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On (i)-Curves in Blowups of $\mathbb{P}^r$

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Abstract: In this paper, we study (i)-curves with $i \in \{-1, 0, 1\}$ in the blown-up projective space $\mathbb{P}^r$ in general points. The notion of (-1)-curves was analyzed in the early days of mirror symmetry by Kontsevich, with the motivation of counting curves on a Calabi–Yau threefold. In dimension two, Nagata studied planar (-1)-curves in order to construct a counterexample to Hilbert's 14th problem. We introduce the notion of classes of (0)- and (1)-curves in $\mathbb{P}^r$ with s points blown up, and we prove that their number is finite if and only if the space is a Mori Dream Space. We further introduce a bilinear form on a space of curves and a unique symmetric Weyl-invariant class, F (which we will refer to as the *anticanonical curve class*). For Mori Dream Spaces, we prove that (-1)-curves can be defined arithmetically by the linear and quadratic invariants determined by the bilinear form. Moreover, (0)- and (1)-Weyl lines give the extremal rays for the cone of movable curves in $\mathbb{P}^r$ with $r+3$ points blown up. As an application, we use the technique of movable curves to reprove that if $F^2 \leq 0$ then Y is not a Mori Dream Space, and we propose to apply this technique to other spaces.

Keywords: projective geometry; birational geometry; curves; rational curves

MSC: 14E07; 14E30

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1. Introduction

1.1. Historical Background

The concept of (-1)-curves on a complex threefold was introduced and studied by Clemens [1], Friedman [2] (Section 8), and Kontsevich [3] (Section 1.4) and [4] (Section 2.3) as a *smooth rational curve with normal bundle isomorphic to* $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The interplay between mathematics and physics, in the early days of Gromov–Witten theory, and the role of (-1)-curves on a 3-dimensional Calabi–Yau is largely exposed in [5]. The connection to enumerative geometry started from the influential paper [6] via the count of rational curves on a quintic threefold by solving Picard–Fuchs equations on its mirror pair. While developing the theory of mirror symmetry, in [3] Kontsevich predicts that there are infinitely many (-1)-curves on a Calabi–Yau threefold with prescribed numbers in each degree.

In this paper, we study a natural generalization of this concept to higher-dimensional varieties and to other normal bundles; in particular, we make the following definition.

Definition 1. *Fix $i \in \{-1, 0, 1\}$. An (i)-curve on a smooth r-dimensional variety X is a smooth irreducible rational curve with normal bundle isomorphic to $\mathcal{O}(i)^{\oplus(r-1)}$.*

We focus on the case of Y^r_s, which is the blowup of $\mathbb{P}^r$ at s general points, which are referred to as the *base points*. For this, we are motivated by understanding the structure of the set of classes of (i)-curves in the Chow ring, in hopes of obtaining numerical criteria. In the two-dimensional case, this is closely related to finite generation for the Cox ring of Y^2_s, whose study was initiated by Nagata ([7,8]) to understand the finite generation of rings of invariants and Hilbert's 14th problem. Nagata's counterexample directly used the infinity of (-1)-curves and their classes to prove that for $s = 16$ (later shown for $s \geq 9$) the corresponding Cox ring (isomorphic to the relevant ring of invariants) was not finitely generated, see [9] (Theorem 2a); his pioneering work contributed to the development of birational geometry. Nagata's correspondence between *planar* (-1)-*curves and* (-1)-*Weyl lines* plays a key role there, and also in this work.

This theme of research continued in Mori's work on establishing the Minimal Model Program, and led to the identification of *Mori Dream Spaces*: those for which the Cox ring is finitely generated.

Questions regarding Mori Dream Spaces were analyzed before via the theory of divisors, more precisely via the generators of the Cox Ring. In this paper we use the notions of rigid and movable curves in $\mathbb{P}^r$, and we use it to answer similar questions for Y^r_s, the blow-up of $\mathbb{P}^r$ in s points. We believe that the theory of curves developed in this paper may be applied to open problems in $\overline{M}_{0,n}$.

Rationality questions for varieties of arbitrary dimension are difficult. In recent work, ref. [10] introduces a new invariant to study rationality for smooth connected algebraic varieties via equivariant birational theory, emphasizing that modern techniques of rationality arguments will involve Gromov-Witten theory. We hope that understanding (i)-curves will prove useful in questions regarding rationality.

1.2. Main Results

This paper is organized as follows.

In Section 2, we present the main elements needed from the Chow ring $A^*(Y^r_s)$ and its intersection form $(-,-)$. In Section 2.2, we give a straightforward description of the action of the standard Cremona transformations on curve classes $A^{r-1}(Y^r_s)$ in Proposition 2. Finally, in Section 2.3 we present the Coxeter group theory that results in the basic analysis of the Weyl group.

This leads us to define the bilinear form $\langle -,- \rangle$ on the curve classes $A^{r-1}(Y^r_s)$ and an anticanonical curve class $F \in A^{r-1}$ (meant to be dual in some sense to the anticanonical class $-K_Y \in A^1$), which plays a central role in the analysis. We denote the bilinear form induced by the Coxeter group theory on $A^1(Y^r_s)$ by $\langle -,- \rangle_1$ (known as the Dolgachev–Mukai pairing) and we exploit both forms systematically (For more details, see [11]).

In Section 3.1, we study general properties of (i)-curves on an arbitrary smooth variety of dimension r, before returning to focus on the Y^r_s case. We introduce the concept of (i)-*Weyl lines* and their classes to denote curves and classes in the Weyl orbit of the proper transform of a line in $\mathbb{P}^r$ through $1-i$ of the base points. We make corresponding definitions of (i)-Weyl hyperplanes and observe numerical properties for their classes as well; Corollary 4 of Section 3 proves that (i)-Weyl hyperplanes and (i)-divisorial classes are equivalent in $\mathbb{P}^r$; this remark extends Nagata's correspondence for divisors from $i = -1$ [12] to $i = 1$. Some care must be taken with respect to an assumption of irreducibility for the curves in question; Example 2 emphasizes this in the planar case.

In Section 4, we study numerical conditions that provide a useful tool to prove the finiteness of (i)-curves. In Section 6, we use the computation of Weyl actions on curves to prove that the space Y^r_s is a Mori Dream Space if and only if it has infinitely many classes of (1)-curves. Moreover, there are infinitely many (-1)-curves if the number of points is at least $r+5$. We conclude with applications of the theory of rigid and movable curves. More precisely, we identify extremal rays for the movable cone of curves by exploiting the faces of the effective cone of divisors for Mori Dream Spaces in Section 7.

We close this section with a summary of what we believe are the most important results of this article.

The classification of the Mori Dream Spaces among the spaces Y^r_s is conveniently expressed in terms of the bilinear forms $\langle -,- \rangle_1$ on A^1 and $\langle -,- \rangle$ on A^{r-1} coming from the Coxeter group approach. The existence of the bilinear form $\langle -,- \rangle$ on the curve classes gives us a linear invariant $\langle c, F \rangle$ (which equals $(c \cdot -K_Y)$) and a quadratic invariant $\langle c, c \rangle$ for curve classes c. Although the curve class cannot detect the decomposition of the normal bundle of a smooth rational curve, the linear invariant does detect its anticanonical degree, and we say that a class c is a *numerical* (i)*-class* if $\langle c, F\rangle = 2 + i(r-1)$, which is the value for the class of an (i)-curve. This linear invariant is related to the virtual dimension (namely, χ of the normal bundle) of a curve C in Y^r_s, see (10).

Theorem 1. *If $Y = Y^r_s$, then the following statements are equivalent:*

1. *Y is a Mori Dream space.*
2. *The Coxeter group (and the Weyl group) is finite.*
3. *$F^2 := \langle F, F\rangle > 0$ (which is identical to $\langle K_Y, K_Y\rangle_1 > 0$ and also $(-K_Y \cdot F) > 0$). (This is equivalent to $r = 2, s \le 8; r = 3$ or $4, s \le r+4$; or $r \ge 5, s \le r+3$).*
4. *Y has finitely many classes of (0)-curves (or equivalently, classes of (0)-Weyl lines or (0)-numerical classes).*
5. *Y has finitely many classes of (1)-curves (or equivalently, classes of (1)-Weyl lines or (1)-numerical classes).*

Corollary 2 implies the equivalence of the first three statements, while Theorem 12 and Corollary 9 imply part (4) and (5).

We conjecture (Conjecture 1) that every (i)-Weyl line is an (i)-curve, and it is easy to see that the class of an (i)-curve is a numerical (i)-class. It is natural to ask the following question: when are these notions identical? They are not equivalent in arbitrary dimension. We do prove Conjecture 1 in certain cases (for $i = 0, 1$ and for $i = -1$ if $s \le r+4$).

Example 6 gives a (-1)-curve that is not a (-1)-Weyl line in Y^3_8 (the first case in dimension at least 3 when the space Y is not a Mori Dream Space). The numerical (-1)-classes can also contain classes of curves of different genera. Indeed, the same example of the F class in Y^3_8 is the class of four (-1)-curves (four disjoint lines each through two of the 8 points), but also the class of an elliptic curve that is a complete intersection of two quadrics in $\mathbb{P}^3$. It is, however, also the class of a (-1)-curve (indeed, four disjoint ones, none of which are (-1)-Weyl lines). It is therefore remarkable that for Mori Dream Spaces, (-1) numerical classes represent just one curve in Y^r_s, which is a (-1)-curve.

The main results of Section 4 involve characterizing the (i)-curves and (i)-Weyl lines using the numerical invariants; we have the following when $i = -1$.

Theorem 2. *Assume $r \ge 3$. Suppose that C is a curve in Y with class $c \in A^{r-1}(Y)$. If Y is a Mori Dream Space or $Y = Y^5_9$, then the following are equivalent:*

1. *C is an (-1)-curve.*
2. *C is a (-1)-Weyl line.*
3. *$\langle c, c\rangle = 3 - 2r$ and $\langle c, F\rangle = 3 - r$.*
4. *$c = (1; 1^2)$ or $c = (r, 1^{r+3})$.*

To prove this result, we study the notion of numerical (-1)-classes that in dimension at least 3 for the Mori Dream Space cases are each the class of a unique (-1)-curve. In particular, we prove that in the Mori Dream Space cases there is a finite number of such curves. Theorem 2 extends to irreducible (0)-curves with two exceptions: the F class if $r = 3$ and $s = 7$ and the class $2F$ if $r = 4$ and $s = 8$ (Remark 4). Theorem 10 discusses (1)-curves in even dimensional spaces Y^r_{r+3}. Moreover, we note in particular that Y^5_9 is not an MDS and has an infinite Weyl group but has finitely many (-1)-curves, which may be surprising. In Section 6, we prove that

If $s \ge r+5$ there are infinitely many (-1)-Weyl lines (and hence (-1)-curves) on Y.

Moreover, in Section 3.5 if $\mathcal{Z}_{\ge 0}\langle -1\rangle$ denotes the cone generated by classes of (-1)-curves and classes of curves that meet all (0)-divisorial classes non-negatively, then

The cone of classes of effective curves in Y^r_s is a subcone of $\mathcal{Z}_{\ge 0}\langle -1\rangle$.

Finally, Section 7 presents applications of the theory of (i)-curves to the effective cone of divisors $\mathrm{Eff}_{\mathbb{R}}(Y)$ on $Y = Y^r_{r+3}$. The geometry of Mori Dream spaces Y was previously analyzed via the work of Mukai and techniques from the birational geometry of moduli spaces. We recall that in Y^r_{r+3}, the cone of effective divisors is closed, and a line bundle L on a projective variety is pseudo-effective (effective) if and only if $L \cdot C \geq 0$ for all irreducible curves C that move in a family covering X [13]. Theorems 15 and 16 imply the following.

Theorem 3.

1. *The extremal rays of the cone of movable curves in Y^r_{r+3} are (0)-Weyl lines and (1)-Weyl lines.*
2. *(0)-Weyl lines and (1)-Weyl lines are extremal rays for the cone of movable curves in Y^r_s for arbitrary s.*

As a corollary to part (2), the infinity of (0)- and (1)-Weyl lines give a different approach, via the theory of movable curves, of the following result of Mukai (originally proved via the theory of divisors): *If $F^2 \leq 0$, then Y^r_s is not a Mori Dream Space.* We leave as an open question to investigate if the theory of movable curves can be further applied to prove similar properties for other spaces.

2. The Chow Ring, Cremona Transformations, and the Weyl Group

Let us consider the rational variety $Y = Y^r_s$, defined as the blowup of $\mathbb{P}^r$ at s general points $p_1, \ldots, p_s$, with blowup map $\pi : Y \to \mathbb{P}^r$. Proposition 2 contains one of the main results of this paper: *the computations of the Weyl group orbits for curves* directly on the Chow ring of Y (on which the Weyl group acts naturally) without performing a sequence of flops. Proposition 1 introduces the concept of the *anticanonical curve class F*, which we will use throughout the paper.

There are two complications in using the Chow ring classes to study (i)-curves. First, there is no numerical criterion for the rationality of a curve when $r \geq 3$. (In the planar case, there is the genus formula expressing rationality in terms of the normal bundle and anticanonical degree, emphasized in Proposition 6; this was also exploited in [12] and in Section 3.4 in order to define divisorial (i)-classes).

Second, even if one knows that a given class is represented by a smooth rational curve, the normal bundle summands are difficult to compute in arbitrary dimension. In particular, for a given degree of the normal bundle of a rational curve it is not easy to describe simple sufficient conditions to make the normal bundle balanced. However, we expect that for rational curves through general points, the normal bundle should be as balanced as possible.

Example 1. *For $i \in \{-1, 0, 1\}$, we have the following examples of (i)-curves on Y:*

1. *The proper transform of a line through $1 - i$ points is an (i)-curve, if $s \geq 1 - i$.*
2. *The proper transform of a rational normal curve of degree r through $r + 2 - i$ of the points is an (i)-curve, if $s \geq r + 2 - i$. (This curve exists and is unique if $i = -1$).*

These two examples are immediate, given that the normal bundle of the line is $\mathcal{O}(1)^{\oplus(r-1)}$, the normal bundle of the rational normal curve is $\mathcal{O}(r+2)^{\oplus(r-1)}$, and upon blowing up a point the normal bundle is twisted by -1.

We make the following conjecture.

Conjecture 1. *An (i)-Weyl line (a curve in the orbit of a line through $1 - i$ points) is an (i)-curve on Y.*

We note that the conjecture is trivially true in the planar case $r = 2$; in this case, the Cremona transformation is an isomorphism. We are not able to prove this in general, but we have results in special cases, and in particular it is true for $i = 0, 1$ or with $i = -1$ and $s \leq r + 4$; see Propositions 4 and 19.

In order to generate more examples of (i)-curves, Proposition 2 gives the Weyl group orbit action on curves.

Weyl group actions on curves were considered for $r = 3$ in [14] and $r = 4$ in [15]. This action was used to study closure of the diminished base locus of divisors [16].

2.1. The Chow Ring of Y^r_s

Let H be the hyperplane class in $\mathbb{P}^r$ and E_i, $1 \leq i \leq s$, be the exceptional divisors in Y.

The generators of the Chow ring $A^*(Y)$ of Y are easy to describe; in codimension zero, we have only the class of the entire variety, and in codimension r we have only the point class. In each of the intermediate codimension j, we have the pullback (via π) of the general linear space in $\mathbb{P}^r$ of codimension j (and hence of dimension $r - j$) and, for each $i = 1, \ldots, s$, the class of the general linear space of dimension $r - j$ inside the exceptional divisor E_i.

We will only require the divisor classes (i.e., $A^1(Y)$) and the curve classes (i.e., $A^{r-1}(Y)$) in the rest of this paper. We have already introduced notation above for the generators of A^1; we will use h to denote the class of the pullback of a general line in $\mathbb{P}^r$ and e_i will be the class of the general line inside E_i for each i. The following is standard.

Proposition 1. *With the above notation, we have the following:*

(a) *The Chow group $A^0(Y)$ is one-dimensional, generated by the identity class $[Y]$.*
(b) *The Chow group $A^1(Y)$ has dimension $s + 1$, generated by H and E_i, $1 \leq i \leq s$.*
(c) *The Chow group $A^{r-1}(Y)$ has dimension $s + 1$, generated by h and e_i, $1 \leq i \leq s$.*
(d) *The Chow group $A^r(Y)$ is one-dimensional, generated by the class $[p]$ of a point p.*
(e) *Multiplication in $A(Y)$ is induced from the intersection form on Y, and we have that the pairing $(- \cdot -) : A^1 \times A^{r-1} \to A^r$ is given by*

$$(H \cdot h) = 1; \;\; (H \cdot e_i) = 0; \;\; (E_i \cdot h) = 0; \;\; (E_i \cdot e_j) = -\delta_{ij}$$

where we have abbreviated the multiples of the point class $n[p]$ simply by the integer n.

(f) *The canonical class of Y in $A^1(Y)$ is given by*

$$K_Y = -(r+1)H + (r-1)\sum_{i=1}^{s} E_i$$

and we define the anticanonical curve class $F_Y \in A^{r-1}(Y)$ to be

$$F_Y = (r+1)h - \sum_{i=1}^{s} e_i.$$

Proof. Statements (a–e) are standard facts concerning the Chow rings of blowups, see [17] (Chapter 13). The Chern classes of the tangent bundle for a blowup is a computation using the normal bundle sequence for the exceptional divisors E_i; since on E_i, we have the exact sequence

$$0 \longrightarrow T_{E_i} \longrightarrow T_Y|_{E_i} \longrightarrow N_{E_i/Y} \longrightarrow 0$$

we have that $c(T_Y|_{E_i}) = (1+\alpha)^r(1-\alpha)$, where $\alpha \in A^1(E_i)$ is the class of the hyperplane in E_i. Now, the ring homomorphism $\pi^* : A(\mathbb{P}^r) \to A(Y)$ simply inserts the hyperplane class h into $A(Y)$, and the tangent bundle is isomorphic away from the exceptional divisors, so that (by symmetry) $c_j(T_Y) = \pi^*(c_j(T_{\mathbb{P}^r})) - t_j \sum_j e_{i,j}$ for some integer t_j. The Euler sequence for $\mathbb{P}^r$ gives that $c_j(T_{\mathbb{P}^r}) = \binom{r+1}{j} h_j$. If $f : E_i \to Y$ is the inclusion map, then the projection formula gives that

$$f_*(c_j(T_Y|E_i) \cdot \alpha^{r-1-j}) = c_j(T_Y) \cdot f_*(\alpha^{r-1-j})$$

and since α^{r-1-j} is the class of the codimension $r - 1 - j$ linear space in E_i, it has codimension $r - j$ in Y, and $f_*(\alpha^{r-1-j}) = e_{i,r-j}$. Hence, the RHS of the above is equal to

$-t_j(e_{i,j} \cdot e_{i,r-j}) = t_j e_{i,r} = t_j[p]$. Since $c_j(T_Y|E_i)$ is the α^j term of $(1+\alpha)^r(1-\alpha)$, this is equal to $(\binom{r}{j} - \binom{r}{j-1})\alpha^j$; hence, the LHS of the above is $f_*([\binom{r}{j} - \binom{r}{j-1})]\alpha^{r-1}) = [\binom{r}{j} - \binom{r}{j-1})]e_{i,r}$. Hence, as noted in (f), $t_j = [\binom{r}{j} - \binom{r}{j-1})]$. □

If $\bar{C} \subset \mathbb{P}^r$ is a curve of degree d with multiplicity m_i at p_i for each i, and $C \subset Y$ is the proper transform in Y, then the class $[C] \in A^{r-1}(Y)$ is $[C] = dh - \sum_i m_i e_i$ which can be easily deduced by intersecting with the basis elements of $A^1(Y)$. Similarly for divisors, if D is a divisor on $\mathbb{P}^r$ with multiplicity m_i at p_i for each i, then the proper transform of D in Y has the class $dH - \sum_i m_i E_i$. With this notation, the intersection pairing between divisors and curves can be written as

$$(dH - \sum_i m_i E_i \cdot d'h - \sum_i m'_i e_i) = dd' - \sum_i m_i m'_i.$$

2.2. The Standard Cremona Transformations

The theory in the planar case is well understood and, beyond the planar case, Weyl orbits of curves for $r = 3$ were computed by [14], and by the two authors for $r = 4$ in [15]. The method used in these papers is more difficult as it involves tracing these curves through a sequence of flops of lines and planes, and the difficulty increases significantly for arbitrary dimension r. We will exploit the standard Cremona transformation of $\mathbb{P}^r$ (centered at $r+1$ points), which we describe below. The most important result of this section is Proposition 2 part (a) and (c), which generalize formulas of [14,15] from dimension 3 and 4 to arbitrary dimension.

The standard Cremona transformation of $\mathbb{P}^r$ (inverting the coordinates, i.e., sending $[x_0 : \cdots : x_r]$ to $[x_0^{-1} : \cdots : x_r^{-1}]$) is realized geometrically by blowing up the $r+1$ coordinate points, then the proper transforms of all coordinate lines, then the proper transforms of all coordinate 2-planes, etc., until one blows up the proper transforms of all coordinate $(r-2)$-planes; then, one blows down the exceptional divisors starting with those over the coordinate lines, then those over the coordinate 2-planes, etc., finally ending by contracting the proper transforms of the coordinate hyperplanes. We will index the $r+1$ coordinate points used here by $\{1, \ldots, r+1\}$, considering them as the first $r+1$ of the s points to be blown up to obtain Y^r_s.

After the first stage of performing the blowups, we arrive at an r-fold $\mathbb{X}^r_{r-1}$, which has divisor classes H (the pullback of the hyperplane class on $\mathbb{P}^r$) and E_J for index sets $J \subset \{1, \ldots, r+1\}$ with $1 \le |J| \le r-1$, where E_J is obtained by blowing up the proper transform of the span of the coordinate points indexed by J under the blow up of smaller dimensional linear spans of J. The linear system that defines the Cremona transformation on $\mathbb{X}^r_{r-1}$ is given by the Cremona transformation of the hyperplane class, which is

$$H' = rH - \sum_{i=1}^{r-1}(r-i) \sum_{J;|J|=i} E_J$$

For $2 \le |J| \le r-1$, we have that the Cremona image of E_J is given by

$$E'_J = E_{J'}$$

where J' is the complement of J in $\{1, \ldots, r+1\}$. Each E_i is transformed to the coordinate hyperplane through all other indices, and so

$$E'_i = H - \sum_{J:|J| \le r-1, i \notin J} E_J$$

The subspace in the codimension-one part of the Chow ring of $\mathbb{X}^r_{r-1}$ generated by the divisor classes E_J with $2 \le |J| \le r-1$ (i.e., the exceptional divisors over the positive-dimensional linear coordinate spaces of $\mathbb{P}^r$) is invariant under the Cremona transformation, and the quotient space inherits the action. This quotient space is naturally isomorphic to $A^1(Y^r_{r+1})$, and the action extends (trivially) to an action on $A^1(Y^r_s)$ for any $s \geq r+1$. The formulas above imply that, in the Chow ring of Y^r_s, we have

$$H' = rH - (r-1)\sum_{j=1}^{r+1} E_j \quad \text{and} \quad E'_i = H - \sum_{j=1, j\neq i}^{r+1} E_j \quad \text{if} \quad i \leq r+1 \tag{1}$$

while $E'_i = E_i$ if $i > r+1$.

Since the s points are general, any set of $r+1$ of them can be the base points of a corresponding Cremona transformation. For any subset I of $r+1$ indices, we will denote by ϕ_I the corresponding Cremona transformation, which induces an action on the codimension-one Chow space $A^1(Y^r_s)$. The formulas given in (1) describe $\phi_{\{1,2,\ldots,r+1\}}$ acting on $A^1(Y^r_{r+1})$.

Dually, the subspace of the curve classes in the Chow ring of $\mathbb{X}^r_{r+1}$ spanned by the general line class and the general line classes inside of each E_i is also invariant under the Cremona action. This subspace is naturally isomorphic to $A^{r-1}(Y^r_{r+1})$, and therefore we have a Cremona action there, which extends to $A^{r-1}(Y^r_s)$.

In the next proposition, we describe these actions explicitly, and leave the details of checking the formulas to the reader.

Proposition 2. *Fix any $(r+1)$-subset $I \subset \{1, 2, \ldots, s\}$.*

(a) *The action of ϕ_I on $A^{r-1}(Y^r_s)$ is given by sending $dh - \sum_i m_i e_i$ to $d'h - \sum_i m'_i e_i$, where*

$$d' = rd - (r-1)\sum_{i\in I} m_i = d + (r-1)t_{r-1}$$
$$m'_i = d - \sum_{j\in I, j\neq i} m_j = m_i + t_{r-1} \quad \textit{for} \quad i \in I$$
$$m'_i = m_i \quad \textit{for} \quad i \notin I$$

for $t_{r-1} = d - \sum_{i\in I} m_i$. It has order two. In particular, $\phi_I(h) = rh - \sum_{i\in I} e_i$, $\phi_I(e_j) = (r-1)h - \sum_{i\in I, i\neq j} e_j$ if $j \in I$, and $\phi_I(e_j) = e_j$ if $j \notin I$.

(b) *The action of ϕ_I on $A^1(Y^r_s)$ is given by sending $dH - \sum_i m_i E_i$ to $d'H - \sum_i m'_i E_i$, where*

$$d' = rd - \sum_{j\in I} m_j = d + t_1$$
$$m'_i = (r-1)d - \sum_{j\in I, j\neq i} m_j = m_i + t_1 \quad \textit{for} \quad i \in I$$
$$m'_i = m_i \quad \textit{for} \quad i \notin I$$

for $t_1 = (r-1)d - \sum_{i\in I} m_i$. It has order two. In particular, $\phi_I(H) = rH - (r-1)\sum_{i\in I} E_i$, $\phi_I(E_j) = H - \sum_{i\in I, i\neq j} E_j$ if $j \in I$, and $\phi_I(E_j) = E_j$ if $j \notin I$.

(c) *The intersection pairing between $A^1(Y)$ and $A^{r-1}(Y)$ is ϕ_I-invariant, i.e., for any class $D \in A^1(Y)$ and $C \in A^{r-1}(Y)$, we have*

$$(D \cdot C) = (\phi_I(D) \cdot \phi_I(C)).$$

We will abbreviate $\phi = \phi_I$ for $I = \{1, 2, \ldots, r+1\}$.

2.3. The Weyl Group

In this section, Equation (4) introduces a bilinear form on A^{r-1} (and on A^1) and Corollary 2 exploits the properties of these forms in the language of Mori Dream Spaces.

Note that the symmetric group on the indices of the s points acts on all these spaces, and also preserves the intersection form pairing; moreover, if σ is the permutation taking the subset I to the subset J, then $\sigma\phi_I\sigma^{-1} = \phi_J$. The group W generated by the Cremona transformations and the symmetric group is called the *Weyl group* of these Chow spaces.

The canonical class $K_Y = -(r+1)H + (r-1)\sum_i E_i$ is the only symmetric ϕ-invariant class (up to scalars) in A^1 (and it is therefore invariant under all ϕ_I). Dually, the anticanonical curve class $F = (r+1)h - \sum_{i=1}^{s} e_i$ is the only symmetric ϕ-invariant class (up to scalars) in $A^{r-1}(Y)$, and again it is invariant under all ϕ_I.

The symmetric group is generated by the transpositions $\sigma_i = (i, i+1)$ for $1 \le i \le s-1$, and so the Weyl group generated by the Cremona transformation and the symmetric group is generated by the elements ϕ and σ_i. Each of these elements have order two. Moreover, it is easy to see that σ_i and σ_j commute if and only if $|i-j| > 1$, and ϕ and σ_i commute for all $i \neq r+1$. Finally, we have that $(\sigma_i\sigma_{i+1})^3 = 1$ for each i, and $(\phi\sigma_{r+1})^3 = 1$.

We therefore see that the Weyl group actions on A^1 and on A^{r-1} give representations of the Coxeter group associated with the $T_{2,r+1,s-r+1}$ graph:

$$\begin{array}{ccc} (1) - (2) - \cdots - (r) - & (r+1) & -(r+2) \cdots - (s-1) \\ & | & \\ & (0) & \end{array} \tag{2}$$

where the (0) vertex corresponds to ϕ and the (i) vertex corresponds to σ_i for $i \ge 1$.

We will describe this a bit further in the rest of this section, but this is relatively well known; see [18–20], for example. The relevant theory from Coxeter groups can be found in [21–23].

There are W-invariant quadratic forms on A^1 and A^{r-1}, defined by

$$q_1(dH - \sum_i m_i E_i) = (r-1)d^2 - \sum_i m_i^2 \quad \text{and} \quad q_{r-1}(dh - \sum m_i e_i) = d^2 - (r-1)\sum_i m_i^2; \tag{3}$$

We leave it to the reader to check that these are ϕ-invariant (they are clearly symmetric). These give rise to associated bilinear forms: for $\alpha = 1$ or $\alpha = r-1$, we have

$$\langle x, y\rangle_\alpha = \frac{1}{2}(q_\alpha(x+y) - q_\alpha(x) - q_\alpha(y)) \tag{4}$$

and as usual $q_\alpha(x) = \langle x, x\rangle_\alpha$ for all x and both values of α.

We usually abbreviate the bilinear form on the curve classes as simply $\langle -, -\rangle$ without the subscript.

In the divisor case, $\langle x, y\rangle_1$ is the Dolgachev–Mukai pairing that has $\langle H, H\rangle_1 = r-1$, $\langle E_i, E_i\rangle_1 = -1$, and all other values on the given basis elements for A^1 equal to zero. In the curve case, when $r = 3$ this quadratic invariant was observed and used in [14]; its formula is

$$\langle dh - \sum_i m_i e_i, d'h - \sum_i m_i' e_i\rangle = dd' - (r-1)\sum_i m_i m_i'.$$

Proposition 3. *With the above notation, we have the following:*

1. *The action of W on A^{r-1} induces an action on $K^\perp = \{c \in A^{r-1} \mid (c \cdot K) = 0\}$, which has dimension s.*
2. *For $c \in A^{r-1}$, we have $(-K \cdot c) = \langle F, c\rangle$. Hence, the orthogonal space to F (with respect to the pairing given by the bilinear form $\langle -, -\rangle$) is equal to $K^\perp$.*
3. *The action of W on $K^\perp$ is isomorphic to the standard geometric representation of the Coxeter group.*
4. *The standard bilinear form on the standard geometric representation corresponds, under this isomorphism, to $(-1/2)$ times the restriction of the $\langle -, -\rangle$ pairing on A^{r-1}.*

5. *Since the $\langle -,- \rangle$ pairing on A^{r-1} has signature $(1,s)$, the standard bilinear form of the Coxeter group is positive definite if and only if $q_{r-1}(F) > 0$; this corresponds to having $(r+1)^2 > s(r-1)$, i.e., if and only if*

$$r = 2 \text{ and } s \leq 8; \text{ or } r = 3,4 \text{ and } s \leq r+4; \text{ or } r \geq 5 \text{ and } s \leq r+3. \quad (5)$$

Since Coxeter groups are finite exactly when the standard bilinear form is positive definite, we have the following:

Corollary 1. *The Weyl group, acting on either A^1 or A^{r-1}, is finite if and only if (r,s) satisfy (5).*

The pairs (r,s) satisfying (5) exactly describe the cases when Y^r_s is a Mori Dream Space (see [24,25]). Hence:

Corollary 2. *$Y = Y^r_s$ is a Mori Dream Space if and only if the Weyl group is finite.*

It turns out that the action of W on the curve classes is isomorphic to the action on the divisor classes. Hence, all of the above statements could have been reformulated with the divisor classes as well. For example, the standard bilinear form of the Coxeter group is positive definite if and only if $q_1(K) > 0$.

The abstract group given by the graph (2) has one generator for each vertex, with relations that each generator has order two, that generators commute if they are not connected with an edge, and that the generators satisfy $(ab)^3 = 1$ if they are connected. If we set $\sigma_0 = \phi$ acting on A^1 and A^{r-1}, we see that this abstract group maps onto the Weyl group generated by ϕ and the permutations of the points by sending the generator for vertex i to σ_i for each $i = 0, \ldots, s-1$.

The general theory of such Coxeter groups now constructs, and exploits to great effect, a faithful representation of the abstract group on a real vector space E of dimension equal to the number of vertices, as follows. Take a basis vector w_i for each vertex i as a basis for the space E. Define a bilinear form $B(-,-)$ by

$$B(w_i, w_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \text{ and } j \text{ are not connected with an edge} \\ -1/2 & \text{if if } i \text{ and } j \text{ are connected with an edge.} \end{cases}$$

Define the action of the group element σ_i on E by $\sigma_i(x) = x - 2B(x, w_i)w_i$, which is a reflection. One checks that the relations in the group hold for this action, so that the action extends to the abstract Coxeter group. Moreover, for each i, σ_i preserves the bilinear form B: $B(x,y) = B(\sigma_i(x), \sigma_i(y))$, and hence every element of the group does. It is a basic theorem in the theory that this action is faithful, and gives an injective representation of the abstract Coxeter group into $GL(E)$, indeed into the orthogonal group of the bilinear form B.

This representation of the Coxeter group is called the *standard geometric representation*. A basic part of the theory is that

the Coxeter group is finite if and only if the bilinear form B is positive definite. (6)

We now relate this representation to the action on the divisor classes A^1. The ϕ-invariant symmetric anticanonical curve class $F = (r+1)h - \sum_i e_i \in A^{r-1}(Y)$ gives a functional on the divisor classes $A^1(Y)$ via the intersection pairing; the kernel of this functional is a codimension-one subspace $E^1 \subset A^1$ on which ϕ and the symmetric group act irreducibly. A divisor class $dH - \sum_i m_i E_i$ is in E^1 if and only if $(r+1)d = \sum_i m_i$.

We have $\dim(E^1) = s$, and a basis for E^1 is given by the divisor classes $\{W_i\}_{0 \leq i \leq s-1}$, where $W_0 = H - \sum_{i=1}^{r+1} E_i$ and $W_i = E_i - E_{i+1}$ for $1 \leq i \leq s-1$.

One now checks that the quadratic form q_1 on A^1 defined by

$$q_1(dH - \sum_i m_i E_i) = (r-1)d^2 - \sum_i m_i^2$$

is symmetric and invariant under the action of ϕ (and hence under all the Cremona actions ϕ_I). It gives rise to a (symmetric, ϕ-invariant) bilinear form B_1 on A^1 by setting

$$B_1(x,y) = \frac{-1}{4}(q_1(x+y) - q_1(x) - q_1(y))$$

which takes on the following values:

$$B_1(H,H) = \frac{1-r}{2}; \quad B_1(E_i,E_i) = 1/2; \quad B_1(H,E_i) = 0; \quad B_1(E_i,E_j) = 0 \text{ for } i \neq j$$

and hence has signature $(s,1)$ on A^1.

This bilinear form is a multiple of the Dolgachev–Mukai pairing on divisor classes, which we denote by $\langle\langle -,- \rangle\rangle$:

$$B_1(x,y) = -2\langle\langle x,y \rangle\rangle.$$

It has the following values on the W_i basis vectors for E^1:

$$\begin{aligned} B_1(W_i,W_i) &= 1 \text{ for all } i \\ B_1(W_0,W_i) &= \begin{cases} 0 \text{ if } i \geq 1, i \neq r+1 \\ -1/2 \text{ if } i = r+1 \end{cases} \\ B_1(W_i,W_j) &= 0 \text{ if } 1 \leq i,j \leq s-1, |i-j| > 1 \\ B_1(W_i,W_{i+1}) &= -1/2 \text{ if } i \geq 1 \end{aligned}$$

The above computation is the crucial element in showing the following, which we leave to the reader.

Theorem 4. *The linear map Ψ sending $w_i \in E$ to $W_i \in E^1$ is an isomorphism preserving the actions of the Coxeter groups and the bilinear forms*

$$\Psi(\sigma w) = \sigma\Psi(w) \text{ for all group elements } \sigma \text{ and all } w \in E, \text{and}$$

$$B_1(\Psi(x),\Psi(y)) = B(x,y) \text{ for all } x,y \in E.$$

Next, we note that E^1, defined as the set of divisor classes w such that $(w \cdot F) = 0$, is also, equivalently, the set of classes w such that $B_1(w, K_Y) = 0$, since if we write $w = dH - \sum_i E_i$, we have

$$(w \cdot F) = (dH - \sum_i m_i E_i \cdot (r+1)h - \sum_i e_i) = d(r+1) - \sum_i m_i$$

and

$$\begin{aligned} B_1(w,K_Y) &= B_1(dH - \sum_i m_i E_i, -(r+1)H + (r-1)\sum_i E_i) \\ &= -d(r+1)\frac{1-r}{2} - (r-1)\sum_i m_i(1/2) = \frac{r-1}{2}(d(r+1) - \sum_i m_i). \end{aligned}$$

Since the bilinear form B_1 has signature $(s,1)$ on A^1, and since E^1 is now seen to be the perpendicular space to K_Y (under the bilinear form B_1), the signature of the form restricted to E^1 is positive definite if and only if $B_1(K_Y, K_Y) < 0$. We have that $B_1(K_Y, K_Y) = \frac{1-r}{2}((r+1)^2 - s(r-1))$. This proves the following, by (6).

Theorem 5. *The Weyl group acting on A^1 and on A^{r-1} is finite if and only if $(r+1)^2 > s(r-1))$, i.e., if and only if*

$$r = 2, s \leq 8; r = 3, s \leq 7; r = 4, s \leq 8; r \geq 5, s \leq r+3. \tag{7}$$

Corollary 3. *$Y = Y^r_s$ is a Mori Dream Space if and only if the Weyl group acting on A^1 is finite.*

Proof. Observe that Equation (7) classifies all cases when the space Y^r_s is a Mori Dream Space. One direction follows from the work of [25]: if the Weyl group acting on A^1 is not finite, then there are infinitely many (-1)-Weyl classes that give generators for the Cox ring of Y, so Y is not a Mori Dream Space. On the other hand, if the action of the Weyl group on A^1 is finite, then the conclusion follows from [24]. □

Via the intersection pairing, the curve classes A^{r-1} form the dual space to A^1, and therefore the dual space to the subspace E^1 (defined as the perpendicular space to the anticanonical curve class F) is the quotient space $E^{r-1} = A^{r-1}/\langle F \rangle$. It is not difficult to compute that the natural action of the Coxeter group on this dual space is the given action of the Cremona transformation and the symmetric group. Indeed, one can apply this Coxeter group theory in either direction, and show instead that the standard geometric representation is isomorphic to the appropriate subspace of the curve classes and derive all the relevant statements that way.

For our purposes, we simply note that the action on the curve classes (mod F) is isomorphic to the contragradient action of the standard geometric representation. The corresponding quadratic invariant of the curve classes is

$$q_{r-1}(dh - \sum_i m_i) = d^2 - (r-1)\sum_i m_i^2.$$

and the finiteness of the Weyl group is equivalent to having $q_{r-1}(F) = (r+1)^2 - s(r-1) > 0$. This quadratic invariant on the curve classes was observed in [14] for the $r = 3$ case. We introduce the notation $\langle -, - \rangle$ to denote the bilinear form corresponding to the quadratic form q_{r-1}, so that $q_{r-1}(x) = \langle x, x \rangle$.

We have proved that a blown-up projective space is a Mori Dream Space if and only if the intersection product between the *anticanonical curve class* and *anticanonical divisor* is positive. In the cases above, which are Mori Dream Spaces, this curve class intersects the effective cone of divisors positively. Interesting geometry happens when the effective cone of divisors is tangential to this curve class. Moreover, for Mori Dream Spaces, this curve class describes the central ray of the cone of moving curves (i.e., the curve classes sweeping out the ambient space). This property is analogous to the anticanonical divisor being the central ray of the effective cone for toric varieties.

In recent preprints [26,27], we discovered that this curve class is important in the birational geometry of these spaces and that it previously appeared in particular cases in the work of Dolgachev, Casagrande [26], and the recent work of Xie (for spaces of dimension 4 and 8 general points).

A key problem in this direction would be, for a smooth algebraic variety X, to define its *anticanonical curve class*. We would of course want to prove that any new definition agrees with the one we introduce here (see also [11]) using the Coxeter theory of blown-up projective spaces.

According to Mori's theorem, on an smooth algebraic variety X, we know little about the part of the cone of curves $\overline{NE(X)}$ where the intersection with the canonical divisor K_X is non-negative, but in the complementary half-space, the cone is spanned by some countable collection of curves, which are quite special: they are rational, and their 'degree' is bounded very tightly by the dimension of X. Moreover, away from the hyperplane $\{C|C \cdot K_X = O\}$, the extremal rays of the cone cannot accumulate.

It would be extremely interesting to generalize the geometric properties of the anticanonical curve class and its interplay with the anticanonical divisor on smooth varieties. An analogue of Mori's theorem for blown-up projective spaces and for general varieties X using the anticanonical curve class definition and its consequences in birational geometry would be a future goal.

3. Basics for Curves in Y^r_s

In this section, we introduce the the main terminology for curves that we will use in this paper and we discuss general properties. In Section 3.4, we explain that Weyl group orbits on divisors can be expressed algebraically using the bilinear form $\langle , \rangle_1$ and we apply these actions to the description of the Mori cone of curves in Section 3.5.

3.1. General Facts

Assume that C is a smooth irreducible curve of genus g in a smooth variety Y of dimension r. The tangent bundle/normal bundle sequence for $C \subset Y$ gives

$$\deg c_1(N_{C/Y}) = 2g - 2 - (C \cdot K_Y). \tag{8}$$

and therefore by Riemann–Roch, we have

$$\chi(N_{C/Y}) = (r-3)(1-g) - (C \cdot K_Y). \tag{9}$$

In the case when $\deg c_1(N_{C/Y}) = i(r-1)$ (as is the case with an (i)-curve), the two equations above reduce to

$$(C \cdot K_Y) = 2g - 2 - i(r-1) \quad \text{and} \quad \chi(N_{C/Y}) = (g-1-i)(1-r)$$

which proves the following.

Lemma 1. *For a smooth curve C in Y with* $\deg(N_{C/Y}) = i(r-1)$*, the following are equivalent:*

(a) *The genus g of C is zero, so that C is a smooth rational curve.*
(b) $\chi = \chi(N_{C/Y}) = (r-1)(i+1)$.
(c) $(C \cdot K_Y) = -2 - i(r-1)$.

Without the hypothesis on the degree of the normal bundle, both (8) and (9) allow us to easily conclude the following.

Lemma 2. *Assume $r \geq 4$. For a smooth curve C, any two of the above (a), (b), and (c) imply the third. For $r = 3$, if (a) holds, then (b) and (c) are equivalent.*

The only difficulty comes in assuming (b) and (c) with $r = 3$; indeed, (b) and (c) together imply that $(r-3)g = 0$, and we have no conclusion for g.

Definition 2. *Fix $i \in \{-1, 0, 1\}$ and let $Y = Y^r_s$.*

1. *An (i)-curve in Y is a smooth rational irreducible curve whose normal bundle splits as a direct sum of $\mathcal{O}(i)$ line bundles.*
2. *A numerical (i)-curve in Y is an smooth rational curve C such that $(K_Y \cdot C) = -2 - i(r-1)$*
3. *An (i)-Weyl line is a curve that is the Cremona image (under the Weyl group) of the proper transform of a line through $1 - i$ of the s points.*
4. *An (i)-curve class is the class c of an (i)-curve.*
5. *A numerical (i)-class is a curve class $c \in A^{r-1}$ such that $(K_Y \cdot c) = -2 - i(r-1)$.*
6. *An (i)-Weyl class is a class $c \in A^{r-1}$, which is in the orbit of the class of an (i)-Weyl line (equivalently, a class of an (i)-Weyl line). Hence,*

(a) *a (-1)-Weyl class is in the orbit of $h - e_1 - e_2$;*
(b) *a (0)-Weyl class is in the orbit of $h - e_1$;*
(c) *a (1)-Weyl class is in the orbit of h.*

Every (i)-curve class is a numerical (i)-class. The numerical (i)-curves are transformed into one another via Cremona transformations; hence, so are the numerical (i)-classes.

Basic questions in this situation are whether all (i)-curves are (i)-Weyl lines, and whether all numerical (i)-curves are (i)-curves. With the curve classes, we want to investigate whether the numerical (i)-classes are all realized by classes of (i)-curves, and whether the (i)-curve classes are (i)-Weyl classes.

Motivation for these definitions may be provided by the following. Suppose C is a smooth rational curve in $Y = Y^r_s$; then, we define the virtual dimension of C to be

$$vdim(C) := (r+1)(d+1) - (r-1)\sum_{i=1}^{s} m_i - 4. \tag{10}$$

This comes from considering the parametrization of the map from $\mathbb{P}^1$ to $\mathbb{P}^r$ defining the image of C in $\mathbb{P}^r$, which is defined using $r+1$ polynomials of degree d (having then $(r+1)(d+1)$ coefficients), and noting that a multiplicity m point imposes $(r-1)m$ conditions. (The last -4 term comes from the three-dimensional family of automorphisms of $\mathbb{P}^1$ and the homogeneity in $\mathbb{P}^r$). This is just a naive dimension count, assuming that all the multiplicity conditions are independent; the actual number of parameters is at least this.

Even if this number is non-negative, one cannot conclude that an irreducible curve exists, though; the solutions to the equations may lie at the boundary of the parameter space.

Of course, this virtual number of parameters is just χ of the normal bundle: by (9), we have $\chi(N_{C/Y}) = r - 3 - (C \cdot K_Y) = r - 3 + (r+1)d - (r-1)\sum_i m_i = (r+1)(d+1) - (r-1)\sum_{i=1}^{s} m_i - 4$ as claimed.

If we assume that the number of parameters for such a curve C is non-negative (so that in general we can hope that such a rational curve C with these multiplicities is expected to exist) but that it is isolated and is a reduced point in the Hilbert scheme (it does not move in a family, even infinitesimally), then we are imposing that $H^0 = H^1 = 0$ for the normal bundle, which is equivalent to having $N_{C/Y}$ split as a direct sum of $\mathcal{O}(-1)$s, and hence C will be a (-1)-curve.

We then expect a finite number of smooth rational (-1)-curves representing a numerical (-1)-class.

The following classification enables us to prove Conjecture 1 for $i = -1$ in some special cases of interest.

Proposition 4.

(a) *If $r \geq 3$ and $s \leq r+4$, then every (-1)-Weyl line is either a line through two points or the rational normal curve of degree r through $r+3$ points.*
(b) *If $r = 2$, or $r \geq 3$ and $s \leq r+4$, then every (-1)-Weyl line is a (-1)-curve.*

Proof. To prove (a), start with the line through two points (which we may assume to be p_1 and p_2). Apply the first standard Cremona transformation by choosing $r+1$ base points. If p_1 and p_2 are both among those base points, the line is contracted, and we do not produce a (-1)-Weyl line. If only one of the two points are among those base points, the line is transformed to a line through two points. If neither of the two points are among those base points, the line is transformed to a rational normal curve through all of the $r+3$ points.

Now, analyze the rational normal curve through $r+3$ points. Again, choose $r+1$ base points and apply the standard Cremona transformation. If those $r+1$ points are a subset of the $r+3$ points on the curve, then the curve is transformed to a line through the remaining two points. If not, then since $s \leq r+4$, we must have $s = r+4$, and the $r+1$ base points consist of r points on the curve, and the extra point is not among the $r+3$ points on the curve. In that case, the curve is transformed to a rational normal curve again.

This proves (a), and statement (b) is clear for $r=2$. For $r \geq 3$, it follows by first noting that the line starts with normal bundle equal to $\mathcal{O}(1)^{\oplus(r-1)}$ in $\mathbb{P}^r$ and the rational normal curves starts with normal bundle equal to $\mathcal{O}(r+2)^{\oplus(r-1)}$ in $\mathbb{P}^r$. Since blowing up a point twists the normal bundle by $\mathcal{O}(-1)$, both become (-1)-curves in Y^r_s. □

In general, we have five concepts here, for the case $i=-1$: the Cremona lines (orbits of the lines through two points); the (-1)-curves (smooth rational curves in Y with normal bundle splitting as a sum of $\mathcal{O}(-1)$s); Weyl classes (the orbit of the line class through two points); classes of (-1)-curves; and (-1)-classes (classes c with $(c \cdot K_Y) = r-3$).

Every Cremona line is a (-1)-curve; every Weyl class and every class of a (-1)-curve is a (-1)-class. The class of a Cremona line is a Weyl class. Conversely, every Weyl class is the class of a Cremona line; the sequence of transformations that exhibit the Weyl class in the orbit of the class of the line through two points, when applied to that line, produces an irreducible curve that is the Cremona line with that Weyl class (see Corollary 8).

$$\begin{array}{ccc}
\underline{\text{Curves in } Y} & \longrightarrow & \underline{\text{Classes in } A^{r-1}(Y)} \\
\text{Cremona lines} & \longrightarrow & \text{Weyl classes} \\
\cap & & \cap \\
(-1)\text{-curves} & \longrightarrow & \text{classes of } (-1)\text{-curves} \\
& & \cap \\
& & (-1)\text{-classes}
\end{array}$$

The classes of irreducible curves have additional constraints on them, some of which are captured in the following.

Lemma 3. *Suppose $c = dh - \sum_i m_i e_i$ is the class of an irreducible curve C not inside one of the exceptional divisors E_i. Then, the following are true:*

(a) *For each i, $d \geq m_i \geq 0$;*

(b) *If $d \geq 2$, then for each $i \neq j$, $d \geq m_i + m_j$;*

(c) *If the multiplicities are in descending order $m_1 \geq m_2 \geq \ldots \geq m_s$, and if for some $k \leq r$ we have $d < \sum_{i=1}^k m_i$, then $m_{k+1} = \cdots = m_s = 0$.*

Proof. If such a curve C exists with this class, then the intersection with E_i must be non-negative, so each $m_i \geq 0$. In addition, intersecting C with divisors in the class $H - E_i$ shows that $d \geq m_i$ since $|H - E_i|$ has no base locus. This proves (a). For (b), if $d < m_i + m_j$, then the intersection of C with every hyperplane through p_i and p_j is negative, so that if C is irreducible it must be inside every such hyperplane, and therefore inside their intersection, which is the line joining the two points; hence, $d = 1$ (and c must be the class of that line). Finally, (c) is similar: if the inequality holds, then C lies inside the linear space spanned by those first k points, and so cannot have positive multiplicity at any other point outside that linear space (where all other points are by the generality). □

3.2. Projections

Projections offer an interesting perspective and tool to study these curves. If C is an irreducible curve in $\mathbb{P}^r$, whose proper transform in Y^r_s has class $c = dh - \sum_{i=1}^s m_i e_i \in A^{r-1}(Y^r_s)$, then we may consider the projection π from any one of the multiple points (say, p_1 with multiplicity m_1) and obtain the curve $\pi(C) \subset \mathbb{P}^{r-1}$, whose proper transform in Y^{r-1}_{s-1} has class $\pi(c) = (d - m_1)h - \sum_{i=2}^s m_i e_i \in A^{r-2}(Y^{r-1}_{s-1})$.

We have seen above that $\chi(N_{C/Y})$ is determined by the class of C, and it can happen that $\chi(N)$ is non-negative for C but negative for $\pi(C)$. This is an indication that, although the $\chi(N)$ computation suggests that a rational curve exists with that class, the projection is not expected to exist (and hence the original curve actually will not either).

For example, suppose c is a numerical (-1)-class so that $(r+1)d - (r-1)\sum_{i=1}^{s} m_i = 3 - r$; this is the condition that $\chi(N_{C/Y}) = 0$ if the irreducible curve C exists. In this case, $\chi(N_{\pi(C)}) = r(d - m_1) - (r-2)\sum_{i=2}^{s} -3 + (r-1)$, and one computes that

$$\chi(N_C) - \chi(N_{\pi(C)}) = d + m_1 + 1 - \sum_{i=2}^{s-1} m_s.$$

Therefore, if this quantity is positive we do not expect $\pi(C)$ to exist, and hence should not expect C to exist. We would generally apply this when m_1 is the largest multiplicity, obtaining the criterion that

$$d + m_1 \leq \sum_{i=2}^{s} m_s \tag{11}$$

for C to be expected to exist. This essentially says that, for a given degree, no single multiplicity should be too large.

It is easy to see that projections commute with the Cremona transformation.

Proposition 5. *Let $c = dh - \sum_i m_i e_i$ be a class in $A^{r-1}(Y_s^r)$. We have the Cremona transformation $\phi : A^{r-1}(Y_s^r) \to A^{r-1}(Y_s^r)$ on Y_s^r (based at the first $r+1$ points) and also the Cremona transformation $\phi : A^{r-2}(Y_{s-1}^{r-1}) \to A^{r-2}(Y_{s-1}^{r-1})$ (based at the first r points). If we denote by π the projection from the first point, we have*

$$\pi(\phi(c)) = \phi(\pi(c)).$$

Proof. Using the formulas from Proposition 2(b), we see that if we define $t = d - \sum_{i=1}^{r+1} m_i$, then

$$\phi(c) = (d+t)h - \sum_{i=1}^{r+1}(m_i + t)e_i - \sum_{i>r+1} m_i e_i,$$

and so

$$\pi(\phi(c)) = (d + t - (m_1 + t))h - \sum_{i=2}^{r+1}(m_i + t)e_i - \sum_{i>r+1} m_i e_i.$$

Now, $\pi(c) = (d - m_1)h - \sum_{i=2}^{s} m_i e_i$, so that if we define $t' = (d - m_1) - \sum_{i=2}^{r+1} m_i$, we have

$$\phi(\pi(c)) = (d - m_1 + t')h - \sum_{i=2}^{r+1}(m_i + t')e_i - \sum_{i>r+1} m_i e_i.$$

The result follows by noting that $t' = t$. □

We note that if I is any subset of $r+1$ indices for the s points of Y_s^r, and $i \in I$, then we may denote the Cremona transformation based at the points with indices in I by ϕ_I. If we denote the projection from the i-th point by π_i, then the same proof as above shows that

$$\pi_i(\phi_I(c)) = \phi_{I-\{i\}}(\pi_i(c)).$$

We present the following Lemma, which will be useful later.

Lemma 4. *Assume that $r \geq 3$ and $s \leq r+4$. Let C be a (-1)-Weyl line in Y_s^r. Since C is a (-1)-Weyl line, there is a Cremona transformation A that takes C to a line through two points. Factor A as $a_1 a_2 \ldots a_k$ minimally, so that each a_i reduces the degree. Then, there is a base point of a_k such that projecting from that point gives a (-1)-Weyl line class in Y_{s-1}^{r-1}.*

Proof. Let us argue by induction on k. If $k = 1$, then C is obtained from the line through two points by applying one standard Cremona transformation based at $r+1$ of the s points. If the two points of the line are among the $r+1$ points, we have a contradiction: the line is contracted. If exactly one of the two points of the line is among the $r+1$ points, then C

is again a line through two points, and this means that $k = 0$ not $k = 1$. Hence, we may suppose that neither of the two points of the line are among the $r + 1$ points. In this case, C is the rational normal curve of degree r in $\mathbb{P}^r$ through $r + 3$ points. Hence, the projection of C from one of those $r + 3$ points with multiplicity one is a rational normal curve of degree $r - 1$ in $\mathbb{P}^{r-1}$, as is well known.

Suppose now the statement is true for k, and let us show it for $k + 1$.

Let π be the projection from one of the base points of the first standard Cremona a_{k+1} in the factorization of A. If c is the class of C, then Proposition 5 shows that $B(\pi(c)) = \pi(a_{k+1}(c))$, where B is the standard Cremona transformation of $\mathbb{P}^{r-1}$ based at the remaining r points, and the images of the base points of a_{k+1} which are not the projection points. Since B is an involution, we therefore have $\pi(c) = B(\pi(a_{k+1}(c)))$. Now, by induction on k, the class $a_{k+1}(c)$ represents a curve D in $\mathbb{P}^r$, which satisfies the Lemma, i.e., the projection from one of the $r + 1$ base points of a_k is a (-1)-Weyl line. Since $s \leq r + 4 \leq 2r + 1$, there is at least one point of intersection between the $r + 1$ base points of a_{k+1} and of a_k. This is the point we choose to project from with the projection π.

In that case, by induction $\pi(a_{k+1}(c))$ is the class of a (-1)-Weyl line. Therefore, so is $B(\pi(a_{k+1}(c)))$. Therefore, so is $Q(c)$. ☐

3.3. The Planar Case

We now analyze (i)-curves in the planar case $Y = Y_s^2$, when $r = 2$.

We note that in this case, $A^1 = A^{r-1}$ and all three bilinear forms $(- \cdot -)$, $\langle -, - \rangle_1$, and $\langle -, - \rangle = \langle -, - \rangle_{r-1}$ are equal, and the curve class $F = -K$.

For a plane curve class c, the common invariants can be determined from the intersection form: the arithmetic genus $p_a(c) = (2 + (c \cdot c) + (c \cdot K_Y))/2$ and the Euler characteristic $\chi(c) = \chi(\mathcal{O}_Y(c)) = ((c \cdot c) - (c \cdot K_Y))/2$. With these formulas, we immediately have the following.

Proposition 6. *Let $c \in Pic(Y)$ be an arbitrary curve class and fix $i \in \mathbb{Z}$. Then, any two of the following statements imply the others:*

1. $p_a(c) = 0$
2. $\chi(c) = 2 + i$
3. $(c \cdot c) = i$
4. $(c \cdot K_Y) = -2 - i$.

A class satisfying the fourth condition is, by definition, a numerical (i)-class. If C is an irreducible curve with class c, then the first condition says that C is smooth and rational, and the third condition on the self-intersection would say that C is an (i)-curve.

However, these numerical conditions do not imply that a curve C with a class satisfying the above conditions must be irreducible.

Example 2. *Consider the planar divisor of degree* 5 *with two triple points and eight simple points. It satisfies the conditions of Proposition 6 with $i = -1$ but it is not represented by an irreducible curve: the proper transform of the line through the two triple points splits off from this system.*

Remark 1. *If $i \geq -1$, and if C is any class satisfying any two of the above equivalent conditions, with positive degree, then we have*

$$\chi(C) = h^0(C) - h^1(C) = 2 + i \geq 1$$

(because the H^2 term must vanish if C has positive degree). We conclude that any such class is effective.

The next result is the planar case of Theorem 4, whose proof is a simple consequence of the Max Noether inequality.

Theorem 6. *Let C be an irreducible curve in Y and fix $i \in \{-1, 0, 1\}$. Then, C is an (i)-curve if and only if C is an (i)-Weyl line.*

The next result shows that for planar Mori Dream Spaces the irreducibility assumption may be relaxed.

Theorem 7. *Let $s \leq 8$, so that Y^2_s is a del Pezzo surface, and fix $i \in \{-1, 0, 1\}$. Then, an effective divisor C is linearly equivalent to an (i)-curve if and only if the conditions of Proposition 6 hold, i.e.,*

$$(C \cdot C) = i \quad \text{and} \quad (C \cdot K_Y) = -2 - i. \tag{12}$$

If $i = -1$, then C is a (-1)-curve itself.

Proof. In the del Pezzo case, $-K$ is ample, and so the $(C \cdot K)$ condition says that C has anticanonical degree equal to 1, 2, or 3. If $(C \cdot -K) = 1$ (the $i = -1$ case), then C must be irreducible and smooth, and then the $(C \cdot C)$ condition implies that C is rational, and hence a (-1)-curve.

In case C has anticanonical degree greater than 1, we conclude in a similar way if C is irreducible. Hence, assume C is not irreducible, with anticanonical degree 2 or 3. In that case, C must split as $C = kG + J$, where G has anticanonical degree 1, (and hence as above is a (-1)-curve), $1 \leq k \leq 2$, and G and J are distinct. If $k = 2$, then J is a (-1)-curve as well, and in that case $(C \cdot C) = 4(G^2) + (J^2) + 4(G \cdot J) = 4(G \cdot J) - 5$, which cannot be equal to 0 or 1 as required.

We can therefore assume $k = 1$. In the case $i = 0$ with anticanonical degree 2, then $C = G + J$ with G and J distinct (-1)-curves, and the self-intersection condition implies $(G \cdot J) = 1$. In that case, C moves in a pencil whose general member is a (0)-curve.

In the case $i = 1$ with anticanonical degree 3, then J has anticanonical degree 2; by the above analysis, we may assume that J moves in a pencil whose general member is a (0)-curve. In that case, we have $1 = (C \cdot C) = (G + J \cdot G + J) = 2(G \cdot J) - 1$, which gives $(G \cdot J) = 1$, and then C moves in a linear system whose general member is an (1)-curve. □

Proposition 7. *Suppose that $i \in \{-1, 0\}$ and C is an irreducible curve in Y^2_s, whose class is a numerical (i)-class. Then, the possibilities are as follows:*

1. *($i = -1$ case): C is either a (-1)-Weyl curve or C is a cubic with class F in Y^2_8.*
2. *($i = 0$ case): C is either a (0)-Weyl curve, C is a cubic with class F in Y^2_7, or C is a sextic with class $2F$ in Y^2_8.*

Proof. Suppose that C is an irreducible curve on Y with positive degree, whose class is a numerical (i)-class, i.e., $(C \cdot K_Y) = -2 - i$. Write the class of C as $c = (d; m_1, m_2, \ldots, m_s)$ with the multiplicities in decreasing order. If $d = 1$, then $-3 + \sum_i m_i = -2 - i$ so $\sum m_i = 1 - i \leq 2$; hence, we must have an (i)-Weyl line. We may therefore assume that $d \geq 2$; if $d < m_1 + m_2 + m_3$, we may perform a Cremona transformation and reduce the degree. Hence, we may assume that $d \geq m_1 + m_2 + m_3$. In this case, the numerical condition on $(C \cdot K)$ gives $0 \leq (m_1 - m_4) + (m_2 - m_5) + (m_2 - m_6) + (m_3 - m_7) + (m_3 - m_8) + m_1 \leq 2 + i$ If $m_1 = 0$, then all $m_i = 0$ and this is impossible. If $m_1 = 1$, then assume that $m_k = 1$ but $m_{k+1} = 0$; then, the intersection with K gives that $3d - k = 2 + i$, so that the only solutions that respect the inequalities are $(1; 1)$ and $(3; 1^7)$ for $i = 0$ and $(3; 1^8)$ for $i = -1$. The case $m_1 \geq 2$ gives the unique solution $(6; 2^8)$ for $i = 0$. □

Theorem 11 of Section 6 in the planar case reproves via the theory of movable curves that if Y is not a del Pezzo surface then it is not a Mori Dream Space, because there are infinitely many (0)- and (1)-Weyl lines.

3.4. The Divisor Case

Certain algebraic equations for the Weyl orbits of hyperplanes using the Dolgachev–Mukai intersection pairing were established in [12]. In this section, Corollary 4 extends results from (-1)-Weyl hyperplanes to (1)-Weyl hyperplanes. As we will observe in this paper, (-1)-Weyl lines in higher dimensions cannot be characterized by the quadratic and linear bilinear form.

Since there is no arithmetic formula to express a rationality condition for divisors or curves in a projective space of higher dimension, in [12] the definition of divisorial classes was formulated algebraically similarly to conditions (3) and (4) of Proposition 6. We introduce below $(0)-$ and $(1)-$ Weyl hyperplanes.

Definition 3. *Let $i \in \{-1, 0, 1\}$.*

1. *An (i)-*Weyl hyperplane *is a divisor in the Weyl orbit of a hyperplane through $r-1-i$ points.*
2. *A* numerical (i)-divisorial class *is a divisor class $[D] \in A^1$ such that $\langle [D], [D] \rangle_1 = i$ and $\frac{1}{r-1}\langle [D], -K_Y \rangle_1 = 2 + i$.*

It is obvious that the class of an (i)-Weyl hyperplane is a numerical (i)-divisorial class. They are equivalent for irreducible effective divisors. This is based on the Noether inequality for divisors; for the proof of the following, see [12] (Theorem 0.4):

Proposition 8. *Assume $i \in \{-1, 0, 1\}$. Let D be an irreducible effective divisor whose class is a numerical (i)-divisorial class, and assume it is not a hyperplane through $n-1$ or $n-2$ base points. Then, the divisor D is not Cremona-reduced, i.e., if we write $[D] = dH - \sum_i m_i E_i$ and order the multiplicities in decreasing order, then $(r-1)d < \sum_{i=1}^{r+1} m_i$.*

Corollary 4. *Assume $i \in \{-1, 0, 1\}$. Let D be an irreducible effective divisor whose class is a numerical (i)-divisorial class. Then, D is an (i)-Weyl hyperplane.*

3.5. Applications to Mori Cone of Curves

Let us introduce $\mathcal{Z}_{\geq 0}$ as the cone of curve classes in $A^{r-1}(Y)$ that meet all numerical (0)-divisorial classes non-negatively. Define $\mathcal{Z}_{\geq 0}\langle -1 \rangle$ to be the cone generated by $\mathcal{Z}_{\geq 0}$ and all (-1)-curves in Y^r_s.

Then, we have the following inclusion of the Mori cone of curves on Y.

Theorem 8. *The cone of classes of effective curves in Y^r_s is a subcone of $\mathcal{Z}_{\geq 0}\langle -1 \rangle$.*

4. (i)-Curves in Mori Dream Spaces Y^r_s, $r \geq 3$

We now turn our attention to (i)-curves in Y^r_s with $r \geq 3$, with attention on the Mori Dream Space cases of (7); for $r \geq 3$, these cases are $s \leq r+4$ for $r = 3, 4$ and $s \leq r+3$ for $r \geq 5$. We note that in all these cases, Proposition 19 holds, and every (i)-Weyl line is an (i)-curve for each $i \in \{-1, 0, 1\}$.

We will assume that $s \geq r+2$; for lower values of s, there is at most one Cremona transformation, which commutes with the symmetric group, and the situation is easy to analyze.

For $s = r+2$, it is elementary to see that the only (-1)-Weyl lines are the lines through two of the points. The only (0)-Weyl lines are the lines through one of the points and the conics through all $r+2$ points; the latter only happens for $r = 2$, though, since a conic must live in a plane. The only (1)-Weyl lines are the general lines, and again conics through $r+1$ points (again, only in the case $r = 2$).

This case, $s = r+2$, is satisfactorily treated in Proposition 9, and shows that there are no numerical (i)-classes other than the ones noted above.

We will concentrate, therefore, on the cases $s \geq r+3$. We will introduce here the terminology that we will use throughout the section.

We will denote by $c = (d; m_1, \ldots, m_s)_r$ a class in A^{r-1} and we will employ exponential notation to indicate repeated multiplicities.

Definition 4. *We say that the class $c \in A^{r-1}(Y)$ with $d \geq 2$ is* Cremona-reduced *if the multiplicities are arranged in decreasing order and $d \geq m_1 + m_2 + \cdots + m_{r+1}$.*

The inequality above is the condition that no Cremona transformation centered at $r+1$ of the points will reduce the degree of the class. (In the case of the line through two points, the Cremona transformation contracts it).

We recall that the canonical divisor is fixed under the Weyl group action, and that the irreducibility and effectivity of curves of degree at least 2 are preserved under the Weyl group action. Throughout this section, we will use the following observation.

Remark 2. *If $c \in A^{r-1}$ is the class of an effective and irreducible curve C of degree at least* 2*, then we can reduce C to a Cremona-reduced effective, irreducible curve with class $f \in A^{r-1}$ and $(K_Y \cdot c) = (K_Y \cdot f)$.*

This focuses our attention on the Cremona-reduced classes. The following will be useful in our analyses.

Lemma 5. *Suppose $j \in \{-1, 0, 1\}$ and that c is a Cremona-reduced numerical (j)-class with positive degree and non-negative multiplicities. Then,*

$$(r-1) \sum_{i=r+2}^{s} m_i \geq -2 - j(r-1) + 2 \sum_{i=1}^{r+1} m_i.$$

Proof. Write $c = (d; \underline{m})$ with multiplicities in decreasing order. The numerical condition is that

$$(r+1)d - (r-1) \sum_{i=1}^{s} m_i = 2 + j(r-1)$$

and therefore

$$(r-1) \sum_{i=1}^{s} m_i = -2 - j(r-1) + (r+1)d \geq -2 - j(r-1) + (r+1) \sum_{i=1}^{r+1} m_i$$

using the Cremona reduced assumption. Subtracting $(r-1) \sum_{i=1}^{r+1} m_i$ from both sides gives

$$(r-1) \sum_{i=r+2}^{s} m_i \geq -2 - j(r-1) + 2 \sum_{i=1}^{r+1} m_i.$$

□

We remark that (1)-classes in Y^r_{r+3} consist of (1)-Weyl lines together with multiples of vector F, $(n+4)/4 \cdot F$.

4.1. The Case $s = r + 3$

We first address the case of $s = r + 3$ with $r \geq 2$. It is an easy computation to show that the only (-1)-Weyl lines in Mori Dream Spaces and Y^5_9 are the lines through two points and the rational normal curves of degree r through all $r + 3$ points (and all permutations). To see this, one computes from the bottom up: applying Cremona transformations to these classes gives these classes back, no matter which $r + 1$ points one chooses. In particular, on these spaces the (-1)-curves are the $\binom{s}{2}$ lines through two points and rational normal curves of degree r passing through $r + 3$ points. Similarly, the (0)-Weyl lines are just the lines through one of the points, and the rational normal curves through all but one of the points.

Proposition 9. *Let $j \in \{-1, 0\}$, $r \geq 3$ and $s \leq r+3$. Let C be an irreducible curve in Y^r_s. Then, $[C] \in A^{r-1}$ is a numerical (j)-class if and only if C is a (j)-Weyl line.*

Proof. Since the class of every (j)-Weyl line is a numerical (j)-class, it suffices to prove the other direction.

Let $c = [C]$ have the form $dh - \sum_i m_i e_i$; if $d = 1$, we are done. We assume that $d \geq 2$, with multiplicities ordered decreasingly.

By Remark 2, we can assume that c is Cremona-reduced, and that it satisfies the inequalities $d \geq m_i \geq 0$ since C is irreducible.

Now, Lemma 5 gives that

$$(r-1)(m_{r+2} + m_{r+3}) \geq 2\sum_{i=1}^{r+1} m_i + j(1-r) - 2 = 2\sum_{i=3}^{r+1} m_i + 2m_1 + 2m_2 + j(1-r) - 2$$

and since $\sum_{i=3}^{r+1} m_i \geq (r-1)m_{r+2} \geq (r-1)m_{r+3}$, the right-hand side of the above is at least $(r-1)(m_{r+2} + m_{r+3}) + 2m_1 + 2m_2 + i(1-r) - 2$. Subtracting $(r-1)(m_{r+2} + m_{r+3})$ from both sides now gives

$$0 \geq 2m_1 + 2m_2 + j(1-r) - 2.$$

If $j = -1$, this says that $3 - r \geq 2m_1 + 2m_2$, which implies all multiplicities are non-positive if $r \geq 3$, a contradiction if $d \geq 2$. (If $r = 2$, we must have $2m_1 + 2m_2 \leq 1$, so again we must have all multiplicities $m_i = 0$, which again gives a contradiction: we would then have to have $3d = 1$, and this is impossible).

If $j = 0$, then we have $2(m_1 + m_2) \leq 2$, and therefore $m_1 = 1$ and $m_2 = 0$. This forces $d = 1$, and we have the (0)-Weyl line. □

It is tempting to try to use this proposition to conclude that, in these ranges of parameters, all numerical (-1)-classes are (-1)-Weyl classes. However, this is false: we may indeed be able to reduce the degree, but the result may be a class whose degree becomes negative (or a multiplicity that becomes negative).

The example below shows that the irreducibility assumption is important: it ensures that Cremona transforms stay with non-negative parameters.

Example 3. *Consider the example of the class $(13; 4, 3^6)_4$ of degree 13 in $\mathbb{P}^4$ with one point of multiplicity 4 and six points of multiplicity 3. This is an effective class in $\mathbb{P}^4$ with seven points and $(-K \cdot C) = 5 \cdot 13 - 3 \cdot (4 + 3 \cdot 6) = -1$, so C is a numerical (-1)-curve. However, it is not a Weyl class because it is not irreducible. Applying a Cremona transformation to five of the points, including the multiplicity 4 point, gives the class $(4; 3^2, 1, 0^4)_4$, which cannot be the class of an irreducible curve, by Lemma 3. Indeed, any curve C with this class must contain a line through the two triple points, and the residual class is $(3; 2^2, 1, 0^4)_4$, which also must contain that line; the second residual is $(2; 1^3, 0^4)_4$, which is the class of a net of conics in the plane spanned by the three points of multiplicity one.*

We note that for the example above, $c := (13; 4, 3^6)_4$; then, $\langle c, c \rangle = -41$, which is not equal to $3 - 2r = -5$. Hence, it is not a (-1)-Weyl line class using this criterion, either.

We can replace the assumption that the class comes from an irreducible curve by using the quadratic invariant $\langle c, c \rangle$ in this case, and obtain a purely numerical condition for a class to be a (-1)-Weyl line class.

Proposition 10. *Suppose $r \geq 3$ and $s = r+3$. Let $c \in A^{r-1}$ be a class with positive degree and non-negative multiplicities. Then, c is a (-1)-Weyl line class if and only if*

$$\langle c, F \rangle = 3 - r \text{ and } \langle c, c \rangle = 3 - 2r.$$

Proof. Since the bilinear form and the class F are invariant under the Cremona transformations, one implication holds, since these are the values for a line through two points.

Therefore, it is enough to prove that a class with the given quadratic and linear invariants is a (-1) Weyl line. Let $c = (d; \underline{m})_r$ be such a class, and write $N_1 = \sum_i m_i$ and $N_2 = \sum_i m_i^2$. Our assumptions are then that $(r+1)d - (r-1)N_1 = 3 - r$ and $d^2 - (r-1)N_2 = 3 - 2r$.

We first claim that $d \le r$. For a fixed degree d, the quantity N_1 is then determined, namely $N_1 = ((r+1)d + (r-3))/(r-1)$. If all of the multiplicities are equal, then each one of them would be equal to $m = N_1/(r+3)$. The value of N_2 would be minimized if all multiplicities were equal, i.e., $N_2 \ge \sum_{i=1}^{r+3} m^2 = (r+3)(N_1/(r+3))^2 = N_1^2/(r+3)$. Therefore,

$$
\begin{aligned}
3 - 2r = d^2 - (r-1)N_2 &\le d^2 - (r-1)N_1^2/(r+3) \\
&= d^2 - \frac{r-1}{r+3}(((r+1)d + (r-3))/(r-1))^2 \\
&= d^2 - \frac{1}{(r+3)(r-1)}((r+1)^2 d^2 + 2(r+1)(r-3)d + (r-3)^2) \\
&= \frac{1}{(r+3)(r-1)}(((r+3)(r-1) - (r+1)^2)d^2 - 2(r+1)(r-3)d - (r-3)^2) \\
&= \frac{-1}{(r+3)(r-1)}(4d^2 + 2(r+1)(r-3)d + (r-3)^2).
\end{aligned}
$$

Now, the upper bound quantity here when $d = r$ is $\frac{-1}{(r+3)(r-1)}(4r^2 + 2r(r+1)(r-3) + (r-3)^2) = \frac{-1}{(r+3)(r-1)}((r-1)(r+3)(2r-3) = 3 - 2r$ and the quadratic function in d only increases for $d \ge r$, so that if $d > r$, then $\langle c, c\rangle < 3 - 2r$, which is a contradiction. Hence, we conclude that $d \le r$.

Since all the multiplicities are non-negative integers, we have $m_i \le m_i^2$ for every i, so that $N_1 \le N_2$ (with equality only if all multiplicities are 0 or 1). Since $(r-1)N_1 = (r+1)d + r - 3$ and $(r-1)N_2 = d^2 + 2r - 3$, we conclude that $(r+1)d + r - 3 \le d^2 + 2r - 3$, which simplifies to $(d-1)(d-r) \ge 0$. Hence, d cannot lie in the open interval $(1, r)$, and so the only possibilities are narrowed down to $d = r$ and $d = 1$.

If $d = r$, then the values for $\langle c, F\rangle$ and $\langle c, c\rangle$ imply that $N_1 = N_2 = r + 3$. Hence, all multiplicities are either 0 or 1; since the sum is the number of points $r + 3$, they must all be 1, and we have the class of the rational normal curve.

Similarly, if $d = 1$ we must have $N_1 = N_2 = 2$, and this leads to the line through two points. □

We can summarize the situation for irreducible (-1)- and (0)-curves when $s = r + 3$ in the following.

Theorem 9. *Let $i \in \{-1, 0\}$, $r \ge 3$, and $c \in A^{r-1}(Y^r_{r+3})$ be the class of an irreducible curve. The following are equivalent:*

1. *c is the class of a (i)-curve.*
2. *c is a (i)-Weyl line class.*
3. *$\langle c, c\rangle = 1 + (i-1)(r-1)$ and $\langle c, F\rangle = 2 + i(r-1)$.*
4. *c is a numerical (i)-class.*

Proof. The invariance of the bilinear form gives us that (2) implies (3), and we always have that either (1) or (3) implies (4). Theorem 19 proves that (2) implies (1).

Since $s = r + 3$, Proposition 9 proves that (4) implies (2), which completes the equivalencies. □

We study now (1)-curves in Mori Dream Spaces with $s = r + 3$ and we recall the anticanonical curve class $F := (r+1)h - \sum_{i=1}^{s} e_i$.

Proposition 11. *Let $r \ge 3$, $s = r + 3$, and let C be an irreducible curve in Y^r_s. Then, $c = [C] \in A^{r-1}$ is a numerical (1)-class if and only if C is a (1)-Weyl line or $2|(r+1)$ and* *c is in the Weyl orbit of a class of the form $c = mF + c'$, where $m \leq (r+1)/4$ and $c' = (e; \underline{n})$, where $e = (r+1)/2 - 2m$, $n_i = 0$ for $i \geq r-1$, and $e = \sum_i n_i$.*

Proof. We first remark that one direction is clear. Certainly, if C is a (1)-Weyl line, we have the numerical condition. Suppose c has the form $mF + c'$, as in the statement. If we write $c = (d; \underline{m})$, then $d = (r+1)m + \sum_i n_i = (r+1)m + e$ and $m_i = m + n_i$ for each i. Hence,

$$\begin{aligned}(-K \cdot c) &= (r+1)d - (r-1)\sum_i m_i \\ &= (r+1)^2 m + (r+1)\sum_i n_i - (r-1)((r+3)m + \sum_i n_i) \\ &= 4m + 2\sum_i n_i = 4m + 2e = r+1\end{aligned}$$

as required.

To prove the other direction, we assume we have the numerical condition, and that the class c is Cremona-reduced with decreasing multiplicities. If $d = 1$, then the only possibility is the class h, so we may assume $d \geq 2$, and then we must have $m_1 \geq 1$ as well.

We will prove in this case that $c = mF + c'$, as in the statement of the proposition. The condition for a numerical (1)-class is

$$\langle c, F \rangle = (-K \cdot c) = (r+1)d - (r-1)\sum_{i=1}^{r+3} m_i = r+1$$

which can be written as

$$(r-1)(d - 1 - \sum_{i=1}^{r+1} m_i) + (d - 1 - (r-1)m_{r+2}) + (d - 1 - (r-1)m_{r+3}) = 0. \quad (13)$$

Since c is Cremona-reduced, we have $d \geq \sum_{i=1}^{r+1} m_i$, which is at least $m_1 + m_2 + (r-1)m_{r+2}$ since the multipliciities decrease. Therefore, in (13), the last two parenthesis are non-negative and therefore $d - 1 - \sum_{i=1}^{r+1} m_i$ is non-positive. However, this forces $t = d - \sum_{i=1}^{r+1} m_i \in \{0, 1\}$, since c is Cremona-reduced.

We distinguish two cases:

Case $t = 1$:

Now (13) gives that all three terms are zero, and hence we must have $d = 1 + (r-1)m_{r+2} = 1 + (r-1)m_{r+3}$.

We have

$$\begin{aligned}0 = t - 1 = d - 1 - \sum_{i=1}^{r-1} m_i - (m_r + m_{r+1}) \\ \leq d - 1 - (r-1)m_{r+2} - (m_r + m_{r+1}) = -(m_r + m_{r+1})\end{aligned}$$

which forces $m_r = 0$, and hence $m_{r+2} = 0$ also, so that $d = 1$, a contradiction.

Case $t = 0$:

In this case, (13) implies

$$(\sum_{i=1}^{r+1} m_i - (r-1)m_{r+2}) + (\sum_{i=1}^{r+1} m_i - (r-1)m_{r+3}) = r+1$$

which can be reorganized as

$$\sum_{i=1}^{r-1}(2m_i - m_{r+2} - m_{r+3}) + 2m_r + 2m_{r+1} = r+1. \quad (14)$$

Since the m_is are decreasing, all the terms in the sum are non-negative and decreasing. Hence, the last one, $2m_{r-1} - m_{r+2} - m_{r+3}$, is the smallest. We distinguish three cases:

Case A: $2m_{r-1} - m_{r+2} - m_{r+3} \geq 2$. Now,

$$r + 1 = 2m_r + 2m_{r+1} + \sum_{i=1}^{r-1}(2m_i - m_{r+2} - m_{r+3}) \geq 2(r-1)$$

which forces $r = 3$, $m_r = m_{r+1} = 0$ and $2m_i - m_{r+2} - m_{r+3} = 2$ for all $i \leq r-1$. Since $m_r = 0$, so are m_{r+2} and m_{r+3}, and hence we have $m_i = 1$ for all $i \leq r-1$; all other $m_j = 0$. Now, $t = 0$ gives $d = 2$ and we have $c = (2; 1^2, 0^4)_3$, which is not a numerical (1)-class.

Case B: $2m_{r-1} - m_{r+2} - m_{r+3} = 1$. In this case, the sum in (14) is at least $r-1$, so that we must have $2m_r + 2m_{r+1} \leq 2$. This forces $m_{r+1} = 0$, and hence $m_{r+2} = m_{r+3} = 0$ as well. But then our Case B assumption gives $2m_{r-1} = 1$, an impossibility.

Case C: $2m_{r-1} - m_{r+2} - m_{r+3} = 0$. In this case, the decreasing order implies that $m_{r-1} = m_{r+2} = m_{r+3}$, so in fact $m_{r-1} = m_r = m_{r+1} = m_{r+2} = m_{r+3}$; call this value m. We then have $m_i \geq m$ for every i, so we may write $m_i = m + n_i$ for decreasing non-negative integers n_i, with $n_i = 0$ for $i \geq r-1$. Now, (14) becomes

$$\sum_{i=1}^{r-2} n_i + 2m = (r+1)/2$$

forcing r to be odd.

Since $t = 0$, we have $d = \sum_{i=1}^{r+1} m_i = (r+1)m + \sum_{i=1}^{r-2} n_i$. Hence, if we define the class $c' = (e; \underline{n})_r$ with $e = \sum_{i=1}^{r-2} n_i$, then we have $c = mF + c'$, where F is the anticanonical curve class $(r+1; 1^s)$. This is the other case of the statement. □

4.2. The Case $s = r + 4$

With $r + 4$ points, a parallel approach works up to $\mathbb{P}^5$, at least for the (-1)-curve case.

Proposition 12. *Suppose $3 \leq r \leq 5$ and $s = r + 4$. Let C be an irreducible curve in Y_s^r. Then, $[C]$ is a numerical (-1)-class if and only if C is a (-1)-Weyl line.*

Proof. It suffices to show that if c is a numerical (-1)-class of the form $dh - \sum_i m_i e_i$ with decreasing multiplicities representing an irreducible curve with $d \geq 2$, then c is not Cremona-reduced. Assume by contradiction that c is Cremona-reduced; then, Lemma 5 applies and we have

$$(r-1)(m_{r+2} + m_{r+3} + m_{r+4}) \geq (r-3) + 2\sum_{i=1}^{r+1} m_i.$$

Now, the left-hand side is at most $3(r-1)m_{r+1}$ and the right-hand side is at least $(r-3) + 2(r+1)m_{r+1}$, so that $3(r-1)m_{r+1} \geq (r-3) + 2(r+1)m_{r+1}$ or $(r-5)m_{r+1} \geq r-3$.

For $r = 4, 5$, this is an immediate contradiction, since $m_{r+1} \geq 0$. For $r = 3$, we must have $-3m_4 \geq 0$, and hence $m_4 = 0$; then, the original inequality gives $0 \geq 2(m_1 + m_2 + m_3 + m_4)$, forcing all multiplicities to be equal to zero, a contradiction. □

We cannot extend the above statement for $r > 5$:

Example 4. *Consider $\mathbb{P}^6$ with $s = 10$ points and the class $c = 3F = (21; 3^{10})_6$. This is a Cremona-reduced numerical (-1)-class, and hence is not a (-1)-Weyl line class.*

Remark 3. *For $s = r + 4$, we are in a similar situation: the only (-1)-Weyl lines are the proper transforms of the lines through two points and the proper transform of the rational normal curve of degree r through $r + 3$ of the points. There are $\binom{r+4}{2}$ such lines and $r + 4$ RNCs. However, for $r \geq 6$ there are (-1)-curves that are not (-1)-Weyl classes.*

For dimension at most 5 and $s = r+4$ points, we have the same result as for $s = r+3$, parallel to Proposition 10:

Proposition 13. *Suppose $r \leq 5$ and $s = r+4$. Let $c \in A^{r-1}$ be a class with positive degree and non-negative multiplicities. Then, c is a (-1)-Weyl line class if and only if*

$$\langle c, F\rangle = 3 - r \text{ and } \langle c, c\rangle = 3 - 2r.$$

Proof. The proof parallels that of the previous Proposition 10; it suffices to prove that a class with a given linear and quadratic invariant is a (-1)-Weyl line. We use the same notation: let $c = (d; \underline{m})_r$ be such a class, and write $N_1 = \sum_i m_i$ and $N_2 = \sum_i m_i^2$ as above.

For a fixed degree d, the quantity N_1 is again fixed to be $N_1 = ((r+1)d + (r-3))/(r-1)$. Again, N_2 would be minimized if all multiplicities were equal (to $m = N_1/(r+4)$), so that $N_2 \geq \sum_{i=1}^{r+4} m^2 = (r+4)(N_1/(r+4))^2 = N_1^2/(r+4)$. Therefore,

$$\begin{aligned}
3 - 2r = d^2 - (r-1)N_2 &\leq d^2 - (r-1)N_1^2/(r+4) \\
&= d^2 - \frac{r-1}{r+4}(((r+1)d + (r-3))/(r-1))^2 \\
&= d^2 - \frac{1}{(r+4)(r-1)}((r+1)^2 d^2 + 2(r+1)(r-3)d + (r-3)^2) \\
&= \frac{1}{(r+4)(r-1)}(((r+4)(r-1) - (r+1)^2)d^2 - 2(r+1)(r-3)d - (r-3)^2) \\
&= \frac{-1}{(r+4)(r-1)}((5-r)d^2 + 2(r+1)(r-3)d + (r-3)^2).
\end{aligned}$$

For $r = 2$, this gives $-1 \leq \frac{-1}{6}(3d^2 - 6d + 1)$ or $3d^2 - 6d - 5 \leq 0$, which forces $d \leq 2$ and leads to the line and the conic.

For $r = 3$, this gives $-3 \leq \frac{-1}{14}(2d^2)$ or $d^2 \leq 21$, giving $d \leq 4$. Now, in this case, since $-3 = d^2 - 2N_2$, the degree cannot be even. Hence, $d = 1$ or $d = 3$, leading again to the line or the twisted cubic, which is the RNC.

For $r = 4$, the inequality is $-5 \leq \frac{-1}{24}(d^2 + 10d + 1)$, forcing $d \leq 7$. However, the $\langle c, F\rangle_{r-1}$ value gives us $5d - 3N_1 = -1$, so that d must be 1 mod 3. If $d = 7$, then $N_1 = 12$ and $\frac{-1}{24}(d^2 + 10d + 1) = -5$, so that the inequalities are equalities, and all multiplicities are in fact equal to $3/2$, which is not possible. Hence, $d = 1$ or $d = 4$, and we have the line or the RNC again.

Finally, when $r = 5$ the inequality is $-7 \leq \frac{-1}{36}24d + 4$, giving $d \leq 9$, and $\langle c, F\rangle_{r-1} = -2$ forces d to be odd.

If $d = 9$, then the equations give $N_1 = 14$ and $N_2 = 22$. Again, for fixed N_1, N_2 is minimized by having all multiplicities as equal as possible, and for $r + 4 = 9$ points we have that N_2 is minimized with multiplicities $(2^5, 1^4)$ with sum equal to 14. However, for this set of multiplicities $N_2 = 24$, and so $N_2 = 22$ is impossible.

If $d = 7$, then again $N_1 = 11$ and $N_2 = 14$; for this, N_1, N_2 is minimized with multiplicities $(2^2, 1^7)$, but this gives $N_2 = 15$.

If $d = 3$, this implies $N_1 = 5$ and $N_2 = 4$, a contradiction since $N_2 \geq N_1$ always.

If $d = 5$, we have the RNC, and if $d = 1$ we have the line. □

Proposition 14. *Suppose $r \leq 5$ and $s = r+4$. Let c be a Cremona-reduced numerical (0)-class with non-negative multiplicities. Then, c is either the line class $h - e_1$; $r = 3$, $s = 7$, and $c = F = (4; 1^7)$; or $r = 4$, $s = 8$, and $c = 2F = (10; 2^8)_4$.*

Proof. We start with Lemma 5.

For $r = 5$ and $s = 9$, this gives $4\sum_{i=7}^{9} m_i \geq -2 + 2\sum_{i=1}^{6} m_i$ which we can re-write as

$$(m_1 - m_7) + (m_2 - m_7) + (m_3 - m_8) + (m_4 - m_8) + (m5 - m_9) + (m_6 - m_9) \leq 1.$$

Hence, at most one of these non-negative differences is one. If all are equal to zero, then all multiplicities are equal, say, to m; in this case, the numerical condition gives that $6d - 4 \cdot 9m = 2$, which is impossible.

Hence, exactly one of these differences is one, all the other five are zero. We cannot have the second, fourth, or sixth equal to one, since they are at most the first, third, and fifth, respectively. We conclude that these are zero, which implies that there exists m such that $m_1 = m + 1$ and $m_i = m$ for $i \geq 2$. Now, the numerical condition gives $6d - 4 \cdot (9m + 1) = 2$, implying that $d = 6m + 1$. The Cremona-reduced conditions then force $m = 0$, and we have the line class $h - e_1$.

For $r = 4$ and $s = 8$, we have $3\sum_{i=6}^{8} m_i \geq -2 + 2\sum_{i=1}^{5} m_i$ which we re-write as

$$m_1 + (m_1 - m_6) + 2(m_2 - m_6) + 2(m_3 - m_7) + (m_4 - m_7) + (m_4 - m_8) + 2(m_5 - m_8) \leq 2.$$

Since $m_1 \geq 1$, at most one of the differences here can be positive, and all of the doubled ones must be zero. Hence, $m_2 = m_6$, forcing all m_i for $i \geq 2$ to be equal, say, to m. This then reduces to $2m_1 - m \leq 2$, leading to having either $m_1 = m \leq 2$ or $m_1 = 1$ and $m = 0$.

If all m_i are equal to m, then the numerical condition gives $5d - 3 \cdot 8m = 2$, so we must have $d = 10$ and $m = 2$, since $m \leq 2$. In the other case, we have the line through one point: $h - e_1$.

For $r = 3$ and $s = 7$, we have $2\sum_5^7 m_i \geq -2 + 2\sum_1^4 m_i$, which we may rewrite as

$$2m_1 + 2(m_2 - m_5) + 2(m_3 - m_6) + 2(m_4 - m_7) \leq 2$$

and since m_1 is positive, each of the differences above must be zero, and $m_1 = 1$. The two cases are then when $m_2 = 0$ (giving the line class $h - e_1$) or when $m_2 = 1$ (which gives the F class).

For $r = 2$ and $s = 6$, we have $\sum_4^6 m_i \geq -2 + 2\sum_1^3 m_i$, which also leads only to the line class $h - e_1$. $\square$

We note that the exceptional case $c = (10; 2^8)$ in $\mathbb{P}^4$ has $\langle c, c \rangle = 10^2 - 3 \cdot 8 \cdot 2^2 = 4$, and the other exceptional case $c = (4; 1^7)$ in $\mathbb{P}^3$ has $\langle c, c \rangle = 4^2 - 2 \cdot 7 \cdot 1^2 = 2$, both of which are different from the (0)-Weyl line class values (which are -2 and -1, respectively). Hence, we have the following.

Corollary 5. *Suppose $r \leq 5$ and $s = r + 4$; a Cremona-reduced class c with non-negative multiplicities is a (0)-Weyl line if and only if $\langle c, c \rangle = 2 - r$ and $\langle c, F \rangle = 2$.*

In the case of $r + 4$ points, for $r \geq 6$, it is no longer the case that the two invariants $\langle c, F \rangle$ and $\langle c, c \rangle$ pick out the Weyl classes.

Example 5. *Consider again the class $c = 3F = (21; 3^{10})_6$ of curves of degree 21 in $\mathbb{P}^6$ with ten points of multiplicity 3. We have $\langle c, F \rangle = 7 \cdot 21 - 5(10 \cdot 3) = -3$ and $\langle c, c \rangle_{r-1} = 21^2 - 5 * 10 * 9 = -9$, as is the case for the line through two points, but this is not a (-1) Weyl class. In fact, the class c is Cremona-reduced, as well.*

It is also true that there are counterexamples in $\mathbb{P}^3$ for a larger number of points. The class $c = (7; 4, 1^{10})$ has $\langle c, F \rangle_{r-1} = 0$ and $\langle c, c \rangle_{r-1} = -3$ as a (-1)-Weyl line does; however, it is not in the Cremona orbit of $h - e_1 - e_2$.

4.3. The Proof of Theorem 2 and Conclusions

Using Propositions 10 and 13, we have the following.

Corollary 6. *If $r \geq 3$ and $Y = Y_s^r$ is a Mori Dream Space (i.e., $r = 3, 4$ and $s \leq r + 4$ or $r \geq 5$ and $s \leq r + 3$), the only classes $c \in A^{r-1}(Y)$ with positive degree and non-negative multiplicities that satisfy the equations*

$$\langle c, F \rangle = 3 - r \text{ and } \langle c, c \rangle = 3 - 2r$$

are either the proper transform of a line through two points or the rational normal curve of degree r through r + 3 points.

Proof of Theorem 2. Assume Y is Y^5_9 or is a Mori Dream Space. The first part of the previous proof applies here as well. For (-1)-curves, the equivalence of conditions (3) and (2) in Theorem 2 follows again from Corollary 6. If $r = 2$, the proof of Theorem 2 for the del Pezzo surfaces follows from Theorem 6 and Propositions 6 and 7. In higher dimensions, $r > 2$, Proposition 12 proves that (4) implies (2), proving the equivalences (1), (2), and (4). □

Corollary 7. *If Y is Mori Dream Space and $r \geq 3$ or $Y = Y^5_9$, then the only (-1)-curves are the ones described in Example 1.*

Theorem 10. *Theorem 9 for $i = 1$ holds if r is even.*

Indeed, the equivalence between numerical (1)-classes and a (1)-Weyl line in even dimensional spaces Y^r_{r+3} follows from Proposition 11, and this completes the cycle.

Remark 4. *Theorem 2 holds for irreducible (0)-curves in Mori Dream cases or Y^5_9, with two exceptions. The proof follows from Proposition 14; one exception is the anticanonical class F in Y^3_7 that contains a (0)-curve that is not a (0)-Weyl line. The only other candidate for a (0)-curve that is not a (0)-Weyl line in the above hypothesis is $r = 4$, $s = 8$, and $2F$.*

Similarly, Theorem 10 does not hold if r is not even. Indeed, consider $i = 1$, $r = 3$, and $s = 6$; then, F is a (1)-curve that is not a (1)-Weyl line in Y^3_6 (as in Example 6).

Next, the proposition also holds for (0)-curves and $vdim(C) = r - 1$, with part (3) including the (0)-curves in F in Y^3_7 and $2F$ in Y^4_8.

Proposition 15.

1. *Any (-1)-curve has $vdim(C) = 0$.*
2. *If Y is a Mori Dream Space or Y^5_9, the only irreducible curve classes with $vdim(C) = 0$ are classes of (-1)-curves and $-K_{Y^2_8}$.*

Proof. Part (1) follows from the observation that $vdim(C) = (-K_Y \cdot C) + (r - 3)$ (Equation (10)) is stable under the Weyl group action, since the anticanonical divisor is also stable. Part (2) follows from Theorem 2, since the condition $vdim(C) = 0$ is equivalent to C being a (-1)-numerical class or the anticanonical class in Y^2_8. □

We ask the following question:

Question 1. *Is it true that a smooth, rational, irreducible, non-degenerate curve in Y^r_s is rigid if and only if $vdim(C) = 0$?*

We now have the following for the Mori Dream Space cases.

Proposition 16. *A Mori Dream space has finitely many classes of (i)-curves for $i \in \{-1, 0, 1\}$.*

Proof. If $i = -1$, the statement follows from Theorem 2. Remark 4 implies that (0)-curves in a Mori Dream Space are either (0)-Weyl lines or F in Y^3_7 or $2F$ in Y^4_8. Assume now that C is a (1)-curve. If Y is a del Pezzo surface, then a (1)-curve is in the Weyl orbit of a line and we conclude, since the Weyl group is finite by Corollary 2. If $s \leq r + 3$, Proposition 11 implies that a numerical (1)-class is either a (1)-Weyl line or is in the Weyl orbit of the curve $mF + C'$, where m and the degree of C' are bounded above by $\frac{r+1}{2}$. Since the Weyl group is finite, we have finitely many possibilities for a (1)-curve if $s \leq r + 3$. If $Y = Y^4_8$, a numerical (1)-class satisfies the equality $5d - 3\sum_{i=1}^{8} m_i = 5$. Suppose the class is Cremona-reduced, i.e., $d \geq \sum_{i=1}^{5} m_i$; then, all multiplicities are bounded above by 5, and this implies that the degree $d \leq 25$. We conclude again, since the Weyl group is finite. Similarly, if $Y = Y_7^3$ we obtain that all multiplicities $m_i \leq 2$, while the linear invariant gives that $d \leq 8$, and we conclude again via the finiteness of the Weyl group. $\square$

Example 6 (The F-class in Y_8^3)**.** *In order to construct examples of $(-1)-$curves that are not $(-1)-$ Weyl lines, we study the F-class in Y_8^3. The F-class in Y_8^3 represents smooth rational curves and is also the class of an elliptic curve.*

An interesting example of the type of phenomena that can appear is afforded by the F class in $\mathbb{P}^3$ with $s = 8$, which is the class of quartics passing simply through eight points. This class is Cremona-reduced and, as we have noted, is invariant under the Weyl group. It is a numerical (-1)-class: $(-K, F) = 0 = 3 - r$ in this case. It is the class of an effective curve; the easiest way to see this is to note that if you break up the eight points into pairs, then the four lines joining the four pairs are a (disconnected, reducible) curve in $\mathbb{P}^3$ in this class.

It is also the class of an irreducible curve of genus one. The linear system of quadrics in $\mathbb{P}^3$ has dimension 9 and the quadrics through the eight points is a pencil $\mathcal{P}$. The base locus of this pencil, which is the intersection of any two of the quadrics in the pencil, is (in general) a smooth curve E of degree four and genus one through the eight points. If one chooses a smooth quadric in the pencil, and considers it as being isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then the genus one curve has bidegree $(2, 2)$.

Since the eight points are general, the general quadrics in the pencil will be smooth and exactly four members will be cones over smooth conics; none of the four vertices are among the eight given points. In each case, there are lines on the quadrics: in the smooth case there are the two rulings, and in the cone case there is the one system of lines through the vertex. Consider the incidence correspondence $\mathcal{I} = \{(Q, R) \mid Q \in \mathcal{P}, R = \text{ a ruling on Q}\}$. We note that $\mathcal{I}$ is in 1-1 correspondence with the set of g_2^1s on the base curve E; a ruling on one of the quadrics restricts to E as a g_2^1 and inversely a g_2^1 gives a quadric in the pencil as the union of the secants, with the secants forming the ruling.

For a pair $(Q, R) \in \mathcal{I}$, if Q is smooth, we may consider the linear system $3R + R'$, where R' is the other ruling; this system is of bidegree $(3, 1)$ and has dimension 7. Hence, there is a unique member $C \in 3R + R'$ passing through the first seven of the given original base points$\{p_i\}$. We ask that, as we vary Q in the pencil, this curve C pass also through the last (eighth) point.

Note that $3R + R' \equiv 2R + H$, where H is the hyperplane class on Q, and that these systems therefore restrict to $2g_2^1 + H|_E$ on the genus one curve E. The entire construction depends on the choice of the g_2^1 on E, and we therefore ask how many g_2^1s are needed such that $|2g_2^1 + H|_E - (\sum_{i=1}^{7} p_i)|$ contains the point p_8.

We may always write a g_2^1 as $p_8 + p$ for a variable point $p \in E$. This condition is then that $2p + 2p_8 + H|_E - (\sum_{i=1}^{7} p_i)) = p_8$ in the Jacobian of E; there are four such points p of course (any solution plus the two-torsion points). This gives four curves C of bidegree $(1, 3)$ on a quadric through all eight points. Each of these is a smooth rational curve, and is indeed a (-1)-curve in Y_8^3 with class F. (The normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ using the results of [28]).

Example 7 (F-curves)**.** *The moduli stack of n-pointed stable curves of genus g, $\overline{\mathcal{M}}_{g,n}$, for $2g - 2 + n > 0$ has a stratification given by topological type; the 1-dimensional strata are also called F-curves.*

The question of describing the ample and the effective cones of $\overline{\mathcal{M}}_g$ goes back to Mumford.

The F-conjecture of Fulton states that for the moduli spaces $\overline{\mathcal{M}}_{0,n}$ (the moduli spaces of rational pointed curves), their Mori cone of curves is polyhedral, generated by the F-curves; this was proved for $n \leq 7$ by Hu and Keel.

It is easy to check, via our theory of curves developed in this paper, that the F-curves in $\overline{\mathcal{M}}_{0,n}$ are movable for $n \geq 9$, sweeping out subvarieties of $\overline{\mathcal{M}}_{0,n}$ of dimension at least 2. A counterexample to the F-conjecture would have to be a rigid (non-movable) curve; we would expect that such a curve, if it exists, would be a (-1)-curve. Hence, finding such (-1)-curves in $\overline{\mathcal{M}}_{0,n}$ would be a first step in disproving the F-conjecture, and would be of great interest.

5. Nine Points in $\mathbb{P}^4$

We will prove that, for the case of nine points in $\mathbb{P}^4$, the orbit of a line through two of the points is infinite.

We will use the notation of $(d; m_1, m_2, \ldots, m_9)$ to denote the class of a curve of degree d with multiplicity m_i at the point p_i. The line through two points has the class $(1; 0, 0, 0, 0, 0, 0, 0, 1, 1)$. The approach will be to take this class and repeatedly apply the Cremona transformation using the five points with the lowest multiplicity. If those five points are the first five, then we have seen that $\phi(d; m_1, m_2, \ldots, m_9) = (d'; m'_1, m'_2, \ldots, m'_9)$ where

$$d' = d + 3t; \quad m'_i = m_i + t \quad \text{for} \quad i \leq 5; \quad m'_i = m_i \quad \text{for} \quad i \geq 6 \tag{15}$$

where $t = d - m_1 - m_2 - m_3 - m_4 - m_5$. We will analyze the process of applying the Cremona to the five lowest multiplicity points, and then re-ordering so that the multiplicities are in ascending order, in order to simplify the notation.

Lemma 6. *Suppose that the class $C = (d; m_1, m_2, \ldots, m_9)$ satisfies the following:*

(a) $m_i \leq m_{i+1}$ *for every* $i = 1, \ldots, 8$;
(b) $d > m_3 + m_4 + m_7 + m_8 + m_9$.

Then, the class $(e; n_1, n_2, \ldots, n_9)$, which is the (re-ordered) Cremona image $C' = \phi(C)$ of C based at the first five points, equal to (using the notation of (15))

$$(d'; m_6, m_7, m_8, m_9, m_1 + t, m_2 + t, m_3 + t, m_4 + t, m_5 + t), \quad \textit{i.e.,} \tag{16}$$

$$e = d'; n_1 = m_6; n_2 = m_7; n_3 = m_8; n_4 = m_9; \tag{17}$$

$$n_5 = m_1 + t; n_6 = m_2 + t; n_7 = m_3 + t; n_8 = m_4 + t; n_9 = m_5 + t \tag{18}$$

also satisfies (a) and (b) above. Moreover, $e > d$.

Proof. First, we claim that the multiplicities, in this order, are ascending. It is clear that $n_1 \leq n_2 \leq n_3 \leq n_4$ and $n_5 \leq n_6 \leq n_7 \leq n_8 \leq n_9$ using (a). We need only check that $n_4 \leq n_5$, therefore. This is equivalent to having $m_9 \leq m_1 + t$ or $d \geq m_2 + m_3 + m_4 + m_5 + m_9$. This is implied by (b) and the ascending order of the m_is. Therefore, we have the multiplicities in the correct (ascending) order, and so (a) is satisfied for C'.

We next check (b) for C': this is

$$e = d + 3t > n_3 + n_4 + n_7 + n_8 + n_9 = m_8 + m_9 + (m_3 + t) + (m_4 + t) + (m_5 + t)$$

which is equivalent to having $d > m_3 + m_4 + m_5 + m_8 + m_9$. This follows from (b) for C, since $m_7 \geq m_5$.

Finally, $e > d$ if and only if $t > 0$, which follows from (b), since $m_1 \leq m_9$. □

Corollary 8. *With nine or more points in $\mathbb{P}^4$, the orbit of a line through two points is infinite.*

Proof. Using the notation above, we start with the line class $L_1 = (1; 0, 0, 0, 0, 0, 0, 0, 1, 1)$ and iterate, and set $L_{i+1} = \phi(L_i)$. We cannot immediately apply the Lemma above because the inequality (b) is not satisfied for L_1. We have $L_2 = (4; 0, 0, 1, 1, 1, 1, 1, 1, 1,)$, $L_3 = (7; 1, 1, 1, 1, 1, 1, 2, 2, 2)$, $L_4 = (13; 1, 2, 2, 2, 3, 3, 3, 3, 3)$, $L_5 = (22; 3, 3, 3, 3, 4, 5, 5, 5, 6)$, and $L_6 = (40; 5, 5, 5, 6, 9, 9, 9, 10)$. The class L_6 does satisfy the conditions of the Lemma, and therefore so does L_i for all $i \geq 6$. Since the degrees strictly increase, they increase without bound and the orbit is infinite. □

6. Infinity of (i)-Curves

In this section, we will prove the following.

Theorem 11. *There are finitely many classes of* (0)*-curves or equivalently finitely many classes of* (1)*-curves if and only if* Y^r_s *is a Mori Dream Space.*

The proof is presented after Corollary 9 in the first subsection below.
Although Y^5_9 is not a Mori Dream Space, we can determine all (-1)- and (0)-curves.

Proposition 17.

1. *In* Y^5_9*, all* (0)*-curves are* (0)*-Weyl lines (or have numerical* (0)*-classes).*
2. *In* Y^5_9*, the only* (-1)*-curves are ones in Example 1 (i.e., the proper transforms of lines through two points and the proper transform of the rational normal curves through* $n+3$ *points).*

Proof. Indeed, Theorem 2 implies that there are *finitely* many (-1)-curves; therefore, the notion of (-1)-curves does not determine the finite generation of the Cox Ring of Y. Moreover, Proposition 14 implies that all Cremona-reduced (0)-curves are lines through two points. Therefore, every (0)-curve is a (0)-Weyl line (and there are infinitely many), see Remark 4. □

Question 2. *Does Theorem 11 hold for other varieties?*

6.1. Infinity of Movable Curves in $\mathbb{P}^r$

In this subsection, we analyze (i)-Weyl lines in Y^r_s, which is not a Mori Dream Space; namely, in Section 6.1 we begin with an analysis of (1)-Weyl lines, which are the Cremona images of the general line h in $\mathbb{P}^r$, and in Section 6.2 we study rigid curves, $i=-1$.

We will analyze the process of applying the Cremona to the $r+1$ lowest multiplicity points, and then re-ordering so that the multiplicities are in ascending order, in order to simplify the notation. We begin with the $s=r+4$ case. It is more useful for this section to place the multiplicities in ascending order, and we will systematically perform that here.

Lemma 7. *Let* $L=(d; m_1 \leq m_2 \leq \ldots \leq m_{r+4})$ *be a curve class in* $A^{r-1}(Y^r_{r+4})$ *with positive degree and non-negative multiplicities in nondecreasing order. Let* $I \subset \{1,\ldots,r+4\}$ *have size* $r+1$*, so that there are three indices missing; assume that those indices are* $1, k, \ell$ *with* $4 \leq k$ *and* $k+3 \leq \ell \leq r+2$*. (This implies in particular that* $1 \notin I$*;* $2,3 \in I$*;* $m_{r+3}, m_{r+4} \in I$*; and* $r \geq 5$*). Assume that*

$$d > \sum_{i \in I} m_i, \text{ or } d = \sum_{i\in I} m_i \text{ and } m_{r+4} > m_1. \tag{19}$$

Then the Cremona image $\phi(L)$ *(where* ϕ *is based at the first* $r+1$ *points, those with the lowest multiplicities) has the form* $\phi(L)=(d'; \{m'_i\})$ *(where the* m'_i *are also placed in increasing order) and these parameters also satisfy (19). Moreover, in this case* $d' > d$*.*

Proof. We define $t = d - \sum_{i=1}^{r+1} m_i$, and note that the degree of $\phi(L)$ is then $d' = d + (r-1)t$ and the multiplicities are

$$m_1+t, m_2+t, \ldots, m_{r+1}+t, m_{r+2}, m_{r+3}, m_{r+4}$$

in some order, by Proposition 2a.

We first claim that $m_{r+4} \leq m_1 + t$. This is equivalent to having $d \geq \sum_{i=2}^{r+1} m_i + m_{r+4}$, which is implied by (19) and the assumptions that $1 \notin I$ and $m_{r+4} \in I$.

Since this is true, the re-ordering of the multiplicities of $\phi(L)$ to be in increasing order gives

$$m'_1 = m_{r+2}, m'_2 = m_{r+3}, m'_3 = m_{r+4}, \text{ and for } 4 \le i \le r+4, m'_i = m_{i-3} + t.$$

Therefore (19) for $\phi(L)$, which is $d' > (\text{ resp. } \ge) \sum_{i \in I} m'_i$, is equivalent to

$$d + (r-1)t > (\text{ resp. } \ge) \, m_{r+3} + m_{r+4} + \sum_{i \in I, i \ge 4} (m_{i-3} + t)$$

since $1 \notin I$ and $2, 3 \in I$. Note that since $|I| = r+1$, the sum above contains $r-1$ indices; hence, we may subtract $(r-1)t$ from both sides of this inequality to obtain

$$d > (\text{ resp. } \ge) \sum_{i \in I, i \ge 4} m_{i-3} + m_{r+3} + m_{r+4}.$$

To prove this, it suffices to show that the right side of (19) is at least the right side of the above, i.e., that

$$\sum_{i \in I} m_i \ge \sum_{i \in I, i \ge 4} m_{i-3} + m_{r+3} + m_{r+4}.$$

Now, the sum of $r+1$ multiplicities on the right side of this is exactly that of the indices in the set $I' = \{1, \ldots, r+4\} - \{k-3, \ell-3, r+2\}$. If we denote the j'th index in I by $I(j)$, and similarly for I', it will suffice to show that $I(j) \ge I'(j)$ for each j.

The set of indices in I increases by one at the $1, k, \ell$ points; that of I' increases at the $k-3, \ell-3, r+2$ points. Since $1 \le k-3$, the first increase in I is no later than that in I'; since $\ell \le r+2$, the third increase in I is no later than that in I'. The only failure then would be if the second increase in I would be later than the second in I', and that will only happen if $\ell - 3 < k$. This is forbidden by the assumption that $k+3 \le \ell$.

This proves that (19) also holds for $\phi(L)$.

To finish and show $d' > d$, we must show that $t > 0$, or that $d > \sum_{i=1}^{r+1} m_i$. This follows immediately from (19) if the inequality is strict; if not, but $m_{r+4} > m_1$, we obtain the result since $r+4 \in I$. Finally, we must show that if the inequality (of the parameters for $\phi(L)$ and hence also for L) is an equality, then $m'_{r+4} > m'_1$, i.e., that $m_{r+1} + t > m_{r+2}$; this is equivalent to $d > \sum_{i=1}^{r} m_i + m_{r+2}$. The indices in I dominate those in this sum, and so the only way this could fail is if we have equality here, and all m_js are equal. This is a contradiction, since $m_{r+4} > m_1$. □

Lemma 8. *Assume $r \ge 5$. Let $L = (d; m_1 \le m_2 \le \ldots \le m_{r+4})$ be a curve class in A^{r-1} with positive degree and non-negative multiplicities. Assume that*

$$d > m_2 + m_3 + m_5 + m_6 + \sum_{i=8}^{r+4} m_i. \tag{20}$$

Then the Cremona image $\phi(L)$ (where ϕ is based at the first $r+1$ points, those with the lowest multiplicities) has the form $\phi(L) = (d'; \{m'_i\})$ (where the m'_i are also placed in increasing order) and these parameters also satisfy (20). Moreover, in this case $d' > d$.

Proof. We define $t = d - \sum_{i=1}^{r+1} m_i$, and note that the degree of $\phi(L)$ is then $d' = d + (r-1)t$ and the multiplicities are

$$m_1 + t, m_2 + t, \ldots, m_{r+1} + t, m_{r+2}, m_{r+3}, m_{r+4}$$

in some order, by Proposition 2a.

We first claim that $m_{r+4} \le m_1 + t$. This is equivalent to having $d \ge \sum_{i=2}^{r+1} m_i + m_{r+4}$, which is implied by (20) since the multiplicities are in increasing order.

Since this is true, the re-ordering of the multiplicities of $\phi(L)$ to be in increasing order gives

$$m_1' = m_{r+2}, m_2' = m_{r+3}, m_3' = m_{r+4}, \text{ and for } 4 \le i \le r+4, m_i' = m_{i-3} + t.$$

Therefore, (20) for $\phi(L)$, which is $d' > m_2' + m_3' + m_5' + m_6' + \sum_{i=8}^{r+4} m_i'$, is equivalent to

$$d + (r-1)t > m_{r+3} + m_{r+4} + (m_2 + t) + (m_3 + t) + \sum_{i=5}^{r+1} (m_i + t).$$

Subtracting $(r-1)t$ from both sides, we see that this is equivalent to

$$d > m_2 + m_3 + \sum_{i=5}^{r+1} m_i + m_{r+3} + m_{r+4}.$$

In order to prove this, it therefore suffices to show that the right side of (20) is at least the right side of this, i.e., that

$$m_2 + m_3 + m_5 + m_6 + \sum_{i=8}^{r+4} m_i \ge m_2 + m_3 + \sum_{i=5}^{r+1} m_i + m_{r+3} + m_{r+4}.$$

For $r = 5$, these are exactly the same set of multiplicities, and so the inequality is an equality, and holds. For $r \ge 6$, subtracting $m_2 + m_3 + m_5 + m_6 + m_{r+3} + m_{r+4}$ from both sides gives

$$\sum_{i=8}^{r+2} m_i \ge \sum_{i=7}^{r+1} m_i;$$

this inequality is true since the m_is are in increasing order.

This proves that the inequality (20) is preserved under the Cremona transformation, as claimed.

To show that $d' > d$, we must prove that $t > 0$. This is equivalent to $d > \sum_{i=1}^{r+1} m_i$, which immediately follows from (20). □

In case of $r = 3, 4$, a completely parallel lemma can be proved, but we need to have one more point ($s = r + 5$). In particular, using the same notation, if

$$d > m_3 + m_4 + \sum_{i=7}^{r+5} m_i, \text{ or } d = m_3 + m_4 + \sum_{i=7}^{r+5} m_i \text{ and } m_{r+5} > m_1 \tag{21}$$

then we have the same conclusion: the Cremona image $\phi(L)$ has parameters also satisfying (21), and the degree increases. We leave it to the reader to check the details, which are parallel in all respects to those of Lemma 7.

For $r = 2$ and $s = 9$, the recursive argument requires the inequality

$$d > m_3 + m_6 + m_9, \text{ or } d = m_3 + m_6 + m_9 \text{ and } m_9 > m_1 \tag{22}$$

and again the same argument goes through, with the same conclusion.

Corollary 9. *If $r \ge 5$ and $s \ge r + 4$, $r \ge 3$ and $s \ge r + 5$, or $r = 2$ and $s \ge 9$, there are infinitely many (1)-Weyl line classes and (0)-Weyl line classes.*

Proof. For $r \ge 5$, it suffices to prove this for $s = r + 4$. For the (1)-Weyl line classes, apply Lemma 7 repeatedly starting with $L = h$, the general line class. Since the degrees increase without bound (using the strict inequality of (19)), we have the result. For the (0)-Weyl line classes, we apply the same lemma starting with $L = h - e_1$, which, using the $(d; \underline{m})$

notation, is given by $(1; 0^{r+3}, 1)$. Here, we do not have the strict inequality in (19), but we do have $m_{r+4} = 1 > m_1 = 0$; the result follows.

For $r = 3, 4$, it suffices to prove this for $s = r + 5$; the same argument holds, this time using the recursive assumption of (21).

Finally, for $r = 2$, it suffices to prove this for $s = 9$, and again this is carried out via the recursion supplied by (22). □

We can now prove the statement at the beginning of the section.

Proof of Theorem 11. One direction follows from Proposition 16. The other direction follows from Corollary 9. □

One can also be very explicit about the recursion that produces the infinite family of curves in these cases with low r. For example, for $r = 3$ and $s = 8$, one can easily see (by induction) that the sequence of (0)-Weyl line classes $\phi^i(h - e_8)$ has the form $d = i^2 + i + 1$, and multiplicities $((k^2)^4, (k^2 + k)^3, (k^2 + k + 1)^1)$ if $i = 2k$ and $((k^2 + k)^3, (k^2 + k + 1)^1, (k^2 + 2k + 1)^4)$ if $i = 2k + 1$.

For $r = 2$ and $s = 9$, the sequence of (0)-Weyl line classes $\phi^i(h - e_9)$ has the form $d = 1 + i(i + 1)/2$, and multiplicities $((k + 3k(k - 1)/2)^3, (2k + 3k(k - 1)/2)^3, (3k(k + 1)/2)^2, (1 + 3k(k + 1)/2)^1)$ if $i = 3k$; $((3k(k + 1)/2)^2, (1 + 3k(k + 1)/2)^1, ((k + 1)(1 + 3k/2))^3, (2k + 3k(k - 1)/2)^3)$ if $i = 3k + 1$; and $((3k(k + 1)/2)^2, (1 + 3k(k + 1)/2)^1, ((k + 1)(1 + 3k/2))^3, ((k + 1)(2 + 3k/2))^3)$ if $i = 3k + 2$.

We now observe that the results of Section 4 imply that, for the Mori Dream Spaces Y_s^r ($r = 2, s \le 8; r = 3, 4, s \le r + 4; r \ge 5, s \le r + 3$), there are only finitely many numerical (1)-classes and numerical (0)-classes. Hence, this gives a criterion for these spaces Y_s^r being Mori Dream Spaces.

Theorem 12. *The space Y_s^r is a Mori Dream Space if and only if there are finitely many numerical (1)-classes in A^{r-1}, and if and only if there are finitely many numerical (0)-classes in A^{r-1}.*

The following Lemma can be used to generate infinitely many (0)-Weyl classes in A^{r-1}.

Lemma 9. *Assume $r \ge 6$. Let $L = (d; m_1 \le m_2 \le \ldots \le m_{r+4})$ be a curve class in A^{r-1} with positive degree and non-negative multiplicities. Assume that*

$$d > m_2 + m_3 + m_4 + \sum_{i=6}^{r+1} m_i + m_{r+3} + m_{r+4}. \tag{23}$$

Then the Cremona image $\phi(L)$ (where ϕ is based at the first $r + 1$ points, those with the lowest multiplicities) has the form $\phi(L) = (d'; \{m'_i\})$ (where the m'_i are also placed in increasing order) and these parameters also satisfy (23). Moreover, in this case $d' > d$.

Proof. We define $t = d - \sum_{i=1}^{r+1} m_i$, and note that the degree of $\phi(L)$ is then $d' = d + (r - 1)t$ and the multiplicities are

$$m_1 + t, m_2 + t, \ldots, m_{r+1} + t, m_{r+2}, m_{r+3}, m_{r+4}$$

in some order, by Proposition 2a.

We first claim that $m_{r+4} \le m_1 + t$. This is equivalent to having $d \ge \sum_{i=2}^{r+1} m_i + m_{r+4}$, which is implied by (23) since the multiplicities are in increasing order.

Since this is true, the re-ordering of the multiplicities of $\phi(L)$ to be in increasing order gives

$$m'_1 = m_{r+2}, m'_2 = m_{r+3}, m'_3 = m_{r+4}, \text{ and for } 4 \le i \le r + 4, m'_i = m_{i-3} + t.$$

Therefore, (23) for $\phi(L)$, which is $d' > m'_2 + m'_3 + m'_4 + \sum_{i=6}^{r+1} m'_i + m'_{r+3} + m_{r+4'}$, is equivalent to

$$d + (r-1)t > m_{r+3} + m_{r+4} + (m_1 + t) + \sum_{i=3}^{r-2} (m_i + t) + (m_r + t) + (m_{r+1} + t).$$

Subtracting $(r-1)t$ from both sides, we see that this is equivalent to

$$d > m_1 + \sum_{i=3}^{r-2} m_i + m_r + m_{r+1} + m_{r+3} + m_{r+4}.$$

In order to prove this, it therefore suffices to show that the right side of (23) is at least the right side of this, i.e., that

$$m_2 + m_3 + m_4 + \sum_{i=6}^{r+1} m_i + m_{r+3} + m_{r+4} \geq m_1 + \sum_{i=3}^{r-2} m_i + m_r + m_{r+1} + m_{r+3} + m_{r+4}.$$

For $r = 5$, this is

$$m_2 + m_3 + m_4 + m_6 + m_8 + m_9 \geq m_1 + m_3 + m_5 + m_6 + m_8 + m_9$$

which is equivalent to $m_2 + m_4 \geq m_1 + m_5$, which may not hold. However, if $r \geq 6$, subtracting the common terms from both sides, we see that this is equivalent to

$$m_2 + m_{r-1} \geq m_1 + m_5$$

which is true since the m_is are in increasing order.

This proves that the inequality (23) is preserved under the Cremona transformation, as claimed.

To show that $d' > d$, we must prove that $t > 0$. This is equivalent to $d > \sum_{i=1}^{r+1} m_i$, which immediately follows from (23). □

We make an additional observation here, concerning these infinite series of (0)- and (1)-Weyl classes. Each imposes a condition on divisors to be effective: for any such class c, it is a necessary condition for D to be effective that $(D \cdot c) \geq 0$. We note below that these conditions are independent.

Proposition 18. *For the infinite series $\{c_k\}$ of (0)-Weyl classes constructed above, we have that c_{k+1} is not in the convex hull of $\{c_j\}_{j \leq k}$ for all k.*

Proof. We argue by contradiction: suppose that $c_{k+1} = \sum_{j=1}^{k} a_j c_j$ for non-negative real numbers a_j. Since each class c_j is an (0)-Weyl class, we have that $\langle c_j, F \rangle = 2$ for every j; applying this to c_{k+1} and dividing by 2 gives that $1 = \sum_{j=1}^{k} a_j$. This would then imply that $\deg(c_{k+1}) = \sum_{j=1}^{k} a_j \deg(c_j)$. However, this is not possible, since the degrees of the classes c_j increase monotonically. □

We remark that the same proof applies for (1)-Weyl classes unless $\langle h, F \rangle = 0$, which only happens if $r = 3$.

6.2. Infinity of Rigid Curves in $\mathbb{P}^r$

In this section, we discuss the question of an infinite number of (-1)-curves in the sense of Kontsevich [3], in Y_s^r, where $s \geq r + 5$ via Corollary 10. For $r = 3, 4$, the Weyl group with $r + 4 = 7, 8$ points is finite, and so there are only finitely many (-1)-Weyl lines. However, we can prove a similar statement for $r + 5$ (or more) points.

Lemma 10. *Assume $r \geq 3$. Let $L = (d; m_1 \leq m_1 \leq \ldots \leq m_{r+5})$ be a curve class in $A^{r-1}(Y^r_{r+5})$ with positive degree and non-negative multiplicities. Assume that*

$$d > m_3 + m_4 + \sum_{i=7}^{r+5} m_i. \tag{24}$$

Then the Cremona image $\phi(L)$ (where ϕ is based at the first $r+1$ points, those with the lowest multiplicities) has the form $\phi(L) = (d'; \{m'_i\})$ (where the m'_i are also placed in increasing order) and these parameters also satisfy (24). Moreover, in this case $d' > d$.

Lemma 11. *Suppose that the class $c = (d; m_1, m_2, \ldots, m_{r+5})$ satisfies the following:*

(a) $m_i \leq m_{i+1}$ *for every* $i = 1, \ldots, r+4$;

(b) $d > m_3 + m_4 + m_7 + m_8 + \cdots m_{r+5}$.

Then the class $(e; n_1, n_2, \ldots, n_{r+5})$, which is the (re-ordered) Cremona image $c' = \phi(c)$ of c based at the first $r+1$ points, is equal to

$$(e; m_{r+2}, m_{r+3}, m_{r+4}, m_{r+5}, m_1 + t, m_2 + t, \ldots, m_{r+1} + t), \quad \text{i.e.,}$$
$$n_1 = m_{r+2}; n_2 = m_{r+3}; n_3 = m_{r+4}; n_4 = m_{r+5}; n_j = m_{j-4} + t \quad \text{for} \quad 5 \leq j \leq r+5$$

where $t = d - m_1 - m_2 - \cdots - m_{r+1}$ and $e = d + (r-1)t$. In addition, c' also satisfies (a) and (b) above, and $e > d$.

Proof. We note that (24) implies that

$$d > m_{i_1} + m_{i_2} + \ldots + m_{i_{r+1}} \quad \text{if} \quad i_1 \leq 3, i_2 \leq 4, \text{ and } i_j \leq j+4 \text{ for } 3 \leq j \leq r+1. \tag{25}$$

We again define $t = d - \sum_{i=1}^{r+1} m_i$, and note that the degree of $\phi(L)$ is then $d' = d + (r-1)t$ and the multiplicities are

$$m_1 + t, m_2 + t, \ldots, m_{r+1} + t, m_{r+2}, m_{r+3}, m_{r+4}, m_{r+5}$$

in some order, by Proposition 2a.

We claim that $m_{r+5} \leq m_1 + t$. This is equivalent to having $d \geq \sum_{i=2}^{r+1} m_i + m_{r+5}$, which is implied by (25).

Since this is true, the re-ordering of the multiplicities of $\phi(L)$ to be in increasing order gives

$$m'_1 = m_{r+2}, m'_2 = m_{r+3}, m'_3 = m_{r+4}, m'_4 = m_{r+5}, \text{ and for } 5 \leq i \leq r+5, m'_i = m_{i-4} + t.$$

We next check (24) for $\phi(L)$: this is $d' > m'_3 + m'_4 + m'_7 + \cdots + m'_{r+5}$ which is equivalent to

$$\begin{aligned} d + (r-1)t &> m_{r+4} + m_{r+5} + (m_3 + t) + \cdots + (m_{r+1} + t) \\ &= m_3 + \ldots + m_{r+1} + m_{r+4} + m_{r+5} + (r-1)t \end{aligned}$$

which is equivalent to having $d > m_3 + \ldots + m_{r+1} + m_{r+4} + m_{r+5}$. This follows from (25) if $r \geq 3$.

Finally, $d' > d$ if and only if $t > 0$, which also follows from (25). □

We will now apply Lemma 10 to (-1)-Weyl lines, giving a different perspective on the topic from Corollary 11.

Corollary 10. *For $r \geq 3$, with $r+5$ or more points in $\mathbb{P}^r$, the orbit of the proper transform of a line through two points is infinite. Hence, there is an infinite number of (-1)-Weyl lines and numerical (-1)-classes.*

Proof. We will use exponents to denote repeated multiplicities. Using the notation above, we start with the line class $L_1 = (1; 0^{r+3}, 1^2)$ and iterate, and set $L_{i+1} = \phi(L_i)$. We cannot immediately apply the Lemma above because the inequality of the Lemma is not satisfied for L_1. We have $t = 1$, so $L_2 = (r; 0^2, 1^{r+3})$; this also does not satisfy (24) of the Lemma, so we continue, noting that $t = 1$ again. Hence, $L_3 = (2r-1; 1^2, 2^{r-1}, 1^4) = (2r-1; 1^6, 2^{r-1})$; again, the inequality fails, and now $t = 3$. Continuing, if $r \geq 5$ we have $L_4 = (5r-4; 4^6, 5^{r-5}, 2^4) = (5r-4; 2^4, 4^6, 5^{r-5})$ and this does satisfy the inequality; hence, so does L_i for all $i \geq 4$. Since the degrees strictly increase, they increase without bound and the orbit is infinite.

If $r = 4$, then $L_4 = (13; 1, 2, 2, 2, 3, 3, 3, 3, 3)$, $L_5 = (22; 3, 3, 3, 3, 4, 5, 5, 5, 6)$, and $L_6 = (40; 5, 5, 5, 6, 9, 9, 9, 10)$. The class L_6 does satisfy the conditions of the lemma, and therefore so does L_i for all $i \geq 6$. Since the degrees strictly increase, they increase without bound and the orbit is again infinite.

For $r = 3$, the Lemma is not useful; the inequality is never satisfied. The reader can check, though, that if we write $i = 2j + e$ with $e \in \{0, 1\}$, then $L_i = (2i-1; (j+e-1)^2, j^4, (j+e)^2)$ (for each L_i, we have $h = 2i - 1 - 2(j+e-1) - 2j = 1$), which also gives an infinite set. □

We remark that the infinity of (-1)-Weyl lines does not directly follow from the fact that the Weyl group is infinite: Y_9^5 is not a MDS and has an infinite Weyl group but has finitely many (-1)-curves, so there is a subtlety here.

6.3. Summary of Results in Sections 4 and 6

Case of (-1)-curves: numerical (-1)-classes: $\langle c, F\rangle = 3 - r$; (-1)-Weyl line: $\langle c, c\rangle = 3 - 2r$.

1. For Y_s^2: if C is irreducible, then C is a (-1)-curve $\iff$ C is a (-1)-Weyl line (Proposition 6).
2. For $Y_s^2, s \leq 8$: c is represented by a (-1)-curve $\iff$ $(c, c) = (c, K) = -1$ (Proposition 7).
3. We note that if c is a numerical (-1)-class in Y_s^2, then either c is a (-1)-Weyl line or $s = 8$ and $c = -K$.
4. For Y_{r+3}^r: if C is irreducible, then C is a (-1)-curve $\iff$ C is a (-1)-Weyl line $\iff$ $\langle C, F\rangle = 3 - r$ (Theorem 9).
5. For Y_{r+4}^r, $r \leq 5$: if C is irreducible, then C is a (-1)-curve $\iff$ C is a (-1)-Weyl line $\iff$ $\langle C, F\rangle = 3 - r$ (Proposition 12).
6. For Y_{r+3}^r, $r \geq 2$: a class c with positive degree and multiplicities is a (-1)-Weyl class $\iff$ $\langle c, F\rangle = 3 - r$ and $\langle c, c\rangle = 3 - 2r$ (Proposition 10).
7. For Y_{r+4}^r, $r \leq 5$: a class c with positive degree and multiplicities is a (-1)-Weyl class $\iff$ $\langle c, F\rangle = 3 - r$ and $\langle c, c\rangle = 3 - 2r$ (Proposition 13).
8. If Y_s^r is a Mori Dream Space or Y_9^5, the only (-1)-curves are the ones in Example 1.
9. If $r \geq r + 5$ then Y_s^r has infinitely many (-1)-curves.

Case of (0)-curves: numerical (0)-classes: $\langle c, F\rangle = 2$; (0)-Weyl line: $\langle c, c\rangle = 2 - r$.

10. For Y_s^2: if C is irreducible, then C is a (0)-curve $\iff$ C is a (0)-Weyl line (Proposition 6).
11. For Y_s^2, $s \leq 8$: c is represented by a (0)-curve $\iff$ $\langle c, c\rangle = 0$ and $\langle c, K\rangle = -2$ (Proposition 7).
12. For Y_{r+3}^r: if C is irreducible, then C is a (0)-curve $\iff$ C is a (0)-Weyl line $\iff$ $\langle C, F\rangle = 2$ (Proposition 9).
13. For Y_{r+4}^r, $r \leq 5$: if c is Cremona-reduced and $\langle c, F\rangle = 2$, then $c = h - e_1$, (in the case Y_7^3) $c = F$, or (in the case Y_8^4) $c = 2F$ (Proposition 14).
14. For Y_{r+4}^r, $r \leq 5$: a curve C whose class c is Cremona-reduced with non-negative multiplicities is a (0)-Weyl line if and only if $\langle c, c\rangle = 2 - r$ and $\langle c, F\rangle = 2$ (Corollary 5).
15. If Y_s^r is a Mori Dream Space, then the only (0)-curves are the ones in Example 1 together with (0)-curves in the anticanonical curve class F in Y_7^3 and $2F$ in Y_8^4.
16. Y_s^r is a Mori Dream Space if and only if it has finitely many classes of (0)-curves.

Case of (1)-curves: numerical (1)-classes: $\langle c, F\rangle = r+1$; (1)-Weyl line: $\langle c, c\rangle = 1$.

17. For Y_s^2: if C is irreducible, then C is a (1)-curve $\iff$ C is a (1)-Weyl line (Proposition 6).
18. For Y_s^2, $s \leq 8$: there is a (1)-curve with class c $\iff$ $(c, c) = 1$ and $(c, -K) = 3$ (Proposition 7).
19. For Y_{r+3}^r: if C is an irreducible curve with class c, then $\langle c, F\rangle = r+1 \iff C$ is a (1)-Weyl line or $2|r+1$ and c is in the Weyl orbit of a class of the form $c = mF + c'$, where $m \leq (r+1)/4$ and $c' = (e; \underline{n})$, where $e = (r+1)/2 - 2m$, $n_i = 0$ for $i \geq r-1$, and $e = \sum_i n_i$ (Proposition 11).
20. Y_s^r is a Mori Dream Space if and only if it has finitely many classes of (1)-curves.

7. Applications

A *movable* curve in Y_s^r is a curve that moves in a large enough family so that a general point is contained in at least one member of the family.

In Section 7.1, we give applications of the theory of movable curves to the faces of the effective cone of divisors, while in Section 7.2 we give applications of the theory of rigid curves to the dimensionality problems of linear systems with multiple points in $\mathbb{P}^r$, to resolutions of singularities, and we conclude with some examples of vanishing theorems. The most important results of this section are contained in Theorem 13 (presented directly below) and Theorems 15 and 16, which prove Theorem 3. Theorem 15 gives a one-to-one correspondence between faces of this cone and the collection of (0)- and (1)-Weyl lines. More precisely, a divisor $D \in Pic(Y_{r+3}^r)$ is effective if and only if $D \cdot C \geq 0$ for every curve C that is a (0)- or a (1)-Weyl line on Y. In general, if Y is not a Mori Dream Space (arbitrary s), then Theorem 16 implies that (0)- and (1)-Weyl lines define an infinite set of conditions for the effectivity of a divisor.

We now present the proof of Conjecture 1 for $i = 0, 1$, which we will use later.

Proposition 19. *Conjecture 1 is true for $i = 0$ and $i = 1$: for all r and s, every (i)-Weyl line is an (i)-curve.*

Proof. The proof of (a) is relatively straightforward after we make the following observation. First, the statement is true for the actual line through 1 or 0 points (the initial case of a (0)-Weyl line or a (1)-Weyl line): the normal bundle of a line is a direct sum of $\mathcal{O}(1)$s and, upon blowing up, the normal bundle is twisted by $\mathcal{O}(-1)$.

The argument proceeds by induction on the number of standard Cremona transformations required to arrive at the given (i)-Weyl line. When that number is zero, we have the initial case above. When that number is one, the curve is always a rational normal curve, and has the expected balanced normal bundle too.

If C is an (i)-Weyl line that is obtained by applying k standard Cremona transformations to a line, then we may write $C = \phi(C')$, where C' is obtained by applying $k-1$ such transformations to a line, and ϕ is a standard Cremona transformation. By induction, C' will be an (i)-curve in Y_s^r.

The standard Cremona transformation ϕ on Y_s^r, as explained in Section 2.2, is factored by systematically blowing up and down the proper transforms of the linear subspaces of codimension at least two spanned by subsets of the $r+1$ initial base points. Hence, if these proper transforms are disjoint from the curve C', the curve C will be isomorphic to C', and will have the same normal bundle, proving that C is also an (i)-curve in the transformed Y_s^r. It suffices to show that C' is disjoint from the proper transforms of the codimension-two subspaces, since these contain the others.

Let ψ be the composition of standard Cremona transformations that take C' back to a line. To show that C' is disjoint from the finite number of proper transforms of the codimension-two subspaces $\{L_\alpha\}$ in question, it suffices to show that $\psi(C')$ is disjoint from the transforms $\psi(L_\alpha)$, which are a finite number of codimension-two subvarieties in Y_s^r.

This is obvious for $i = 1$: the line $\psi(C')$ is a general line, and can be chosen to be disjoint from any finite set of codimension-two subvarieties. For $i = 0$, the line $\psi(C')$ is a line through one of the s points, but is otherwise general; in this case, a simple dimension count shows that the general member of the $(r-1)$-dimensional family of such lines can be chosen to be disjoint from the finite set of codimension-two subvarieties as well. □

Corollary 11. *For $r \geq 3$ and $s \geq r+6$, or $r \geq 5$ and $s \geq r+5$, there are infinitely many (-1)-curves in Y_s^r.*

Proof. We know from the above that there are infinitely many (0)-Weyl line classes in these cases, and using Proposition 19, we see that these are all (0)-curves. By imposing an additional point of multiplicity one, we will create infinitely many (-1)-Weyl line classes, all of which are (-1)-curves. □

The following Theorem 13 was first proved by Mukai in [25], using the infinity of (-1)-Weyl divisors. In this section, we will prove this result using the theory of movable curves that we introduced in this paper.

Theorem 13. *If $F^2 = \langle F, F \rangle \leq 0$, (i.e., $r = 2$ and $s \geq 9$; $r = 3, 4$ and $s \geq r+5$; or $r \geq 5$ and $s \geq r+4$) then Y_s^r is not a Mori Dream Space.*

Proof. If $F^2 \leq 0$, Corollary 9 implies that there are infinitely many classes of (0)-Weyl lines. Each such curve class gives a facet of the cone of effective divisors of Y_s^r (see Theorem 16). Effective cone facets given by (0)-classes are independent, via Proposition 18. We conclude that Y can not be a Mori Dream Space because its effective cone is not rational polyhedral. □

Question 3 asks whether the extremal rays of the cone of movable curves in all Mori Dream Spaces Y_s^r are (0)-Weyl lines and (1)-Weyl lines.

Remark 5. *In Y_{r+3}^r, Theorem 9 reveals that (0)-curves are (0)-Weyl lines and in even dimensions and Theorem 10 shows that (1)-curves are (1)-Weyl lines. Remark 4 shows that the F-class in Y_6^3 contains a (1)-curve that is not (1)-Weyl line. However, this (1)-curve is not an extremal ray because it is the sum of two (0)-Weyl lines in Example 1.*

7.1. Effective Cone of Divisors

In this section, we discuss the theory of movable curves and applications to the cone of effective divisors when $s \leq r+3$. There is a vast literature studying the geometry of the space Y_{r+3}^r, and for a more proper list of citations we will refer you to [12] (Section 0). The chamber decomposition of the effective divisorial cone of Y_{r+3}^r is exposed in [29].

Also, Mukai proves that Y_8^4 is isomorphic to the moduli space S of rank two torsion-free sheaves G, with prescribed Chern classes $c_1(G) = -K_Y$ and $c_2(G) = 2$. In [26], the authors use the work of Mukai and Gale duality, which relates spaces Y_8^4,Y_8^2, and S, in order to study the effective cone of divisors for Y_8^4. In particular, Gale duality gives a correspondence between extremal rays for the effective cone divisors on Y_8^4 and curves C in Y_8^2 for which $C \cdot C = 0$ and $C \cdot K_Y = -2$. These curves are (0)-curves on Y_8^2, and we prove they correspond to (0)-Weyl lines. The correspondence is more general, as Corollary 4, Section 3.3, proves that (0)-divisorial classes are equivalent to (0)-Weyl hyperplanes on Y_s^2.

In recent work [26,27], the authors use a different approach to the birational geometry of blown-up projective spaces and give a geometric meaning to the walls of the Mori Chamber Decomposition. From the birational geometry point of view, Theorem 15 proves that the faces of the effective cones of divisors for Mori Dream Spaces are given by hyperplanes corresponding to i-curves, where $i \in \{0, 1\}$ for Mori Dream Spaces. Moreover, all chambers are organized in hyperplane arrangements and each chamber corresponds to some Weyl orbit of a curve class of the form $mh - e_1 - \ldots - e_{m+1}$ for some $1 \leq m \leq r-1$. From this point of view, the birational geometry of these blown-up spaces is completely understood with the tools of Weyl actions on curves developed here. The geometrical meaning behind this observation is that each such curve sweeps out a fixed subvariety of dimension m, which is part of the stable base locus of these blown-up spaces and therefore determines a wall in the Mori Chamber Decomposition. The authors do not know other spaces that satisfy this property.

These observations motivate the following considerations.

In this section, we will prove that in $Y = Y^r_{r+3}$ the collection of (0)- and (1)-curves, $\mathcal{C}$, give all faces of the effective cone of divisors. In order to see this, let us review known results about the faces of effective cones. We recall that in these cases the movable cone of divisors (i.e., divisor classes that do not contain divisorial base components) consists of effective divisors that have positive intersection with all other effective divisors of Y with respect to the Dolgachev–Mukai pairing ([30], Theorem 4.7): if $\mathrm{Eff}_{\mathbb{R}} Y^{\wedge} = \{D \in A^1(Y) \mid \langle D, \mathrm{Eff}_{\mathbb{R}} \rangle_1 \geq 0\}$, then

$$\mathrm{Mov}(Y) = \mathrm{Eff}_{\mathbb{R}} Y \cap \mathrm{Eff}_{\mathbb{R}} Y^{\wedge}.$$

For Y^r_{r+3}, the (-1)-Weyl hyperplanes generate the Cox ring [24] (Theorem 1.2).

We first introduce coordinates for the divisor $D \in \mathrm{Pic}(Y)$, ie $D := dH - \sum_{i=1}^{r+3} m_i E_i$. We recall the following.

Theorem 14 ([31], Theorem 5.1)**.**

Case 1. If the number of points $s = r+2$, a divisor D is effective if and only if

$$(A_i) : d \geq m_i \text{ and } rd \geq \sum_{j=1, j\neq i}^{n+3} m_j$$

Case 2. If the number of points $s = r+3$, then a divisor D is effective if and only if inequalities (A_i) together with $(B_{n,I(t)})$ hold, where

$$(B_{n,I(t)}) : \; k_{t,I(t)} := [(t+1)n - t]d - t\sum_{i=1}^{r+3} m_i - \sum_{i\in I} m_i \geq 0$$

for every set $I(t)$ so that $|I(t)| = r - 2t + 1$, where $-1 \leq t \leq l + \alpha$ and $r = 2l + \alpha$, $\alpha \in \{0, 1\}$.

We recall that in Mori Dream Space cases, the Weyl group and therefore the Weyl orbits of a general line h and a pencil of lines through one point $h - e_i$ are finite.

Denote the collection of all (0)- and (1)-curves on the Mori Dream Space Y by

$$\mathcal{E} := \{\text{(i)-Weyl line for } i \in \{0,1\}\} \subset A^{r-1}(Y). \tag{26}$$

Introduce $\mathcal{E}_{\geq 0}$ to be the collection of divisors that intersect every (0)-curve and (1)-curve positively:

$$\mathcal{E}_{\geq 0} := \{D \in \mathrm{Pic}(Y)_{\mathbb{R}} \text{ so that } (D \cdot C) \geq 0 \text{ for every } C \in \mathcal{E}\} \subset \mathrm{Pic}(Y)_{\mathbb{R}}.$$

Consider also the boundary

$$\mathcal{E}^{\partial} := \{D \in \mathcal{E}_{\geq 0} \text{ so that } (D \cdot C) = 0 \text{ for some } C \in \mathcal{E}\} \subset \mathrm{Pic}(Y)_{\mathbb{R}}.$$

We will now prove the main result of this section.

Theorem 15. *If $s \leq r+3$, then*

$$\mathit{Eff}_{\mathbb{R}} Y^r_s = \mathcal{E}_{\geq 0}.$$

i.e., (0)- and (1)-Weyl lines give the extremal rays for the cone of movable curves in $\mathbb{P}^r$ with $r+3$ points blown up.

Proof. We will first compute the set $\mathcal{E} \subset A^{r-1}(Y)$ of Equation (26).

It is easy to see that (0)-Weyl lines consist only of a pencil of lines through one point and rational normal curves of degree r passing through $r+2$ general points.

$$\alpha_i = h - e_i \text{ and } \alpha_i' = nh - \sum_{j \neq i} e_j.$$

The (0)-Weyl lines α_i give the facets (A_i).

We will leave to the reader to check that the (1)-curves are of the form

$$\beta_{t,I(t)} = (t+1)r - t - \sum_{i \in I(t)} (t+1)e_i - \sum_{i \notin I(t)} te_i$$

for every $-1 \leq t \leq l + \alpha$ and $|I(t)| = r - 2t + 1$. Indeed, ordering the multiplicities in a decreasing order, one can see that performing a standard Cremona transformation on the last $r+1$ points yields

$$Cr(\beta_{t,I(t)}) = \beta_{t+1,I(t+1)}.$$

The (1)-Weyl line of minimal degree corresponds to $t = -1$, i.e., the general hyperplane class $\beta_{-1,I(-1)} = h$, while the (1)-Weyl line of maximal degree corresponds to t being a half of r, i.e., when $\alpha = 0$ (r is even) it is a quasihomogeneous curve $\beta_{l,I(l)}$, and if $\alpha = 1$, a homogeneous curve $\beta_{l+1,I(l+1)}$. The (1)-Weyl line $\beta_{t,I(t)}$ with $|I(t)| = r - 2t + 1$ give facets $(B_{r,I(t)})$. We obtain

$$\begin{aligned} \mathcal{E} &= \{\{\alpha_i, \alpha_i'\}_{1 \leq i \leq r+3}, \text{ and } \{\beta_{t,I(t)}\}_{-1 \leq t \leq l}\}, \\ \mathcal{E}_{\geq 0} &= \{D \in \mathrm{Pic}(Y)_{\mathbb{R}} \text{ so that } D \cdot C \geq 0 \text{ for } C \in \mathcal{E}\}, \\ &= \{D \in \mathrm{Pic}(Y)_{\mathbb{R}} \text{ satisfying } (A_i) \text{ and } (B_{n,I(t)})\}, \\ &= \mathrm{Eff}_{\mathbb{R}} Y. \end{aligned} \tag{27}$$

□

Remark 6. *For the Mori Dream Spaces Y_7^3, we can find 4 types of curves of odd degree up to 7 as (1)-Weyl lines and 8 types of curves of odd degree up to 15 as (0)-Weyl lines. For Y_8^4, there are seven types of (0)-Weyl lines up to degree 19. The Cox ring of Y_7^3 is generated by the (−1)-Weyl divisors together with the anticanonical divisor, and for Y_8^4 the effective cone was considered in [26].*

It is natural to ask the following question:

Question 3. *Does the collection of (0)- and (1)-Weyl lines form all the extremal rays for Y_7^3 and Y_8^4?*

Corollary 12. *The faces of the effective cone of divisors on $\mathbb{P}^r$ with $r+3$ points are the components of $\mathcal{E}^{\partial}$.*

If Y is not a Mori Dream Space, we define the infinite collection of curves

$$\begin{aligned} \mathcal{I} &:= \{\text{(0)-Weyl lines and (1)-Weyl lines}\} \subset A^{r-1}(Y) \\ \mathcal{I}_{\geq 0} &:= \{D \in \mathrm{Pic}(Y)_{\mathbb{R}} \text{ so that } D \cdot C \geq 0 \text{ for } C \in \mathcal{I}\} \subset \mathrm{Pic(Y)}. \end{aligned} \tag{28}$$

i.e., $\mathcal{I}_{\geq 0}$ consists of all divisors that intersect every curve of $\mathcal{I}$ positively. The next theorem gives an infinite set of necessary conditions for the effectivity of a divisor on Y_s^r.

Theorem 16. *Let $Y = Y_s^r$; then $\mathrm{Eff}_{\mathbb{R}} Y \subset \mathcal{I}_{\geq 0}$.*

Proof. We use that if C is an (i)-Weyl line ($i = 0, 1$) and q is a general point in Y, then there is another (i)-Weyl line through q with the same class; these (i)-curves move in an appropriately large family.

Assume by contradiction that an (i)-Weyl line has negative intersection with a divisor D; then, choosing a point q not in D we find an (i)-Weyl line with the same class (and hence also negatively meeting D) through q, which is a contradiction. □

7.2. Base Locus of Effective Divisors

We recall that for the Picard group of Y, $\mathrm{Pic}(Y) =< H, E_1, \cdots, E_s >$, where H is the general hyperplane class and E_i are the exceptional divisors. We recall Definition 3: a $(-1)-$ Weyl hyperplane is the Weyl orbit of the exceptional divisor.

Let G be a (-1)Weyl hyperplane; then, there exists a Weyl group element w with $G = w(E_1)$. We recall that $\langle -, - \rangle_1$ is a bilinear form on the Picard group of Y invariant under the Weyl group action. In particular,

$$\langle G, G\rangle_1 = \langle w(E_1), w(E_1)\rangle_1 = \langle E_1, E_1\rangle_1 = -1.$$

Theorem 17. *If a (-1)-Weyl line C and a (-1)-Weyl hyperplane G are part of the base locus of the linear system $|D|$ of an effective divisor D, then*

$$(C \cdot G) = 0.$$

Proof. Assume by contradiction that there exists an effective divisor D containing in the base locus the Weyl hyperplane G and the (-1) Weyl curve C, with $(G \cdot C) \geq 1$. Let $G = \sigma(E_1)$ for $\sigma \in \mathrm{Weyl}(Y)$, and note that $\langle G, G\rangle_1 = -1$. By [12] (Lemma 7.1), we have for some $p > 0$

$$-p = \langle D, G\rangle_1 = \langle D, \sigma(E_1)\rangle_1 = \langle \sigma^{-1}(D), E_1\rangle_1.$$

The bilinearity implies that

$$\langle D - pG, G\rangle_1 = \langle D, G\rangle_1 - p\langle G, G\rangle_1 = 0.$$

We know that $D - pG$ is an effective divisor, and therefore $\sigma^{-1}(D - pG)$ is also an effective divisor. Moreover,

$$\langle \sigma^{-1}(D - pG), E_1\rangle_1 = \langle D - pG, \sigma(E_1)\rangle_1 = \langle D - pG, G\rangle_1 = 0.$$

Therefore the divisor $\sigma^{-1}(D - pG)$ is based at at most $s - 1$ points, missing the point E_1.

We assume that

$$1 \leq (D \cdot C) = (\sigma(E_1) \cdot C) = (E_1 \cdot \sigma^{-1}(C)).$$

The positive intersection $(E_1 \cdot \sigma^{-1}(C))$ implies that $\sigma^{-1}(C)$ is an effective curve (i.e., not contracted by the Weyl group element σ). Therefore, $\sigma^{-1}(C)$ is a (-1) Weyl line that passes through the point E_1. Moreover, since $p > 0$ we have

$$(\sigma^{-1}(D - pG) \cdot \sigma^{-1}(C)) = ((D - pG) \cdot C) = (D \cdot C) - p(G \cdot C) < 0.$$

We conclude that $\sigma^{-1}(C)$ is the base locus of the effective divisor $\sigma^{-1}(D - pG)$. This is a contradiction, since $\sigma^{-1}(C)$ passes through the point E_1 while the divisor $\sigma^{-1}(D - pG)$ does not pass through the point E_1. Therefore, $\sigma^{-1}(C)$ is a family of curves sweeping out the effective divisor $\sigma^{-1}(D - pG)$, and this is a contradiction. □

Theorem 18. *Let G_1 and G_2 be two (-1)-Weyl hyperplanes in the base locus of an effective divisor D. Then,*

$$\langle G_1, G_2\rangle_1 = 0$$

Proof. By [12] (Lemma 7.1), we know

$$\langle D, G_1\rangle_1 < 0$$
$$\langle D, G_2\rangle_1 < 0$$

and we claim that G_1 and G_2 are orthogonal with respect to the Dolgachev–Mukai pairing.

Let σ be an element of the Weyl group so that $G_1 = \sigma(E_1)$. Then, for some $p > 0$ (the multiplicity of containtment of G_1 in D)

$$-p = \langle D, G_1\rangle_1 = \langle D, \sigma(E_1)\rangle_1 = \langle \sigma^{-1}(D), E_1\rangle_1.$$

Therefore,

$$\langle D - pG_1, G_1\rangle_1 = \langle D, G_1\rangle_1 - p\langle G_1, G_1\rangle_1 = 0.$$

Moreover,

$$\langle \sigma^{-1}(D - pG_1), E_1\rangle_1 = \langle D - pG_1, \sigma(E_1)\rangle_1 = \langle D - pG_1, E_1\rangle_1 = 0.$$

Therefore the effective divisor $\sigma^{-1}(D - pG_1)$ is based at at most $s - 1$ points, missing the point E_1. Assume by contradiction that

$$1 \leq \langle G_1, G_2\rangle_1 = \langle \sigma(E_1), G_2\rangle_1 = \langle E_1, \sigma^{-1}(G_2)\rangle_1.$$

Since $p > 0$, we obtain

$$\langle \sigma^{-1}(D - pG_1), \sigma^{-1}(G_2)\rangle_1 = \langle D - pG_1, G_2\rangle_1 = \langle D, G_2\rangle_1 - p\langle G_1, G_2\rangle_1 < 0.$$

By [12] (Lemma 7.1), we have that the (-1) Weyl divisor $\sigma^{-1}(G_2)$ splits off the divisor $\sigma^{-1}(D - pG_1)$. This gives a contradiction, since the divisor $\sigma^{-1}(G_2)$ passes through the point E_1 and the divisor $\sigma^{-1}(D - pG_1)$ does not pass through the point E_1. □

Let D be an effective divisor on Y. Necessary conditions for effectivity are given in Section 7.1. We remark that Theorem 18 does not hold for curves contained in an effective divisor D, with respect to the bilinear form $\langle -, -\rangle$.

Example 8. *Consider the effective divisor* $D := 6H - \sum_{i=1}^{9} 4E_i$ *in* $\mathbb{P}^5$*. We can see that two types of* (-1) *Weyl lines are contained in D: one is C (the rational normal curve of degree* 5 *passing through first eight points) and the other is* L_{19} *(the line through points* 1 *and* 9*), and both are contained in the base locus of D; however,* $\langle C, L_{19}\rangle_1 = 5 - 4 = 1 > 0$.

Moreover, ref. [15] (Corollary 8.3) shows that if two Weyl surfaces (i.e., the Weyl orbit of a plane through three fixed points) can not be a fixed part of a divisor in $\mathbb{P}^4$ based at eight points, then their intersection in the Chow ring is zero.

This makes us predict that for an effective divisor D on Y, it has a resolution of singularities by blowing up its base locus for D, with the blown-up space $\tilde{Y}$ being smooth. In the next subsection, we show that (-1)-curves create a contribution to the dimension of the linear system of an effective divisor D.

7.3. Resolutions of Singularities

In this subsection, we discuss applications of rigid curves to obtain Riemann–Roch statements for divisors in $\mathbb{P}^r$ interpolating points. The most important result we prove is Corollary 13, which gives an *expected dimension formula* for divisors in $\mathbb{P}^r$ whose base locus contain (-1)-curves together with linear subspaces spanned by the base points, and we pose Question 4.

Let D be an effective divisor on $Y = Y^r_s$; necessary conditions for effectivity are given in Section 7.1. For Mori Dream Spaces Y^r_s and for $\mathbb{P}^2$ and $\mathbb{P}^3$ with an arbitrary number of points, conjectures regarding the dimension $h^0(D)$ of the space of global sections of an effective divisor were formulated ([15,31,32]). For $\mathbb{P}^2$ and $\mathbb{P}^3$, the main observation was that for Cremona-reduced divisors, the Weyl base locus can only be linear. As Examples 9 and 10 suggest, this does not hold in higher dimensions, and therefore an explicit formula for the action of the Weyl group ([15]) must be given first.

Remark 7. *Example 10 shows that in $\mathbb{P}^5$ the Cremona-reduced divisor D can have in its base locus (-1)-curves that are not lines, which contribute to $h^1(D)$. In order to address this problem in this section, we consider more general resolutions of singularities. It has been exploited before that a dimension count can imply vanishing theorems and vanishing theorems imply positivity results for divisors. In Examples 9 and 10, we show explicitly how to obtain vanishing theorems from the dimension count.*

Proposition 20 proves that if C is a (-1)-curve and $(C \cdot D) = -k_C < 0$, then by Bezout C is a fixed part of D at least k times. We blow up the space Y along all (-1)-curves contained in the base locus of an effective divisor D, and we denote by E_C the exceptional divisor created after blowing up the curve C.

The exceptional divisor created after blowing up the (i)-curve C for $i \in \{-1, 0, 1\}$ is

$$E_C \cong \mathbb{P}^1 \times \mathbb{P}^{r-2}$$

and it comes equipped with the information of the normal bundle that

$$E_C|_{E_C} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^{n-2}}(i, -1).$$

Proposition 20. *Let D be an effective divisor and C a (-1)-curve so that $(D \cdot C) = -k < 0$. Then, the curve C is contained in the base locus of the divisor D at least k times.*

Proof. For any $0 \leq s \leq k-1$, we have the short exact sequence

$$0 \to D - (s+1)E_C \to D - sE_C \to (D - sE_C)|_{E_C} \to 0$$

$$(D - sE_C)|_{E_C} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^{r-2}}(s-k, k)$$

Since $s < k$, we have $h^0(\mathcal{O}^1_P(s-k) = 0$, so by the Kunneth formula $h^0(D - sE_C) = 0$. This implies that $h^0(D) = h^0(D - E_C) = \ldots = h^0(D - kE_C)$. □

Let $\mathcal{G}_D := \{C \mid C \text{ be a (-1)-curve on } Y, \text{with } k_C = (-D \cdot C) \geq 1\} \subset A^{r-1}(Y)$. Let $\tilde{Y}$ be the blown up-space Y and denote by $D_{(1)}$ the proper transform of D under the blowup of all its base loci.

$$D_{(1)} := D - \sum_{C \in \mathcal{G}_D} k_C E_C. \tag{29}$$

We will denote the following:

$$h^i(D_{(1)}) := \dim H^i(Y_{(1)}, \mathcal{O}(D_{(1)})).$$

In the planar case $\mathbb{P}^2$, the Segre–Gimigliano–Harbourne–Hirschowitz conjecture predicts that $h^1(D_{(1)}) = 0$.

We recall from [33] (Theorem 2.1) that if D is an effective divisor on $Y = Y^r_s$, and $D_{(1)}$ the proper transform transform of D (29), then

$$
\begin{aligned}
h^1(D) &= \sum_{C\in\mathcal{G}_D} \binom{r+k_C-2}{r} + h^1(D_{(1)}) - h^2(D_{(1)}).\\
h^i(D_{(1)}) &= 0 \text{ for every i } \geq 3\\
\chi(D_{(1)}) &= \chi(D) + \sum_{C\in\mathcal{G}_D} \binom{r+k_C-2}{r}.
\end{aligned}
\tag{30}
$$

The next corollary is a consequence of [32] (Corollary 4.9); we recall from paper [32] (Definition 3.2) the notion of *ldim D*, which is the linear expected dimension, taking into account linear base loci.

Corollary 13. *Let D be an effective divisor on Y, and denote by $\bar{D}$ the proper transform of D under the blow up of its linear base locus of arbitrary dimension and all (-1)-curves. Then,*

$$\chi(\bar{D}) = \mathrm{ldim} D + \sum_{C\in\hat{\mathcal{G}}_D} \binom{r+k_C-2}{r}.$$

If we define $\hat{\mathcal{G}}_D \subset \mathcal{G}_D$ to be the subset of curves of degree bigger than one (the non-line cases), then we recall that [32] implies

$$h^0(D) = \mathrm{ldim} D + \sum_{C\in\hat{\mathcal{G}}_D} \binom{r+k_C-2}{r} + \sum_{\rho=1}^{r} (-1)^{\rho-1} h^\rho(\bar{D}).$$

It is important to remark that linear subsets of dimension k spanned by $k+1$ subsets of the base points do not intersect (-1)-Weyl lines. Indeed, if $k \leq r-2$ this follows from the generality of the points, while if $k = r-1$ this follows from Theorem 17. This observation allows one use the proof of [32] (Corollary 4.9), replacing $D_{(1)}$ by the proper transform of D under the blowup of all (-1)-curves in the base locus.

Question 4. *Let D be an effective divisor on Y. Then, is it true that $h^1(D_{(1)}) = 0$?*

Is it true that the effective divisor $D = 4H - 2\sum_{i=1}^{9} E_i$ in $\mathbb{P}^3$ has $h^1(D) = 1$ but $D \cdot C \geq -1$ for any (-1)-curve C?

Examples

So far, dimensionality problems for divisors and vanishing theorems such as the one predicted by Question 4 have only been computed for divisors that a have linear base locus. The dimension of the space of global sections in Examples 9 and 10 were known, and in this section we show how to prove vanishing theorems. The technique is to start from a divisor $\tilde{F}_0$ whose cohomology is known and to use several short exact sequences to increase some of the multiplicities. As long as the cohomology for the restricted divisors $\tilde{F}_0|_{E_i}$ in the short exact sequences are known, then one can conclude vanishing theorems. Question 4 holds for the next examples.

The first example discusses a divisor in $\mathbb{P}^4$ with 7 points, and in the article [15] we analyzed Cremona images of linear spaces in $\mathbb{P}^4$ with up to eight points. Those involving only seven points are of several types. There are three types of divisors (the linear spaces, the double cone over a conic, and the secant variety to the rational normal quartic curve through seven points). There are also surfaces: the 2-planes and the cone over the twisted cubic. As for curves, there are only the lines and the rational normal curves. In the example below, one only has the curve cases appearing as base loci.

Example 9. *Consider the effective divisor $D := 10H - \sum_{i=1}^{7} 6E_i$ in $\mathbb{P}^4$. One can easily see that it is Cremona-reduced and D contains in its base locus all (-1)-curves on Y_7^4, i.e., the $\binom{7}{2} = 21$ lines passing through two base points and the double rational normal curve passing through all seven points. Each of these double (-1)-curves make $h^0(D)$ increase by $\binom{4+2-1}{4} = 1$; the Formula (13) gives that $h^0(D) = \binom{14}{4} - 7\binom{9}{4} + 21 + 1 = 141$, which also agrees with a computer check. Moreover, one can show by the same method as the next Example 10 that $h^i(D_{(1)}) = 0$ for all $i \geq 1$, where $D_{(1)}$ is the proper transform under the blowup of all 22 (-1)-curves, so it satisfies Question 4.*

Example 10. *We finish this paper with an example that illustrates the utility of this approach in proving vanishing theorems.*

The space Y_9^5 is not a Mori Dream Space, and Corollary 8 applies. We recall that the only (-1)-curves are lines through two points and the rational normal curve of degree 5 passing through eight points. In this example, to answer Question 4, we will use four restriction sequences to divisors E_8 and E_9, and after each such sequence we eliminate the simple base locus created in the kernel divisor.

Consider the effective divisor $D := 6H - \sum_{i=1}^{9} 4E_i$ in $\mathbb{P}^5$, with

$$\dim H^0(\mathbb{P}^5, \mathcal{O}(D)) = 3.$$

As explained in [32] (Example 6.3), if C is the elliptic normal curve of degree 6 through nine points, (which has the anticanonical curve class F), then the secant variety $\sigma_2(C)$ is a threefold, which is a complete intersection of two hypersurfaces of degree 3, $G_1 = 0$ and $G_2 = 0$, and the three sextics $G_1^2 = 0$, $G_1G_2 = 0$, and $G_2^2 = 0$ generate the space of global sections of D.

The divisor D is Cremona-reduced and D contains in its base locus all (-1)-curves on $\mathbb{P}^5$ blown up at nine points with multiplicity 2, i.e., the $\binom{9}{2} = 36$ lines passing through two base points and the nine rational normal curves passing through eight of the nine points. This divisor has $\chi(D) < 0$; however, the set of $36 + 9$ double (-1)-curves will make $h^0(D)$ increase, and we have that Corollary 13 gives

$$\dim H^0(\mathbb{P}^4, \mathcal{O}(D)) = \binom{11}{5} - 9\binom{8}{5} + 36 + 9 = 3.$$

We will now prove that the dimension count will imply vanishing theorems as predicted in the question above. We introduce the following notation:

$$F_0 := 6H - \sum_{i=1}^{7} 4E_i - 2E_8 - 2E_9$$

and $\tilde{F}_0 := F_0 - \sum_{i,j=1}^{7} 2E_{ij}$ is the strict transform under the blowup of its base locus, consisting of $\binom{7}{2} = 21$ lines through the first seven points. This divisor is only linearly obstructed and by [32] (Theorem 5.3) we know that $h^0(\tilde{F}_0) = \binom{11}{5} - 7\binom{8}{5} - 2\binom{6}{5} + \binom{7}{2} = 79$ and $h^i(\tilde{F}_0) = 0$ if $i > 0$ by [34] (Theorem 1.6). Introduce

$$F_1 := \tilde{F}_0 - E_8 = 6H - \sum_{i,j=1}^{7} 4E_i - 3E_8 - 2E_9 - \sum_{i=1}^{7} 2E_{ij}.$$

We have the short exact sequence

$$0 \to F_1 \to \tilde{F}_0 \to \tilde{F}_0|_{E_8} \to 0.$$

The restriction is a quadric divisor in $E_8 \cong \mathbb{P}^4$ and $h^0(\tilde{F}_0|_{E_8}) = h^0(2h) = \binom{6}{4} = 15$. One obtains

$$\begin{aligned} h^0(F_1) &= h^0(\tilde{F}_0) - h^0(\tilde{F}_0|_{E_8}) + h^1(F_1) \\ &\geq h^0(\tilde{F}_0) - h^0(\tilde{F}_0|_{E_8}) = 79 - 15 = 64 \end{aligned} \tag{31}$$

Moreover, F_1 has only simple obstructions, so by a restriction sequence to the exceptional divisor we obtain that $h^i(F_1) = h^i(\tilde{F}_1)$ for any i, where $\tilde{F}_1 := F_1 - \sum_{i=1}^{7} E_{i8} - E_{C_{\hat{9}}}$ is the strict transform under the blowup of seven lines L_{i8} and rational normal curve skipping the ninth point. We further restrict the divisor $\tilde{F}_2$ to E_8 and denote by F_2 the residual divisor

$$F_2 := \tilde{F}_1 - E_8 = 6H - \sum_{i=1}^{8} 4E_i - 2E_9 - 2\sum_{i,j=1}^{7} E_{ij} - \sum_{i=1}^{7} E_{i8} - E_{C_{\hat{9}}}.$$

Running the same argument with

$$0 \to F_2 \to \tilde{F}_1 \to \tilde{F}_1|_{E_8} \to 0,$$

the restriction is a divisor in $\mathbb{P}^4$ and $h^0(\tilde{F}_1|_{E_8}) = h^0(3h - \sum_{i=1}^{8} e_i) = \binom{7}{4} - 8 = 27$.

$$h^0(F_2) \geq h^0(\tilde{F}_1) - h^0(\tilde{F}_1|_{E_8}) = 64 - 27 = 37.$$

Moreover, $h^i(F_2) = h^i(\tilde{F}_2)$ for any i, where $\tilde{F}_2 := F_2 - \sum_{i=1}^{7} E_{i8} - E_{C_{\hat{9}}}$ is the strict transform under the blowup of its simple base locus. We restrict next to E_9 and denote by F_3 the kernel divisor

$$F_3 := F_2 - E_9 = 6H - \sum_{i=1}^{8} 4E_i - 3E_9 - 2\sum_{i,j=1}^{8} E_{ij} - 2E_{C_{\hat{9}}}$$

$$0 \to F_3 \to \tilde{F}_2 \to \tilde{F}_2|_{E_9} \to 0.$$

The restriction has $h^0(\tilde{F}_2|_{E_9}) = h^0(2h) = 15$ so $h^0(F_3) \geq h^0(\tilde{F}_2) - h^0(\tilde{F}_2|_{E_8}) = 37 - 15 = 22$ and $h^i(F_3) = h^i(\tilde{F}_3)$ for any i where $\tilde{F}_3 := F_3 - \sum_{i=1}^{8} E_{i9} - \sum_{i=1}^{8} E_{C_{\hat{i}}}$ is the strict transform under its simple base locus.

Finally, let $F_4 := \tilde{F}_3 - E_9$; consider the short exact sequence

$$0 \to F_4 \to \tilde{F}_3 \to \tilde{F}_3|_{E_9} \to 0.$$

The restriction $\tilde{F}_3|_{E_9}$ has degree 3 and 16 simple points, 8 coming from the restriction $e_i := E_9|_{E_{i9}}$ and the other 8 coming from the restriction $e_{c_{\hat{i}}} := E_9|_{E_{C_{\hat{i}}}}$

$$h^0(\tilde{F}_3|_{E_9}) = h^0(3h - \sum_{i=1}^{8} e_i - \sum_{i=1}^{8} e_{c_{\hat{i}}}) = 35 - 16 = 19$$

$$\begin{aligned} h^0(F_4) &= h^0(\tilde{F}_3) - h^0(\tilde{F}_3|_{E_9}) + h^1(F_4) \\ &\geq h^0(\tilde{F}_3) - h^0(\tilde{F}_3|_{E_9}) = 22 - 19 = 3 \end{aligned} \tag{32}$$

and $h^i(F_4) = h^i(\tilde{F}_4) = h^i(\tilde{D})$ for any i. Since $h^0(F_4) = h^0(D) = 3$, Equation (32) implies that $h^1(F_4) = h^1(\tilde{F}_4) = 0$, and therefore $h^2(\tilde{F}_4) = 0$. We conclude that the divisor D satisfies Corollary 8 and gives a positive answer to Question 4.

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