}_Z = \mathcal{L}_{10\alpha-6a}(6\alpha - 3a, (3\alpha - 2a)^6, [5m - \frac{3}{2}d - a, 5m - \frac{3}{2}d - a]^2)$$

$$\mathcal{L}_V = \mathcal{L}_{9a-18\ell}((4a - 8\ell)^4, [2a - 4\ell, 2a - 4\ell]^2, [\frac{\ell}{2}, \frac{\ell}{2}]^8)$$

$$\mathcal{L}_T = \mathcal{L}_{10m-3d-2a}([5m - \frac{3}{2}d - a, 5m - \frac{2}{3}d - a]^2), \quad \mathcal{L}_{U_i} = \mathcal{L}_0, i = 1, 2, \quad \mathcal{L}_{Y_i} = \mathcal{L}_0, i = 1, \dots, 4.$$

If for all $a \in \mathbb{Z}$ either \mathcal{L}_Z or \mathcal{L}_V is empty, then no limit line bundle of $\mathcal{L}_d(m^{10})$ on X_0 is centrally effective, hence $\mathcal{L}_d(m^{10})$ is empty.

Remark 2.2. As in [4], it is convenient to consider Cremona equivalent models of the linear systems \mathcal{L}_V and \mathcal{L}_Z appearing in Proposition 2.1.

The system \mathcal{L}_V is Cremona equivalent to $\mathcal{L}_{a-2\ell}([\frac{\ell}{2}, \frac{\ell}{2}]^8)$. The position of the eight infinitely near singular points is special: there are two conics Γ_1, Γ_2 intersecting at four distinct points (the contraction of the four quartics), and each of them contains four of the infinitely near points. The conics Γ_1, Γ_2 are the proper transforms of F_1, F_2 . For all this, see [4], Lemma 9.1.

The system \mathcal{L}_Z is Cremona equivalent to $\mathcal{L}_{76d-240m-3a}((13d - 41m - a)^6, (\frac{69}{2}d - 109m - a)^4)$. This reduction follows by Lemma 9.2 of [4], but one has to apply a further quadratic transformation based at the three points of multiplicity $\alpha - \ell - a$ of the system there.

3. PROOF OF THE THEOREM

We can now prove our result:

Theorem 3.1. *If $\frac{d}{m} < \frac{117}{37}$ then the linear system $\mathcal{L}_d(m^{10})$ with ten general base points is empty.*

Proof. Fix d, m and assume $\mathcal{L}_d(m^{10}) \neq \emptyset$. According to Proposition 2.1, there is an integer a such that both \mathcal{L}_V and \mathcal{L}_Z are not empty.

Look at the system \mathcal{L}_V , or rather at its Cremona equivalent form $\mathcal{L}_{a-2\ell}([\frac{\ell}{2}, \frac{\ell}{2}]^8)$ (see Remark 2.2). Consider the curve $\Gamma = \Gamma_1 + \Gamma_2$, i.e. the union of the two conics on which the infinitely near base points are located. Blow up these base points. By abusing notation we still denote by Γ and \mathcal{L}_V the proper transform of curve and system. Then Γ is a 1-connected curve and $\Gamma^2 = 0$. Since \mathcal{L}_V is effective, one has $\mathcal{L}_V \cdot \Gamma \geq 0$, i.e. $a \geq 4\ell$.

Consider then \mathcal{L}_Z , with its Cremona equivalent form $\mathcal{L}_{76d-240m-3a}((13d-41m-a)^6, (\frac{69}{2}d-109m-1)^4)$. Since this is effective, we have $76d-240m \geq 3a \geq 12\ell$, yielding $\frac{d}{m} \geq \frac{117}{37}$. \square

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