

## Exceptional Curves on Rational Numerically Elliptic Surfaces

BRIAN HARBOURNE\*

*Department of Mathematics, University of Nebraska,  
Lincoln, Nebraska 68588-0323*

AND

RICK MIRANDA\*

*Department of Mathematics, Colorado State University,  
Fort Collins, Colorado 80526*

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### 0. INTRODUCTION

A *numerically elliptic surface* is a complete smooth algebraic surface  $X$  over an algebraically closed field  $k$  with a proper morphism  $f: X \rightarrow C$  to a smooth curve  $C$  such that the general fiber of  $f$  is an integral curve of arithmetic genus 1. If the generic fiber is smooth the surface is called *elliptic*; otherwise it is called *quasi-elliptic*. The latter only can occur if  $k$  has characteristic 2 or 3, in which case the general fiber is a rational curve with an ordinary cusp [BM]. If no fiber of  $f$  contains an *exceptional curve* (i.e., a smooth irreducible curve isomorphic to  $\mathbb{P}^1$  and having self-intersection  $-1$ ), then  $X$  is said to be *minimal*; all elliptic surfaces will hereafter be assumed to be minimal.

One says that an elliptic surface  $X$  is *Jacobian* if the smooth points of the generic fiber  $X_n$  comprise the Jacobian curve of  $X_n$ . It is equivalent for  $X_n$  to have a rational point, or for  $f$  to have a section. For a rational Jacobian numerically elliptic surface, the exceptional curves are precisely the sections of the fibration, which provides a tool by which an enumeration of the exceptional curves on a rational Jacobian numerically elliptic surface can be carried out (cf. [MP, HL, MoP]). Whether Jacobian or not, a rational

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elliptic surface is the blowing up of nine points of the projective plane  $\mathbb{P}^2$ , so the set of exceptional curves is always geometrically important in order to understand the structure of  $X$  as a blowing up of  $\mathbb{P}^2$ . The point of this paper is to determine and enumerate the exceptional curves in the non-Jacobian case. Our approach is to recast combinatorially the problem of enumerating exceptional curves on rational minimal numerically elliptic surfaces, which we then solve using generating functions.

Although the complete result is complicated, the “generic result” is simpler to state, for which we recall some well-known facts. Let  $X$  be a rational minimal numerically elliptic surface with finitely many exceptional curves (i.e.,  $X$  is *extremal*; see the definition preceding Proposition 1.5). In this case  $X$  must have reducible fibers, and the intersection graph of such a reducible fiber is always one of those shown in Fig. 1, and since  $X$  is rational the graph can have at most nine vertices.

Given a reducible fiber  $F$ , the vertices of its intersection graph represent the irreducible components of  $F$ , while the number of edges connecting two vertices indicates the number of times the corresponding components of  $F$  meet. Because  $X$  is rational, there is a positive integer  $m$  (which we will refer to as the *multiplicity* of the fibration on  $X$ ) such that every fiber is linearly equivalent to  $-mK_X$ , where  $K_X$  is the canonical class of  $X$ ;  $X$  is Jacobian if and only if  $m = 1$ .

If  $X$  is not Jacobian, then  $m > 1$  and every fiber but one has at least one reduced component. Such fibers are said to have multiplicity 1. The remaining fiber is called a multiple fiber; its multiplicity is  $m$ , in the following sense. As an effective divisor, any fiber  $F$  is a sum of integral multiples of its irreducible components; the multiplicity is the g.c.d. of these multiples.

For nonmultiple fibers, the multiples with which each component must be taken are given in Fig. 1 by the numbers (also often called multiplicities but here referred to as *weights* to avoid ambiguities) written inside each vertex. For a multiple fiber, the weights must be multiplied by the multiplicity  $m$ . The remaining numbers appearing in Fig. 1 are simply for identification of the various vertices of each graph. To each intersection graph  $F$  (and hence to each fiber having graph  $F$ ) we associate the number  $s_F$  of vertices of weight 1; we refer to  $s_F$  as the *discriminant* of  $F$ .

Now for each graph  $F$  shown in Fig. 1, construct the polynomial  $P_F(t)$  in an indeterminate  $t$ ,

$$P_F(t) = (s_F)^{1/2} \prod_{i=0}^n (1 - t^{\mu_i}),$$

where the product is over all of the vertices  $v_0, \dots, v_n$  of  $F$ ,  $\mu_i$  being the weight of  $v_i$ . For each  $j \geq 0$ , let  $(Q_F)_j$  be the coefficient of the term  $t^j$  of the

Taylor series  $Q_F$  of  $(P_F)^{-1}$ . We now give the simplest general statement of our more complete results. The proof is deferred to Section III.

**THEOREM 0.1.** *Let  $X$  be a minimal rational extremal numerically elliptic surface, let  $m$  be the multiplicity of its fibration, and let  $s$  be the product of the discriminants of the nonmultiple reducible fibers of  $X$ . If  $m$  and  $s$  are relatively prime, then the number of exceptional curves on  $X$  is the product over the reducible fibers  $F$  of  $X$  of the coefficients  $(Q_F)_{m/m_F}$ , where  $m_F$  is the multiplicity of the fiber  $F$ .*

It is easy to check that  $(Q_F)_1 = s_F^{1/2}$  for each intersection graph  $F$  appearing in Fig. 1, which together with the theorem gives the following well-known result [HL, MP, MoP].

**COROLLARY 0.2.** *If  $X$  is a minimal rational extremal Jacobian numerically elliptic surface, then the number of exceptional curves on  $X$  is  $s^{1/2}$ , where  $s$  is the product of the discriminants of the reducible fibers of  $X$ .*

We also work out the enumeration of exceptional curves in cases not covered by Theorem 0.1; that is, when  $m$  and  $s$  are relatively prime. The reason we do so is that while  $m$  and  $s$  need not be relatively prime (but often are) when  $X$  is elliptic, if  $X$  is quasi-elliptic and not Jacobian, then  $m$  and  $s$  are relatively prime only if  $X$  has a fiber of type  $E_8$  (cf. Fig. 1) [HL, Sect. 8].

This paper is organized as follows: Section I deals with geometric background; Section II uses this background to formulate the problem combinatorially; Section III examines the foregoing in the context of extremal surfaces; Section IV discusses the existence of the surfaces; and Section V gives examples. Thanks go to David Klarner for consultations on generating functions, to the University of Nebraska at Lincoln College of Engineering for computer support, to Igor Dolgachev for bringing the article [Dy] to our attention, and to the Mountain West Algebraic Geometry Workshop for facilitating several meetings between the authors.

Hereafter,  $X$  will denote a minimal rational numerically elliptic surface, not necessarily with section.

## I. AN INTERSECTION-THEORETIC CHARACTERIZATION OF THE EXCEPTIONAL CURVES ON $X$

We begin with some definitions and notation:

$K_X$ : the canonical class of  $X$ ;

(−1)-class: a class  $E \in \text{Pic } X$  with  $E^2 = -1$  and  $E \cdot K_X = -1$ ;

- $M_1$ : the set of  $(-1)$ -classes of  $X$ ;
- $(-1)$ -curve or exceptional curve: a smooth rational curve  $E$  on  $X$  the class of which in  $\text{Pic } X$  is a  $(-1)$ -class;
- $\mathcal{E}$ : the set of classes of exceptional curves of  $X$ ;
- $(-2)$ -class: a class  $N \in \text{Pic } X$  with  $N^2 = -2$  and  $N \cdot K_X = 0$ ;
- $M_2$ : the set of  $(-2)$ -classes of  $X$ ;
- $(-2)$ -curve or nodal curve: a smooth rational curve  $N$  on  $X$  the class of which in  $\text{Pic } X$  is a  $(-2)$ -class;
- $\mathcal{A}$ : the set of classes of nodal curves of  $X$  (or, more abstractly, the vertices of a disjoint union of not necessarily distinct graphs of Fig. 1—see the remarks after Proposition I.5);
- $\mathcal{A}_+$  or *nodal cone*: the set  $\{D \in \text{Pic } X \mid D \cdot N \geq 0 \text{ for all } N \in \mathcal{A}\}$ ;
- $K^\perp$ : the classes of  $\text{Pic } X$  perpendicular to  $K_X$ ;
- $\Gamma$ : the subgroup of  $K^\perp$  generated by  $\mathcal{A}$  and  $K_X$  (note that  $\Gamma$  is generated by  $\mathcal{A}$  if  $m = 1$  or the multiple fiber of  $X$  is reducible);
- $\mathcal{A}^*$ : the free abelian group of functions from  $\mathcal{A}$  to  $\mathbb{Z}$ ;
- $\mathcal{A}_+^*$ : the semigroup in  $\mathcal{A}^*$  of functions which are nonnegative on  $\mathcal{A}$ ;
- $\mathcal{L}$ : the even unimodular rank 8 lattice of type  $E_8$ .

*Note 1.0.* By sending a curve to its divisor class we get injections (which for convenience we regard as inclusions)  $\mathcal{E} \subset M_1$  and  $\mathcal{A} \subset M_2 \subset K^\perp$  since the divisor classes of distinct reduced irreducible curves of negative self-intersection are distinct.

The following facts are well known.

**LEMMA I.1.** *Let  $X$  be a rational numerically elliptic surface.*

- (a) *The anticanonical class  $-K_X$  of  $X$  is numerically effective (i.e., meets any effective divisor nonnegatively) and effective (i.e., is linearly equivalent to an effective divisor).*
- (b) *If  $C \subset X$  is an irreducible reduced curve with  $C^2 < 0$ , then  $C \in \mathcal{A} \cup \mathcal{E}$ .*
- (c) *Every nodal curve is a component of the numerically elliptic fibration on  $X$ , and every component of a reducible fiber is a nodal curve.*
- (d) *The set  $\mathcal{A}$  is finite.*
- (e) *If  $v \in K^\perp$ , then  $v^2 \leq 0$ ; moreover,  $v^2 = 0$  if and only if  $v$  is a multiple of  $K_X$ .*
- (f)  *$K^\perp$  is a root lattice of type  $E_9$  ( $= \tilde{E}_8$ ) and  $K^\perp/(K_X) \cong \mathcal{L}$ .*
- (g) *For some positive integer  $m$  (called the multiplicity of the*

fibration) all fibers of the numerically elliptic fibration on  $X$  are linearly equivalent to  $-mK_X$ . The fibration has a section if and only if  $m=1$ , in which case every fiber has at least one reduced component. If  $m>1$ , there is always a unique fiber (called the multiple fiber) having no reduced components. A fiber never has more than nine components, and the intersection graph of a reducible fiber is always among those listed in Fig. 1. Moreover, if  $\{v_0, \dots, v_n\}$  are the prime divisors comprising the support of a reducible fiber  $F$ , and  $\{\mu_0, \dots, \mu_n\}$  are the weights of the corresponding vertices of the intersection graph of  $F$ , then  $F=\mu \sum_i \mu_i v_i$ , where  $\mu$  is either  $m$  or 1, according to whether  $F$  is a multiple fiber or not.

*Outline of Proof.* (a) By the Bombieri–Mumford formula for fibers, the class of a fiber on  $X$  is a multiple of  $K_X$ , so  $(K_X)^2=0$ . Using Riemann–Roch for surfaces and Castelnuovo’s criterion  $h^0(X, 2K_X)=0$  for rationality, we see that  $-K_X$  is effective, so fibers are antipluricanonical, whence  $-K_X$  is numerically effective.

(b) This follows from the adjunction formula, numerical effectiveness of  $-K_X$ , and the fact that integral curves have nonnegative genus.

(c) The first statement follows since the fibers are antipluricanonical, and the second statement follows from the classification of reducible fibers on numerically elliptic fibrations [BM].

(d) This follows from (c) since there can be only finitely many reducible fibers.

(e) Since  $K^\perp$  and indeed  $\text{Pic } X$  are isomorphic for all rational minimal numerically elliptic surfaces it suffices to prove this for some such  $X$ . Let  $X$  be Jacobian with a fiber of type  $\tilde{E}_8$  (i.e., the intersection graph of the fiber is the extended Dynkin diagram of type  $E_8$ ; see Fig. 1). Then  $K^\perp$  is generated by the components of the  $\tilde{E}_8$  fiber, and since  $K_X$  is a primitive element of  $\text{Pic } X$  the result follows from the lemma on p. 28 of [BM].

(f) The graphs displayed in Fig. 1 are the extended Dynkin diagrams of the Dynkin diagrams of the finite dimensional simple complex Lie algebras. In particular, the  $E_8$  Dynkin diagram is obtained from the graph shown in Fig. 1 of a fiber of type  $\tilde{E}_8$  by excluding a vertex (in this case unique) of weight 1. The lattice  $\mathcal{L}$  is just the free abelian group on the vertices of this  $E_8$  Dynkin diagram (and hence has rank eight) with the bilinear form  $(\cdot, \cdot)$  induced by the following rule. If  $a$  and  $b$  are different vertices, then  $a \cdot b$  is 1 if  $a$  and  $b$  are adjacent vertices and  $a \cdot b$  is 0 otherwise;  $a \cdot a = -2$ . But  $X$  is obtained by successively blowing up 9 (possibly infinitely near) points of  $\mathbb{P}^2$  so  $\text{Pic } X$  is free of rank 10, generated by  $e_0, \dots, e_9$ , where  $e_0$  is the class of a line and  $e_i$  is the total transform of the  $i$ th point blown up. The intersection form is induced by taking  $e_i \cdot e_j$  to be 0 if  $i \neq j$ , 1 if  $i = j = 0$ , and  $-1$  if  $i = j > 0$ . Now  $-K_X = 3e_0 -$

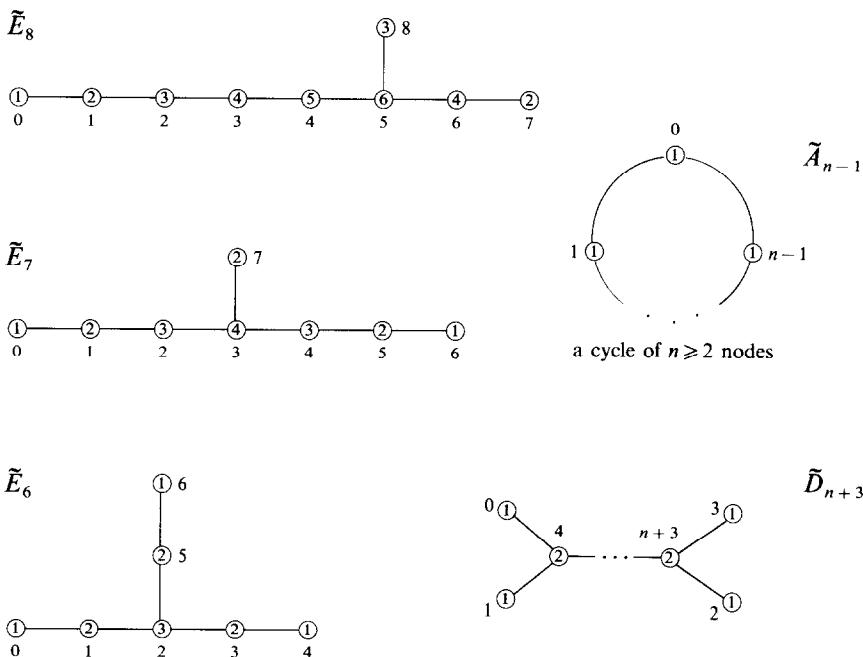


FIGURE 1

$e_1 - \dots - e_9$  and  $K^\perp$  is generated by  $r_0 = e_0 - e_1 - e_2 - e_3$ , and  $r_i = e_i - e_{i+1}$ ,  $i = 1, \dots, 9$ . We also note that  $-K_X = 3r_0 + 2r_1 + 4r_2 + 6r_3 + 5r_4 + 4r_5 + 3r_6 + 2r_7 + r_8$ . By (e) the radical of  $K^\perp$  is generated by  $K_X$ , so the intersection form on  $K^\perp$  descends to  $K^\perp/(K_X)$  and it is now easy to check that sending  $r_0, \dots, r_7$  to the vertices 0 through 7 of the  $E_8$  Dynkin diagram (see Fig. 1) induces an isomorphism of  $K^\perp/(K_X)$  with  $\mathcal{L}$ .

(g) See [HL] for proofs and references to original sources. ■

To distinguish which elements of  $M_1$  lie in  $\mathcal{E}$  we need to know:

LEMMA I.2. *Any  $(-1)$ -class is effective.*

*Proof.* Take  $E \in M_1$ . Riemann-Roch and Serre duality give

$$h^0(X, E) + h^0(X, K_X - E) \geq (1/2)(E^2 - E \cdot K_X) + 1 = 1.$$

But Lemma I.1(a) and  $-E \cdot -K_X = -1$  shows that  $-E = -K_X + (K_X - E)$  is not effective, implying that  $K_X - E$  is not effective; i.e.,  $h^0(X, K_X - E) = 0$ . Therefore  $h^0(X, E) \geq 1$  so  $E$  is effective. ■

We can now give a criterion due to Looijenga [L] for a  $(-1)$ -class to be exceptional.

**PROPOSITION I.3.** *The set  $\mathcal{E}$  of exceptional curves is precisely  $\mathcal{A}_+ \cap M_1$ .*

*Proof.* Since every exceptional class meets every nodal class non-negatively, we of course have  $\mathcal{E} \subseteq \mathcal{A}_+ \cap M_1$ . To show equality, take  $E \in \mathcal{A}_+ \cap M_1$ . By Lemma I.2  $E$  is effective so we may write  $E$  as a positive linear combination of integral curves. Since  $E^2 < 0$ , one of these curves  $C$  has  $E \cdot C < 0$  and so necessarily satisfies  $C^2 < 0$ . Since  $E$  lies in  $\mathcal{A}_+$ ,  $C$  cannot be nodal so by Lemma I.1(b)  $C$  is exceptional. Now  $E - C$  is effective and perpendicular to  $K_X$ , so  $E - C$  is a sum of components of fibers; i.e.,  $E - C \in \Gamma$ . Thus  $(E - C)^2 \leq 0$  and equality holds iff  $E - C$  is a multiple of  $K_X$  by Lemma I.1(e). On the other hand,  $-1 = E^2 = ((E - C) + C)^2 = (E - C)^2 + 2E \cdot C - C^2$ , and since  $E^2 = C^2 = -1$  we find that  $(E - C)^2 = -2E \cdot C - 2 \geq 0$ . Thus  $(E - C)^2 = 0$ , whence  $E - C$  is a multiple, say  $mK_X$ , of  $K_X$ . Finally we derive  $m = 0$ , giving  $E = C \in \mathcal{E}$ :  $-1 = E^2 = (C + mK_X)^2 = C^2 + 2mC \cdot K_X = -1 - 2m$ . ■

To make use of Proposition I.3, we recall a well-known and very useful group action on  $M_1$ . For any elements  $L \in K^\perp$  and  $E \in M_1$ , define  $\tau_L(E)$  to be  $E + L + (\frac{1}{2})(L^2 + 2L \cdot E)K_X$  in  $\text{Pic } X$ . Note that  $L^2$  is even since (as is clear for example from Riemann-Roch for surfaces)  $K^\perp$  is an even (negative semidefinite) lattice, meaning that if  $v \in K^\perp$ , then  $v^2$  is an even (non-positive) integer. Mnemonically,  $\tau_L$  stands for “translation by  $L$ ,” justified by this next lemma.

**LEMMA I.4.** (a)  *$\tau$  defines an action of  $K^\perp$  on  $M_1$ .*

(b)  *$K_X$  acts trivially with respect to this action.*

(c)  *$K^\perp$  acts transitively on  $M_1$ .*

(d) *The induced action of  $K^\perp/(K_X)$  on  $M_1$  is transitive with only the identity fixing any elements of  $M_1$ .*

(e) *If  $\text{rank}(\Gamma) \leq 8$ , then  $\mathcal{E}$  is infinite.*

*Proof.* These facts are well known. The proofs of (a)–(d) are easy. To see (e), note that  $\text{rank}(K^\perp) = 9$  since any rational numerically elliptic surface which is minimal is a blowing up of  $\mathbb{P}^2$  at nine successive points. If  $\text{rank}(\Gamma) \leq 8$ , there is an element  $L$  of  $K^\perp$  in  $\Gamma^\perp$  having no multiple in  $\Gamma$ . But  $\mathcal{E}$  is never empty since  $X$  is a blowing up of  $\mathbb{P}^2$ , and for any  $E \in \mathcal{E}$ , the classes  $\tau_{iL}(E)$ ,  $i > 0$ , are distinct by (d) and lie in  $\mathcal{E}$  by Proposition I.3. ■

**DEFINITION.** If  $\text{rank}(\Gamma) = 9$ ,  $X$  is said to be *extremal*.

The converse (I.5(c)) to Lemma (I.4(e)) is also true. Our proof uses a homomorphism  $\wedge: \text{Pic } X \rightarrow \Delta^*$ : if  $\Delta \neq \phi$ , we define  $H^\wedge$  for  $H \in \text{Pic } X$  by taking  $H^\wedge(N) = H \cdot N$ , where  $N \in \Delta$ .

**PROPOSITION I.5.** *Let  $\mathcal{O}$  be a  $\Gamma$ -orbit in  $M_1$  with respect to the action by  $\tau$ .*

- (a) *If  $\Delta \neq \phi$ , then the restriction of  $\wedge$  to  $\mathcal{O}$  is injective.*
- (b) *The number of  $(-1)$ -curves in  $\mathcal{O}$  is finite and nonempty.*
- (c) *If  $X$  is extremal (i.e.,  $[K^\perp: \Gamma] < \infty$ ), then  $\mathcal{E}$  is finite.*

*Proof.* (a) Let  $E, F \in \mathcal{O}$ , so  $E - F \in \Gamma$ . But  $E^\wedge = F^\wedge$  means that  $E - F$  is perpendicular to  $\Gamma$  so in particular  $(E - F)^2 = 0$ . By Lemma I.1(e) this means  $E$  equals  $F + mK_X$  for some integer  $m$ , but  $-1 = E^2 = (F + mK_X)^2 = -1 - 2m$ , so  $m = 0$ .

(b) If  $\Delta = \phi$ , then  $\mathcal{E} = M_1$  by Proposition I.3 and every element of  $M_1$  is a  $\Gamma$ -orbit by Lemma I.4(b), so the result follows in this case. Now say  $\Delta \neq \phi$ . The  $\{(-1)\text{-curves in } \mathcal{O}\} = \mathcal{O} \cap \Delta_+$ ; this maps injectively via  $\wedge$  by (a) into  $\Delta^*$ . To prove the finiteness it therefore suffices to show that  $(\mathcal{O} \cap \Delta_+)^{\wedge}$  is finite. Let  $m$  be the multiplicity, and let  $E$  be in  $\mathcal{O} \cap \Delta_+$ . Then

$$\begin{aligned} \sum_{N \in \Delta} E^\wedge(N) &= \sum_{\substack{\text{reducible} \\ \text{fibers } F}} \sum_{\substack{N = \text{component} \\ \text{of } F}} E^\wedge(N) \\ &= \sum_{\substack{\text{reducible} \\ \text{fibers } F}} E^\wedge \left( \sum_{\substack{N = \text{component} \\ \text{of } F}} N \right) \\ &\leq \sum_{\substack{\text{reducible} \\ \text{fibers } F}} E^\wedge(F) \\ &\quad \left( \text{since } E^\wedge(N) \geq 0 \ \forall N \text{ and } F \geq \sum_{\substack{N = \text{component} \\ \text{of } F}} N \right) \\ &= \sum_{\substack{\text{reducible} \\ \text{fibers } F}} E^\wedge(-mK_X) \\ &= m \cdot (\# \text{ of reducible fibers}) = mn, \text{ say.} \end{aligned}$$

Therefore  $(\mathcal{O} \cap \Delta_+)^{\wedge}$  maps into  $\{\delta \mid \delta(N) \geq 0 \ \forall N \text{ and } \sum_{N \in \Delta} \delta(N) \leq mn\}$  which is finite.

If  $F$  is an element of  $M_1$ , we can write  $F$  as a sum  $F = \sum_i C_i$  of prime divisors, and we can let  $L = \sum_j C_{i_j}$  where  $C_{i_j}$  is an element of  $\Delta$ . Then  $\tau_{-L}(F)$  is an element of  $\Delta_+ \cap M_1$ , so  $\mathcal{O} \cap \mathcal{E}$  is not empty.

(c) If  $X$  is extremal, then  $\Gamma$  has finite index in  $K^\perp$ . But  $M_1$  comprises one  $K^\perp$ -orbit, hence finitely many  $\Gamma$ -orbits, and by (b) each contains finitely many  $(-1)$ -curves. Thus  $\mathcal{E}$  is finite. ■

Proposition I.5 shows there are two appropriate enumeration problems: for any  $X$ , enumerate  $\mathcal{O} \cap \mathcal{A}_+$ , given a  $\Gamma$ -orbit  $\mathcal{O}$  in  $M_1$ ; and, for  $X$  extremal, enumerate  $\mathcal{E}$ . It is convenient to formulate a more abstract version of the first problem, which we solve in Section II, and we apply the results to solve the second problem in Section III.

To formulate the abstraction, let  $\mathcal{A}$  denote the union of the vertices of a collection  $F_1, \dots, F_n$  of not necessarily distinct graphs from Fig. 1. A case of special interest will of course be when these are the intersection graphs of the reducible fibers on some surface. We will use the notation  $(\mathcal{A})$  for the free abelian group on the elements of  $\mathcal{A}$ ; note that the intersection graphs  $F_1, \dots, F_n$  induce a bilinear form on  $(\mathcal{A})$  by defining it on  $\mathcal{A}$ : if  $N$  and  $M$  are elements of  $\mathcal{A}$ , define  $N \cdot M$  to be  $-2$  if  $N = M$  and to be  $r$  if  $N \neq M$ , where  $r$  is the number of edges directly connecting the two vertices representing  $N$  and  $M$  in the disjoint union of the graphs  $F_1, \dots, F_n$  (and thus either  $0$  or  $1$  unless  $N$  and  $M$  are the two vertices of  $A_1$  where  $N \cdot M = 2$ ). For  $\mathcal{A}^*$  the free abelian group of integer valued functions on  $\mathcal{A}$  we thus have  ${}^\wedge: \mathcal{A} \rightarrow \mathcal{A}^*$  defined as before. We denote by  $\mathcal{A}_+^*$  those elements of  $\mathcal{A}^*$  which are nonnegative on  $\mathcal{A}$ , by  $(\mathcal{A}^\wedge)$  the subgroup of  $\mathcal{A}^*$  generated by the image  $\mathcal{A}^\wedge$  of  $\mathcal{A}$  under  ${}^\wedge$ , and by  $[ ]: \mathcal{A}^* \rightarrow \mathcal{A}^*/(\mathcal{A}^\wedge)$  the quotient homomorphism. We will denote the vertices of a single graph  $F$  by  $\mathcal{A}_F$ , and in a similar way we also have:  $\mathcal{A}_F^*$ ,  $\mathcal{A}_{F+}^*$ , and  $[ ]_F: \mathcal{A}_F^* \rightarrow \mathcal{A}_F^*/(\mathcal{A}_F^\wedge)$ . Note that by restriction of domain we have canonical surjections  $\mathcal{A}^* \rightarrow \mathcal{A}_F^*$  whenever  $F$  is one of the graphs in the collection  $F_1, \dots, F_n$ .

**LEMMA I.6.** Consider a finite collection  $C$  of not necessarily distinct graphs  $F$  of Fig. 1.

(a) The canonical maps  $\mathcal{A}^* \rightarrow \mathcal{A}_F^*$  induce isomorphisms  $\mathcal{A}^* \cong \bigoplus_F \mathcal{A}_F^*$ ,  $\mathcal{A}_+^* \cong \bigoplus_F \mathcal{A}_{F+}^*$ ,  $(\mathcal{A}^\wedge) \cong \bigoplus_F (\mathcal{A}_F^\wedge)$ , and  $\mathcal{A}^*/(\mathcal{A}^\wedge) \cong \bigoplus_F \mathcal{A}_F^*/(\mathcal{A}_F^\wedge)$ , where the sums are taken over all graphs  $F$  in  $C$ .

(b) Suppose  $C \neq \emptyset$  and that  $C$  is the collection of graphs arising from the reducible fibers of some surface  $X$ . If  $\mathcal{O}$  is a  $\Gamma$ -orbit in  $M_1$ , then  $\mathcal{O}^\wedge = [ ]^{-1}[\mathcal{O}^\wedge]$  and the number of  $(-1)$ -curves in  $\mathcal{O}$  is precisely the cardinality  $|([ ]^{-1}[\mathcal{O}^\wedge]) \cap \mathcal{A}_+^*|$  of the set  $([ ]^{-1}[\mathcal{O}^\wedge]) \cap \mathcal{A}_+^*$ .

(c) Let  $\delta = \bigoplus_F \delta_F$  be any element of  $\mathcal{A}^*/(\mathcal{A}^\wedge) \cong \bigoplus_F \mathcal{A}_F^*/(\mathcal{A}_F^\wedge)$ . Then  $|([ ]^{-1}(\delta) \cap \mathcal{A}_+^*)|$  is finite and equals  $\prod_F |\mathcal{A}_{F+}^* \cap [ ]_F^{-1}(\delta_F)|$ , where the product is taken over all  $F \in C$ .

*Proof.* (a) Since  $\mathcal{A}$  is the disjoint union  $\bigcup \mathcal{A}_F$ ,  $\mathcal{A}^* \cong \bigoplus_F \mathcal{A}_F^*$  is obvious and it is clear that it induces a bijection  $\mathcal{A}_+^* \cong \bigoplus_F \mathcal{A}_{F+}^*$ . Since

vertices in distinct graphs are orthogonal,  $(\Delta^\wedge) \cong \bigoplus_F (\Delta_F^\wedge)$  is clear and  $\Delta^\# / (\Delta^\wedge) \cong \bigoplus_F \Delta_F^\# / (\Delta_F^\wedge)$  follows.

(b) Clearly,  $\mathcal{O}^\wedge \subset [\ ]^{-1}[\mathcal{O}^\wedge]$ . On the other hand, let  $f$  be in  $[\ ]^{-1}[\mathcal{O}^\wedge]$ . Then  $[f] = [E^\wedge]$  for some  $E \in \mathcal{O}$ , so  $f - E^\wedge \in (\Delta^\wedge) = \Gamma^\wedge$ . Pick  $G \in \Gamma$  with  $G^\wedge = f - E^\wedge$ . Then  $\tau_G(E) \in \mathcal{O}$ ; since  $\tau_G(E) = E + G + kK_X$  for some  $k$ , we have  $\tau_G(E)^\wedge = E^\wedge + G^\wedge$  (since  $K_X^\wedge = 0$ ), hence  $\tau_G(E)^\wedge = f$ . This proves that  $[\ ]^{-1}[\mathcal{O}^\wedge] = \mathcal{O}^\wedge$ . Since  $^\wedge$  is injective on  $\mathcal{O}$ , then the number of  $(-1)$ -curves in  $\mathcal{O}$  is  $|\mathcal{O} \cap \Delta_+| = |\mathcal{O}^\wedge \cap \Delta_+^\wedge| = |([\ ]^{-1}[\mathcal{O}^\wedge]) \cap \Delta_+^\wedge|$ .

(c) By (a) it is clear that  $\Delta_{F+}^\# \cap [\ ]^{-1}(\delta) \cong \bigoplus_F (\Delta_{F+}^\# \cap [\ ]_F^{-1}(\delta_F))$  under the isomorphism  $\Delta^\# \cong \bigoplus_F \Delta_F^\#$ . Thus the result follows if we merely show that  $\Delta_{F+}^\# \cap [\ ]_F^{-1}(\delta_F)$  is finite. But  $F$  is a nonnegative sum  $\sum a_i N_i$  of elements  $N_i \in \Delta_F$ , and for any  $N \in \Delta_F$ , we have  $F \cdot N = 0$ . Since any two elements  $f$  and  $g$  of  $[\ ]_F^{-1}(\delta_F)$  differ by elements of  $(\Delta_F^\wedge)$ , we see that  $\sum a_i f(N_i) = \sum a_i g(N_i)$ . Since elements of  $\Delta_{F+}^\#$  are nonnegative on  $\Delta$ , there can be at most finitely many for which the sum  $\sum a_i f(N_i)$  is fixed. ■

## II. COMBINATORIAL PROBLEM

In this section we work out the following

*Combinatorial Problem.* Given  $F$  a graph from Fig. 1 and an element  $\delta_F$  of  $\Delta_F^\# / (\Delta_F^\wedge)$ , compute  $|\Delta_{F+}^\# \cap [\ ]_F^{-1}(\delta_F)|$ .

*Remark II.1.* When  $F$  is the collection of graphs arising from the reducible fibers on some surface  $X$  and  $\delta = \bigoplus_F \delta_F$  is the image  $[E^\wedge]$  in  $\Delta^\# / (\Delta^\wedge) \cong \bigoplus_F \Delta_F^\# / (\Delta_F^\wedge)$  of a  $(-1)$ -class on  $X$ , then the number  $|\Delta_{F+}^\# \cap [\ ]^{-1}(\delta)|$  of  $(-1)$ -curves on  $X$  in the same  $\Gamma$ -orbit of  $M_1$  as  $E$  is by Lemma I.6 the product  $\prod_F |\Delta_{F+}^\# \cap [\ ]_F^{-1}(\delta_F)|$  over the graphs  $F$  of  $\Delta$ . Thus a solution of the Combinatorial Problem solves the problem of computing the number of exceptional curves in any given  $\Gamma$ -orbit of  $M_1$ .

We now need to understand for each graph  $F$  of Fig. 1 the homomorphism  $[\ ]_F: \Delta_F^\# \rightarrow \Delta_F^\# / (\Delta_F^\wedge)$ ; we will denote  $\Delta_F^\# / (\Delta_F^\wedge)$  by  $\Delta_F^*$ . Let  $v_i$  be the element of  $\Delta_F^*$  dual to the vertex  $v_i$ , i.e.,  $v_i(v_j) = \text{Kronecker's } \delta_{ij}$ . Recall from our labelling that  $v_0$  has multiplicity one for every  $F$ , i.e.,  $\mu_0 = 1$ . Finally denote by  $z_F$  the class  $\sum \mu_i v_i$  in  $(\Delta_F)$ ; this is the fundamental cycle of the fiber  $F$ .

**LEMMA II.2.** (a) The kernel of  $^\wedge: (\Delta_F) \rightarrow \Delta_F^*$  has rank 1, and is generated by  $z_F$ .

- (b)  $\Delta_F^* \cong \mathbb{Z} \oplus T_F$ , where  $T_F$  is a finite abelian group.
- (c) The “degree map”  $d: \Delta_F^* \rightarrow \mathbb{Z}$  given by  $d(f) = f(z_F)$  is an onto

homomorphism, and  $(\mathcal{A}_F^\wedge) \subseteq \ker(d)$ ; hence  $d$  descends to a map  $d: \mathcal{A}_F^* \rightarrow \mathbb{Z}$ , and  $0 \rightarrow T_F \rightarrow \mathcal{A}_F^* \xrightarrow{d} \mathbb{Z} \rightarrow 0$  is exact.

$$(d) \quad T_F \cong \ker(d)/(\mathcal{A}_F^\wedge).$$

*Proof.* Statement (a) is well known; it is easy to prove using the lemma of Bombieri and Mumford [BM, p. 28]. Statement (b) follows from (a), after noting that  $(\mathcal{A}_F)$  and  $\mathcal{A}_F^*$  have the same rank;  $T_F$  is the torsion part of  $\mathcal{A}_F^*$ . To prove (c), note that  $d(v_0) = 1$ , showing that  $d$  is onto;  $d(\mathcal{A}_F^\wedge) = 0$  since  $z_F$  is orthogonal to each  $v_i$ . Part (d) follows from (c). ■

A splitting of the sequence of II.2(c) is afforded by a choice of a multiplicity one component of  $F$ , as follows. Let  $v_0$  be such a component. Then for each  $i$ ,  $v_i - \mu_i v_0$  is in  $\ker(d)$  and therefore  $g_i = [v_i - \mu_i v_0] \in T_F$ . The map from  $\mathcal{A}_F^*$  to  $T_F$  defined by sending  $[\sum \alpha_i v_i]$  to  $\sum \alpha_i g_i$  is the splitting of the sequence, and exhibits  $\mathcal{A}_F^*$  as a direct product  $\mathbb{Z} \times T_F$ . Note that the projection onto the  $\mathbb{Z}$  factor is given by the degree map  $d$ , and therefore sends  $[v_i]$  to  $\mu_i$  for each  $i$ .

Therefore our problem, slightly restated, is to compute

$$\sigma_n(g) = |\mathcal{A}_{F+}^* \cap [\ ]_F^{-1}(n \oplus g)| \quad \text{for } n \in \mathbb{Z} \text{ and } g \text{ in } T_F,$$

after making the identification of  $\mathcal{A}_F^*$  with  $\mathbb{Z} \oplus T_F$  as described above. Let  $H_F(g) = \sum_{n \geq 0} \sigma_n(g) t^n \in \mathbb{Z}[[t]]$ . We will compute this generating function  $H_F$ .

Let  $\mathbb{Z}[T_F]$  be the group ring of the finite abelian group  $T_F$ , and define  $G_F = \prod_i (1 - g_i t^{\mu_i})^{-1} \in \mathbb{Z}[T_F][[t]]$ . Define

$$\chi_g: T_F \rightarrow \mathbb{Z} \text{ by } \chi_g(h) = \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq g; \end{cases}$$

this is a “characteristic function” for the element  $g$  of  $T_F$ , and is not a group homomorphism. Extend  $\chi_g$  to a map  $\chi_g: \mathbb{Z}[T_F][[t]] \rightarrow \mathbb{Z}[[t]]$  by letting  $\chi_g$  act on the coefficients; this map is additive, but is not a ring homomorphism.

**THEOREM II.3.**  $H_F(g) = \chi_g(G_F)$ .

*Proof.* By expanding the factors of  $G_F$  we have

$$\begin{aligned} G_F &= \prod_i \sum_{k_i} (g_i t^{\mu_i})^{k_i} = \sum_{\substack{(k_0, \dots) \\ \text{all } k_i \geq 0}} \prod_i g_i^{k_i} t^{\mu_i k_i} \\ &= \sum_{\substack{(k_0, \dots) \\ \text{all } k_i \geq 0}} \left( \prod_i g_i^{k_i} \right) t^{\sum \mu_i k_i}. \end{aligned}$$

Note that the terms of the sum are in 1-1 correspondence with the elements of  $\mathcal{A}_{F+}^*: (k_0, \dots)$  corresponds to  $\sum k_i v_i$ . The term  $(k_0, \dots)$

corresponds to an element of  $[\cdot]_F^{-1}(n \oplus g)$  if and only if  $\prod_i g_i^{k_i} = g$  and  $\sum \mu_i k_i = n$ . Hence the coefficient of  $gt^n$  in  $G_F$  is the cardinality of  $\Delta_{F+}^{\#} \cap [\cdot]_F^{-1}(n \oplus g)$ , which is by definition  $\sigma_n(g)$ . The map  $\chi_g$  now eliminates all terms of  $G_F$  other than those of the form  $gt^n$ , and replaces the  $g$  with 1, giving therefore  $H_F(g)$ . ■

To apply this result in a concrete situation, we will need to know the group  $T_F$  and the elements  $g_i$ . This information follows in Table II.4. We have gone to a multiplicative notation for the group  $T_F$ , since addition in  $T_F$  appears in our application as group ring multiplication.

The calculations involved in producing Table II.4 are quite standard, and the results well known. The lattice generated by deleting  $v_0$  from  $\Delta_F$  is a negative definite root lattice of type  $A_n$ ,  $D_n$ , or  $E_n$ . The group  $T_F$  is the *discriminant-form group* (using the notation of [N]), or the *dual quotient group* (using the notation of [CS]) of the root lattice. The computations of the  $g_i$  in each case are straightforward.

Note that for each  $g$  in  $T_F$  there is a unique vertex  $v_i$  with  $\mu_i = 1$  in the graph, such that  $g_i = g$ . This identification of  $T_F$  with the multiplicity one components of the graph depends of course on the choice of  $v_0$ .

Let  $S_F$  be the automorphism group of the graph for  $F$ . Note that  $S_F$  acts transitively on the set of vertices with multiplicity one. We have of course an induced action of  $S_F$  on  $(\Delta_F)$ ,  $\Delta_F^{\#}$ ,  $\Delta_F^*$ , etc., preserving the bilinear forms and the degree map  $d$ .

**LEMMA II.5.** *Assume  $\gcd(n, |T_F|) = 1$ . Then  $S_F$  acts transitively on the set of elements  $\delta$  of  $\Delta_F^*$  with  $d(\delta) = n$ .*

*Proof.* The above set is  $\{n[v_0] \oplus g \mid g \in T_F\}$ . We will show that for each  $g$  in  $T_F$  there is an automorphism  $\sigma_g$  in  $S_F$  which sends  $n[v_0] \oplus 1$  to  $n[v_0] \oplus g$ . Fix the  $g$  in  $T_F$ . Let  $\sigma_g$  be an automorphism of the graph sending  $v_0$  to  $v_i$ , where  $g_i^n = g$ . Then

$$\sigma_g(n[v_0]) = n[v_i] = n[v_0] + n([v_i - v_0]) = n[v_0] \oplus g_i^n = n[v_0] \oplus g$$

as required. ■

TABLE II.4

$F$	$T_F$	Elements $g_i$ ( $g_0 = 1$ in every case)
$\tilde{A}_{n-1}$ :	$\langle \beta \mid \beta^n = 1 \rangle$	$g_i = \beta^i$ for every $i$
$\tilde{B}_{n+3}$ , $n$ odd:	$\langle \beta, \gamma \mid \beta^2 = \gamma^2 = 1 \rangle$	$g_1 = \beta\gamma$ , $g_2 = \gamma$ , $g_3 = \beta$ , $g_{2k} = 1$ , $g_{2k+1} = \beta\gamma$ ( $k \geq 2$ )
$\tilde{B}_{n+3}$ , $n$ even:	$\langle \beta \mid \beta^4 = 1 \rangle$	$g_1 = \beta^2$ , $g_2 = \beta$ , $g_3 = \beta^3$ , $g_{2k} = 1$ , $g_{2k+1} = \beta^2$ ( $k \geq 2$ )
$\tilde{E}_6$ :	$\langle \beta \mid \beta^3 = 1 \rangle$	$g_1 = g_2 = 1$ , $g_3 = g_6 = \beta$ , $g_4 = g_5 = \beta^2$
$\tilde{E}_7$ :	$\langle \beta \mid \beta^2 = 1 \rangle$	$g_1 = g_2 = g_3 = g_5 = 1$ , $g_4 = g_6 = g_7 = \beta$
$\tilde{E}_8$ :	$\{1\}$	$g_i = 1$ for every $i$

Note that the action of  $S_F$  on  $\mathcal{A}_F^\#$  preserves  $\mathcal{A}_{F+}^\#$ , and so by the above lemma, if  $\gcd(n, |T_F|) = 1$ , then  $\sigma_n(g)$  is independent of  $g$ . This leads to the following

**COROLLARY II.6.** *Define  $\chi: T_F \rightarrow \mathbb{Z}$  by  $\chi(g) = 1$  for every  $g$ ; note that  $\chi = \sum_g \chi_g$ . Extend  $\chi$  to  $\chi: \mathbb{Z}[T_F][[t]] \rightarrow \mathbb{Z}[[t]]$  as before. Assume  $\gcd(n, |T_F|) = 1$ . Then  $\sigma_n(g) = (1/|T_F|) \cdot \text{coefficient of } t^n \text{ in } \chi(G_F)$ , and  $\chi(G_F)$  is simply  $\prod_i (1 - t^{n_i})^{-1}$ .*

*Proof.* Follows directly from Theorem II.3 and the above lemma. ■

**Remark II.7.** Let  $F$  be a fiber on a surface  $X$ , and  $E$  a  $(-1)$ -class on  $X$ . Then the image of  $[E^\wedge]$  in  $\mathcal{A}_F^\#/\langle A_F^\wedge \rangle$  can be written in the form  $n \oplus g$ . Interpreting Lemma I.1(g), we see that  $n$  is either 1 or the multiplicity of the fibration on  $X$ , according to whether  $F$  is a multiple fiber or not, and thus the coordinate  $n$  for the images of  $[E^\wedge]$  in  $\mathcal{A}_F^\#$  for all nonmultiple reducible fibers  $F$  must agree, and  $n$  must be 1 for the multiple fiber.

All of the generating functions  $H_F(g)$  we obtain are rational, meaning that they are the Taylor series of a quotient of polynomials. To see why this is so, and to compute them explicitly, we have the following lemma.

**LEMMA II.8.** *Assume  $T = T_1 \oplus \cdots \oplus T_n$ , where the  $T_i$  are finite cyclic groups with generators  $g_i$  of order  $k_i$ ; then  $|T| = \prod k_i$ . Let  $Q(x_1, \dots, x_n, t)$  be a rational function such that  $G_F = Q(g_1, \dots, g_n, t)$ . Let  $g$  be in  $T$ , and write  $g = \prod g_i^{m_i}$ . Then*

$$H_F(g) = \frac{1}{|T|} \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_n=0}^{k_n-1} \zeta_{k_1}^{j_1 m_1} \cdots \zeta_{k_n}^{j_n m_n} Q(\zeta_{k_1}^{-j_1}, \dots, \zeta_{k_n}^{-j_n}, t),$$

where  $\zeta_k = e^{2\pi i/k}$ .

*Proof.* Fix  $g$ , and recall that  $H_F(g) = \chi_g(G_F)$ ; hence both sides of the above expression are additive in the terms of the power series of  $Q$ . Hence it suffices to prove it for  $Q = f(x)t^r$ , where  $f$  is monomial in the  $x_i$ 's; in this case  $G_F = ht^r$ , where  $h = f(g)$ . If  $h = g$ , then every term of the multi-sum above is  $t^r$ , and so the right-hand side sums to  $t^r$ ; this is  $H_F(g)$  also. If  $h \neq g$ , then the sum gives 0, since the sum of the powers of a primitive root is 0. In this case  $H_F(g) = 0$  also. ■

We will now present our computations of the  $H_F$ 's for those  $F$ 's with at most 9 components without much further comment; we leave the verifications to the reader. The reader should be warned that some terms have been collected in the following list.

List II.9 of  $H_F$ 's and  $G_F$ 's:

$E_8$ :  $T = \{1\}$ ;

$$G = 1/(1-t)(1-t^2)^2 (1-t^3)^2 (1-t^4)^2 (1-t^5)(1-t^6)$$

$$\begin{aligned} H(1) &= 1/(1-t)(1-t^2)^2 (1-t^3)^2 (1-t^4)^2 (1-t^5)(1-t^6) \\ &= 1 + t + 3t^2 + 5t^3 + 10t^4 + 15t^5 + 27t^6 \\ &\quad + 39t^7 + 63t^8 + 90t^9 + 135t^{10} + \dots \end{aligned}$$

$E_7$ :  $T = \{1, \beta\}$ ;

$$G = [(1-t)(1-t^2)^2 (1-t^3)(1-t^4)(1-\beta t)(1-\beta t^2)(1-\beta t^3)]^{-1}$$

$$\begin{aligned} H(1) &= (1+t^3+t^4+t^5)/[(1-t)(1-t^2)^3 (1-t^3)(1-t^4)^2 (1-t^6)] \\ &= 1 + t + 4t^2 + 6t^3 + 15t^4 + 22t^5 + 44t^6 \\ &\quad + 64t^7 + 112t^8 + 159t^9 + 254t^{10} + \dots \end{aligned}$$

$$\begin{aligned} H(\beta) &= (t+t^2+t^3+t^6)/[(1-t)(1-t^2)^3 (1-t^3)(1-t^4)^2 (1-t^6)] \\ &= t + 2t^2 + 6t^3 + 10t^4 + 22t^5 + 35t^6 \\ &\quad + 64t^7 + 96t^8 + 159t^9 + 229t^{10} + \dots \end{aligned}$$

$E_6$ :  $T = \{1, \beta, \beta^2\}$ ;

$$G = [(1-t)(1-t^2)(1-t^3)(1-\beta t)(1-\beta^2 t)(1-\beta t^2)(1-\beta^2 t^2)]^{-1}$$

$$\begin{aligned} H(1) &= \frac{1}{3}[G|_{\beta=1} + 2G|_{\beta=\zeta_3}]; \\ &= 1 + t + 3t^2 + 8t^3 + 14t^4 + 26t^5 + 49t^6 + 77t^7 + 124t^8 + 195t^9 + \dots \end{aligned}$$

$$\begin{aligned} H(\beta) &= H(\beta^2) = \frac{1}{3}[G|_{\beta=1} - G|_{\beta=\zeta_3}] \\ &= t + 3t^2 + 6t^3 + 14t^4 + 26t^5 + 45t^6 + 77t^7 + 124t^8 + \dots \end{aligned}$$

$D_{2k}$ ,  $k \geq 2$ :  $T = \{1, \beta, \gamma, \beta\gamma\}$ ;

$$G = [(1-t)(1-t^2)^{k-1} (1-\beta\gamma t^2)^{k-2} (1-\beta t)(1-\gamma t)(1-\beta\gamma t)]^{-1}$$

$$H(1) = \left( -t(1-t^2)^{k-2} + (1+t^2) \sum_{\substack{j=0 \\ j \text{ even}}}^{k-2} \binom{k-2}{j} t^{2j} \right)$$

$$\cdot [(1-t)^2 (1-t^2)^{k+1} (1-t^4)^{k-2}]^{-1}$$

$$H(\beta\gamma) = \left( t(1-t^2)^{k-2} + (1+t^2) \sum_{\substack{j=0 \\ j \text{ odd}}}^{k-2} \binom{k-2}{j} t^{2j} \right)$$

$$\cdot [(1-t)^2 (1-t^2)^{k+1} (1-t^4)^{k-2}]^{-1}$$

$$H(\beta) = H(\gamma) = t \cdot [(1-t)^2 (1-t^2)^{2k-1}]^{-1}.$$

$$H_{D4}(1) = (1 - t + t^2)/[(1 - t)^2 (1 - t^2)^3] \\ = 1 + t + 5t^2 + 6t^3 + 16t^4 + 20t^5 + 40t^6 + 50t^7 + 85t^8 + \dots$$

$$H_{D4}(\beta) = H(\gamma) = H(\beta\gamma) = t/[(1 - t)^2 (1 - t^2)^3] \\ = t + 2t^2 + 6t^3 + 10t^4 + 20t^5 + 30t^6 + 50t^7 + 70t^8 + \dots$$

$$H_{D6}(1) = (1 - t + t^2 + t^3)/[(1 - t)^2 (1 - t^2)^4 (1 - t^4)] \\ = 1 + t + 6t^2 + 8t^3 + 25t^4 + 35t^5 + 80t^6 + 112t^7 + 214t^8 + \dots$$

$$H_{D6}(\beta\gamma) = (t + t^2 - t^3 + t^4)/[(1 - t)^2 (1 - t^2)^4 (1 - t^4)] \\ = t + 3t^2 + 8t^3 + 18t^4 + 35t^5 + 67t^6 + 112t^7 + 192t^8 + \dots$$

$$H_{D6}(\beta) = H_{D6}(\gamma) = (t + t^2 + t^3 + t^4)/[(1 - t)(1 - t^2)^5 (1 - t^4)] \\ = t + 2t^2 + 8t^3 + 14t^4 + 35t^5 + 56t^6 + 112t^7 + 168t^8 + \dots$$

$$H_{D8}(1) = (1 - t + t^2 + 2t^3 + t^4 - t^5 + t^6)/[(1 - t)^2 (1 - t^2)^5 (1 - t^4)^2] \\ = 1 + t + 7t^2 + 10t^3 + 36t^4 + 54t^5 + 140t^6 + 210t^7 + 450t^8 + \dots$$

$$H_{D8}(\beta\gamma) = (t + 2t^2 - 2t^3 + 2t^4 + t^5)/[(1 - t)^2 (1 - t^2)^5 (1 - t^4)^2] \\ = t + 4t^2 + 10t^3 + 28t^4 + 54t^5 + 124t^6 + 210t^7 + 420t^8 + \dots$$

$$H_{D8}(\beta) = H_{D8}(\gamma) \\ = (t + t^2 + 2t^3 + 2t^4 + t^5 + t^6)/[(1 - t)(1 - t^2)^6 (1 - t^4)^2] \\ = t + 2t^2 + 10t^3 + 18t^4 + 54t^5 + 90t^6 + 210t^7 + 330t^8 + \dots$$

$D_{2k+1}$ ,  $k \geq 2$ :  $T = \{1, \beta, \beta^2, \beta^3\}$ ;

$$G = [(1 - t)(1 - \beta t)(1 - \beta^2 t)(1 - \beta^3 t)(1 - t^2)^{k-1} (1 - \beta^2 t^2)^{k-1}]^{-1}$$

$$H(\beta^i) = \frac{1}{4} [(1 - t)^{-4} (1 - t^2)^{2-2k} + x(i)(1 - t^4)^{-k} + (-1)^i (1 - t^2)^{-2k}],$$

$$\text{where } x(i) = \begin{cases} 2 & \text{if } i = 0(4) \\ -2 & \text{if } i = 2(4) \\ 0 & \text{if } i \text{ is odd;} \end{cases}$$

$$H_{D5}(1) = (1 + t^2 + 3t^3 + 3t^4 + 3t^5 + 3t^6 + t^7 + t^9)/[(1 - t)(1 - t^2)^2 (1 - t^4)^3] \\ = 1 + t + 4t^2 + 7t^3 + 18t^4 + 27t^5 + 52t^6 + 77t^7 + 131t^8 + \dots$$

$$H_{D5}(\beta) = H_{D5}(\beta^3) \\ = (t + t^2 + 3t^3 + 3t^4 + 3t^5 + 3t^6 + t^7 + t^8)/[(1 - t)(1 - t^2)^2 (1 - t^4)^3] \\ = t + 2t^2 + 7t^3 + 12t^4 + 27t^5 + 42t^6 + 77t^7 + 112t^8 + \dots$$

$$H_{D5}(\beta^2) = (t + 3t^2 + t^3 + 3t^4 + 3t^5 + t^6 + 3t^7 + t^8)/[(1 - t)(1 - t^2)^2 (1 - t^4)^3] \\ = t + 4t^2 + 7t^3 + 16t^4 + 27t^5 + 52t^6 + 77t^7 + 128t^8 + \dots$$

$$\begin{aligned}
H_{D7}(1) &= (1 + t^2 + 4t^3 + 6t^4 + 4t^5 + 6t^6 + 4t^7 + t^8 + 4t^9 + t^{10})/ \\
&\quad [(1-t)(1-t^2)^3 (1-t^4)^4] \\
&= 1 + t + 5t^2 + 9t^3 + 28t^4 + 44t^5 + 100t^6 + 156t^7 + 306t^8 + \dots
\end{aligned}$$

$$\begin{aligned}
H_{D7}(\beta) &= H_{D7}(\beta^3) \\
&= (t + t^2 + 4t^3 + 4t^4 + 6t^5 + 6t^6 + 4t^7 + 4t^8 + t^9 + t^{10})/ \\
&\quad [(1-t)(1-t^2)^3 (1-t^4)^4] \\
&= t + 2t^2 + 9t^3 + 16t^4 + 44t^5 + 72t^6 + 156t^7 + 240t^8 + \dots
\end{aligned}$$

$$\begin{aligned}
H_{D7}(\beta^2) &= (t + 4t^2 + t^3 + 4t^4 + 6t^5 + 4t^6 + 6t^7 + 4t^8 + t^9 + t^{11})/ \\
&\quad [(1-t)(1-t^2)^3 (1-t^4)^4] \\
&= t + 5t^2 + 9t^3 + 25t^4 + 44t^5 + 100t^6 + 156t^7 + 300t^8 + \dots
\end{aligned}$$

$$\begin{aligned}
A_{k-1}, k \geq 2: T &= \{1, \beta, \beta^2, \dots, \beta^{k-1}\}; \\
G &= [(1-t)(1-\beta t)(1-\beta^2 t) \cdots (1-\beta^{k-1} t)]^{-1}.
\end{aligned}$$

$$G|_{\beta = \zeta_k^j} = (1-t^{k/d})^{-d}, \quad \text{where } d = \gcd(j, k).$$

To come up with the formulas below in these cases, the following result is helpful; it follows from Lemma II.8, and we leave it to the reader to verify it:

$$H_{A_{k-1}}(\beta^i) = \frac{1}{k} \sum_{d|k} x(d, i) (1-t^d)^{-k/d},$$

where

$$x(d, i) = \frac{\varphi(d) \mu(d/(d, i))}{\varphi(d/(d, i))}.$$

(Here  $\mu(N)$  = the Möbius function =  $(-1)^j$  if  $N$  is a product of  $j$  distinct primes, and 0 if there is a prime  $p$  such that  $p^2 \mid N$ ;  $\mu(1) = 1$ .) In particular, if  $k$  is a prime  $p$ , then

$$H(\beta^i) = \frac{1}{p} [(1-t)^{-p} + x(p, i) (1-t^p)^{-1}],$$

where

$$x(p, i) = \begin{cases} p-1 & \text{if } p \mid i \\ -1 & \text{if } p \nmid i. \end{cases}$$

$$A_1: T = \{1, \beta\};$$

$$H_{A1}(1) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + \dots$$

$$H_{A1}(\beta) = t + t^2 + 2t^3 + 2t^4 + 3t^5 + 3t^6 + 4t^7 + 4t^8 + \dots$$

$$A_2: T = \{1, \beta, \beta^2\};$$

$$H_{A2}(1) = 1 + t + 2t^2 + 4t^3 + 5t^4 + 7t^5 + 10t^6 + 12t^7 + 15t^8 + \dots$$

$$\begin{aligned} H_{A2}(\beta) &= H_{A2}(\beta^2) \\ &= t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 9t^6 + 12t^7 + 15t^8 + \dots \end{aligned}$$

$$A_3: T = \{1, \beta, \beta^2, \beta^3\};$$

$$H_{A3}(1) = 1 + t + 3t^2 + 5t^3 + 10t^4 + 14t^5 + 22t^6 + 30t^7 + 43t^8 + \dots$$

$$\begin{aligned} H_{A3}(\beta) &= H_{A3}(\beta^3) \\ &= t + 2t^2 + 5t^3 + 8t^4 + 14t^5 + 20t^6 + 30t^7 + 40t^8 + \dots \end{aligned}$$

$$H_{A3}(\beta^2) = t + 3t^2 + 5t^3 + 9t^4 + 14t^5 + 22t^6 + 30t^7 + 42t^8 + \dots$$

$$A_4: T = \{1, \beta, \beta^2, \beta^3, \beta^4\};$$

$$H_{A4}(1) = 1 + t + 3t^2 + 7t^3 + 14t^4 + 26t^5 + 42t^6 + 66t^7 + 99t^8 + \dots$$

$$\begin{aligned} H_{A4}(\beta) &= H_{A4}(\beta^2) = H_{A4}(\beta^3) = H_{A4}(\beta^4) \\ &= t + 3t^2 + 7t^3 + 14t^4 + 25t^5 + 42t^6 + 66t^7 + 99t^8 + \dots \end{aligned}$$

$$A_5: T = \{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5\};$$

$$H_{A5}(1) = 1 + t + 4t^2 + 10t^3 + 22t^4 + 42t^5 + 80t^6 + 132t^7 + 217t^8 + \dots$$

$$\begin{aligned} H_{A5}(\beta) &= H_{A5}(\beta^5) \\ &= t + 3t^2 + 9t^3 + 20t^4 + 42t^5 + 75t^6 + 132t^7 + 212t^8 + \dots \end{aligned}$$

$$\begin{aligned} H_{A5}(\beta^2) &= H_{A5}(\beta^4) \\ &= t + 4t^2 + 9t^3 + 22t^4 + 42t^5 + 78t^6 + 132t^7 + 217t^8 + \dots \end{aligned}$$

$$H_{A5}(\beta^3) = t + 3t^2 + 10t^3 + 20t^4 + 42t^5 + 76t^6 + 132t^7 + 212t^8 + \dots$$

$$A_6: T = \{1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \beta^6\}$$

$$H_{A6}(1) = 1 + t + 4t^2 + 12t^3 + 30t^4 + 66t^5 + 132t^6 + 246t^7 + 429t^8 + \dots$$

$$\begin{aligned} H_{A6}(\beta^i) &= t + 4t^2 + 12t^3 + 30t^4 + 66t^5 \\ &\quad + 132t^6 + 245t^7 + 429t^8 + \dots \quad \text{for } 7 \nmid i. \end{aligned}$$

$$A_7: T = \{1, \beta, \beta^2, \beta^3, \dots, \beta^7\};$$

$$H_{A7}(1) = 1 + t + 5t^2 + 15t^3 + 43t^4 + 99t^5 + 217t^6 + 429t^7 + 810t^8 + \dots$$

$$\begin{aligned} H_{A7}(\beta^i) \text{ for } i \text{ odd} &= t + 4t^2 + 15t^3 + 40t^4 \\ &\quad + 99t^5 + 212t^6 + 429t^7 + 800t^8 + \dots \end{aligned}$$

$$\begin{aligned} H_{A7}(\beta^2) &= H_{A7}(\beta^6) \\ &= t + 5t^2 + 15t^3 + 42t^4 + 99t^5 + 217t^6 + 429t^7 + 808t^8 + \dots \end{aligned}$$

$$H_{A7}(\beta^4) = t + 5t^2 + 15t^3 + 43t^4 + 99t^5 + 217t^6 + 429t^7 + 809t^8 + \dots$$

$$A_8 : T = \{1, \beta, \beta^2, \dots, \beta^8\};$$

$$H_{A8}(1) = 1 + t + 5t^2 + 19t^3 + 55t^4 + 143t^5 + 335t^6 + 715t^7 + 1430t^8 + 2704t^9 + \dots$$

$$H_{A8}(\beta^i) \text{ if } 3 \nmid i = t + 5t^2 + 18t^3 + 55t^4 + 143t^5 + 333t^6 + 715t^7 + 1430t^8 + 2700t^9 + \dots$$

$$H_{A8}(\beta^3) = H_{A8}(\beta^6) = t + 5t^2 + 19t^3 + 55t^4 + 143t^5 + 335t^6 + 715t^7 + 1430t^8 + 2703t^9 + \dots$$

### III. EXTREMAL SURFACES

Our goal in this section is to compute the number  $|\mathcal{E}|$  of exceptional curves on an extremal (rational minimal numerically elliptic) surface  $X$  in terms of the data assembled in the previous sections. To state the main theorem, we require just a bit more notation. Note that  $[K^\perp]^\wedge / \Gamma^\wedge$  can be viewed as a subgroup of  $\oplus T_F$  in the extremal case; we denote this subgroup by  $R$ . For  $r$  in  $R$ , we write  $r_F$  for the coordinate of  $r$  in  $T_F$ . Recall also the degree function  $d$  on  $A_F^*$ .

**THEOREM III.1.** *Let  $X$  be an extremal rational minimal numerically elliptic surface. Fix any  $(-1)$ -class  $E$  in  $M_1$ . Then*

$$|\mathcal{E}| = \sum_{r \in R} \prod_F (\text{coefficient of } t^{d([E^\wedge]_F)} \text{ in } H_F(\text{torsion part of } ([E^\wedge]_F + r_F))).$$

*Proof.* Note that  $\mathcal{L} = K^\perp / K$  acts transitively on  $M_1$  with trivial stabilizers, and  $\Gamma / K \subseteq \mathcal{L}$  is finite index in the extremal case. Fix any  $E \in M_1$ . Then  $M_1 = \mathcal{L} \cdot E$ . (Here the “ $\cdot$ ” is the  $\mathcal{L}$ -action.) Also,

$$M_1 = \bigcup_{x + \Gamma \in K^\perp / \Gamma} (x + \Gamma) \cdot E = \bigcup_{x + \Gamma \in K^\perp / \Gamma} \Gamma \cdot (x \cdot E),$$

and these unions are disjoint. Hence

$$\mathcal{E} = \left( \bigcup_{x + \Gamma \in K^\perp / \Gamma} \Gamma \cdot (x \cdot E) \right) \cap A_+ = \bigcup_{x + \Gamma \in K^\perp / \Gamma} (\Gamma \cdot (x \cdot E) \cap A_+)$$

and the union is disjoint, so

$$\begin{aligned}
|\mathcal{E}| &= \sum_{x + \Gamma \in K^\perp/\Gamma} |\Gamma \cdot (x \cdot E) \cap \Delta_+| = \sum_{x + \Gamma \in K^\perp/\Gamma} |\Gamma \cdot (x \cdot E)^\wedge \cap \Delta_+^\#| \\
&\quad \text{since } \wedge \text{ is injective on the } \Gamma\text{-orbits} \\
&= \sum_{x + \Gamma \in K^\perp/\Gamma} |[\ ]^{-1}[\Gamma \cdot (x \cdot E)^\wedge] \cap \Delta_+^\#| \quad \text{by Lemma I.6(b)} \\
&= \sum_{x + \Gamma \in K^\perp/\Gamma} |[\ ]^{-1}[(x \cdot E)^\wedge] \cap \Delta_+^\#| \quad \text{since } [\ ] \text{ is modding by } \Gamma \\
&= \sum_{x + \Gamma \in K^\perp/\Gamma} |[\ ]^{-1}[(x^\wedge + E^\wedge)] \cap \Delta_\mu^X \omega \quad \text{since } \langle x \cdot E \rangle^\wedge = x^\wedge + E^\wedge \\
&= \sum_{r \in R} |[\ ]^{-1}([E^\wedge] + r) \cap \Delta_+^\#| \quad \text{using the definition of } R \\
&= \sum_{r \in R} \prod_{F \in F} |[\ ]_F^{-1}([E^\wedge]_F + r_F) a \Delta_{F+}^\#| \\
&= \sum_{r \in R} \prod_{F \in F} (\text{coefficient of } t^{d([E^\wedge]_F)} \text{ in } H_F(\text{torsion part of } ([E^\wedge] + r_F))). \blacksquare
\end{aligned}$$

This is our main theorem. To make this solution of more interest we now show what collections  $F$  of reducible fibers arise and what images  $R = [K^\perp \wedge / \Gamma^\wedge]$  arise.

It is easy but tedious to check that if  $F$  is a particular graph from Fig. 1 then the sublattice of  $\mathcal{L}$  generated by  $\Delta_F$  is the root lattice of the simple complex Lie algebra of the same name (without the  $\sim$ ) as that used for the graph of  $F$  in Fig. 1. If  $\Delta \rightarrow \mathcal{L}$  is an embedding of a disjoint union of such graphs, the lattice generated by  $\Delta$  in  $\mathcal{L}$  is the orthogonal direct sum of the root lattices corresponding to the various graphs comprising  $\Delta$ . The sublattices of  $\mathcal{L}$  corresponding to orthogonal sums of root lattices have been classified. The result of interest in the extremal case is for sublattices of rank 8.

**THEOREM III.2.** *The rank 8 lattices which are sums of root lattices and which embed in  $\mathcal{L}$  are precisely those listed below:*

$$\begin{aligned}
&E_8, A_8, D_8, E_7 \oplus A_1, A_7 \oplus A_1, E_6 \oplus A_2, D_5 \oplus A_3, D_4^{\oplus 2}, A_4^{\oplus 2}, \\
&D_6 \oplus A_1^{\oplus 2}, A_5 \oplus A_2 \oplus A_1, A_3^{\oplus 2} \oplus A_1^{\oplus 2}, A_2^{\oplus 4}, D_4 \oplus A_1^{\oplus 4}, \text{ and} \\
&A_1^{\oplus 8}.
\end{aligned}$$

*Up to reflections through the  $(-2)$ -classes in  $\mathcal{L}$ , each of these lattices embeds in  $\mathcal{L}$  on a unique way.*

*Proof.* See [Dy].  $\blacksquare$

For our applications, we need to appeal to some of the theory of Nikulin [N]. Let  $L$  be one of the above 15 direct sums of root lattices. The

bilinear form (which is even, negative definite, and  $\mathbb{Z}$ -valued) on  $L$  extends to one on the dual lattice  $L^*$  (which is  $\mathbb{Q}$ -valued there), and descends to a quadratic form  $q$  in  $L^*/L$ , defined by  $q(x \bmod L) = \langle x \cdot x \rangle/2 \bmod \mathbb{Z}$ . Any overlattice of  $L$  corresponds to a totally isotropic subgroup  $R$  of  $L^*/L$ ; the overlattice is realized as  $\pi^{-1}(R)$ , where  $\pi$  is the quotient map from  $L^*$  to  $L^*/L$ . The isotropicity ensures that  $\pi^{-1}(R)$  inherits a  $\mathbb{Z}$ -valued bilinear form. Now, according to Nikulin's theory,  $\pi^{-1}(R)$  is unimodular if and only if  $|R|^2 = |L^*/L|$ . Since all these  $L$ 's are negative definite and rank 8, if this size condition on  $R$  is satisfied,  $\pi^{-1}(R)$  must be  $E_8$  and we have realized our embedding of  $L$  into  $E_8$ .

Now in our situation the  $L$  is exactly  $\Gamma/\langle K_X \rangle$  and the  $E_8$  lattice is  $K^\perp/K$ , so that the  $R$  is realized as  $K^\perp/\Gamma$ , or, more precisely in our situation,  $[K^\perp \wedge \Gamma^\wedge]$ . That is, the  $R$  of the discussion above is the previous  $R$ , and the finite group  $L^*/L$  is exactly  $\bigoplus T_F$ .

For each group  $\bigoplus T_F$ , there are only a finite number of possible  $R$ 's satisfying the size condition and the isotropicity. We list them below; it is exactly this information that Theorem III.1 requires.

*List III.3.* Root lattices which embed into the  $E_8$  lattice, with the possible isotropic subgroups  $R$ . For cyclic  $T_F$ , generated by  $\beta$ , we associate the exponent of  $\beta$ ; for  $D_{2k}$ , where  $T$  is the Klein four group  $\{1, \beta, \gamma, \beta\gamma\}$ , we associate  $(1, 0)$  to  $\beta$ ,  $(0, 1)$  to  $\gamma$ , and  $(1, 1)$  to  $\beta\gamma$ .

$$E_8: \quad T = \{1\}; \quad R = \{1\}.$$

$$E_7 \oplus A_1: \quad T = \mathbb{Z}_2 \times \mathbb{Z}_2; \quad q(a, b) = (a^2 - b^2)/4. \\ R = \{(0, 0), (1, 1)\}$$

$$E_6 \oplus A_2: \quad T = \mathbb{Z}_3 \times \mathbb{Z}_3; \quad q(a, b) = (a^2 - b^2)/3. \\ R = \{(0, 0), (1, 1), (2, 2)\} \\ R = \{(0, 0), (1, 2), (2, 1)\}$$

$$D_8: \quad T = \mathbb{Z}_2 \times \mathbb{Z}_2; \quad q(a, b) = ab/2. \\ R = \{(0, 0), (1, 0)\} \\ R = \{(0, 0), (0, 1)\}$$

$$D_6 \oplus A_1^{\oplus 2}: \quad T = \mathbb{Z}_2^{\oplus 4}; \quad q(a, b, c, d) = (a^2 + b^2 - c^2 - d^2)/4. \\ R = \{(0, 0, 0, 0), (0, 1, 1, 0), (1, 0, 0, 1), (1, 1, 1, 1)\} \\ R = \{(0, 0, 0, 0), (0, 1, 0, 1), (1, 0, 1, 0), (1, 1, 1, 1)\}$$

$$D_5 \oplus A_3: \quad T = \mathbb{Z}_4 \times \mathbb{Z}_4; \quad q(a, b) = 3(a^2 - b^2)/8. \\ R = \{(0, 0), (1, 1), (2, 2), (3, 3)\} \\ R = \{(0, 0), (1, 3), (2, 2), (3, 1)\}$$

$$\begin{aligned}
D_4^{\oplus 2}: \quad T &= \mathbb{Z}_2^{\oplus 4}; \quad q(a, b, c, d) \\
&= (a^2 + ab + b^2 + c^2 + cd + d^2)/2. \\
R &= \{(0, 0, 0, 0), (1, 0, 1, 0), (0, 1, 0, 1), (1, 1, 1, 1)\} \\
R &= \{(0, 0, 0, 0), (1, 0, 1, 0), (0, 1, 1, 1), (1, 1, 0, 1)\} \\
R &= \{(0, 0, 0, 0), (1, 0, 0, 1), (0, 1, 1, 0), (1, 1, 1, 1)\} \\
R &= \{(0, 0, 0, 0), (1, 0, 0, 1), (0, 1, 1, 1), (1, 1, 1, 0)\} \\
R &= \{(0, 0, 0, 0), (1, 0, 1, 1), (0, 1, 1, 0), (1, 1, 0, 1)\} \\
R &= \{(0, 0, 0, 0), (1, 0, 1, 1), (0, 1, 0, 1), (1, 1, 1, 0)\}
\end{aligned}$$

i.e.,  $R = \{(0, 0, 0, 0), (1, 0, x), (0, 1, y), (1, 1, z)\}$  where  $\{x, y, z\}$  are the three nonzero elements of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ,

$$\begin{aligned}
D_4 \oplus A_1^{\oplus 4}: \quad T &= \mathbb{Z}_2^{\oplus 6}; \quad q(a, b, c, d, e, f) \\
&= (a^2 + ab + b^2)/2 - (c^2 + d^2 + e^2 + f^2)/4. \\
R &= \{(000000), (001111), (10x_1)(10x_2)(01y_1)(01y_2)(11z_1)(11z_2)\},
\end{aligned}$$

where  $x_i, y_i, z_i \in \mathbb{Z}_2^{\oplus 4}$ , each with exactly two 0's and two 1's, such that  $x_1 + x_2 = y_1 + y_2 = z_1 + z_2 = (1111)$  and  $x_i + y_i + z_i = (0000)$ .

There are six such subgroups  $R$  of  $T$ .

$$\begin{aligned}
A_8: \quad T &= \mathbb{Z}_9; \quad q(a) = 5a^2/9. \\
R &= \{0, 3, 6\}. \\
A_7 \oplus A_1: \quad T &= \mathbb{Z}_8 \times \mathbb{Z}_2; \quad q(a, b) = (a^2 - 4b^2)/16. \\
R &= \{(0, 0), (2, 1), (4, 0), (6, 1)\}. \\
A_4^{\oplus 2}: \quad T &= \mathbb{Z}_5 \times \mathbb{Z}_5; \quad q(a, b) = 3(a^2 + b^2)/5. \\
R &= \{(0, 0), (1, 2), (2, 4), (3, 1), (4, 3)\} \\
R &= \{(0, 0), (1, 3), (2, 1), (3, 4), (4, 2)\}.
\end{aligned}$$

$$\begin{aligned}
A_5 \oplus A_2 \oplus A_1: \quad T &= \mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2; \quad q(a, b, c) \\
&= 7a^2/12 + 2b^2/3 + 3c^2/4 = (7a^2 + 8b^2 + 9c^2)/12. \\
R &= \{(0, 0, 0), (2, 1, 0), (4, 2, 0), (3, 0, 1), (5, 1, 1), (1, 2, 1)\} \\
R &= \{(0, 0, 0), (4, 1, 0), (2, 2, 0), (3, 0, 1), (1, 1, 1), (5, 2, 1)\}. \\
A_3^{\oplus 2} \oplus A_1^{\oplus 2}: \quad T &= \mathbb{Z}_4^{\oplus 2} \times \mathbb{Z}_2^{\oplus 2}; \quad q(a, b, c, d) = (5a^2 + 5b^2 + 6c^2 + 6d^2)/8. \\
R &= \{(0, 0, 0, 0), (0, 2, 1, 1), (2, 0, 1, 1), (2, 2, 0, 0), \\
&\quad (1, 1, 1, 0), (1, 3, 0, 1), (3, 1, 0, 1), (3, 3, 1, 0)\} \\
R &= \{(0, 0, 0, 0), (0, 2, 1, 1), (2, 0, 1, 1), (2, 2, 0, 0), \\
&\quad (1, 1, 0, 1), (1, 3, 1, 0), (3, 1, 1, 0), (3, 3, 0, 1)\}
\end{aligned}$$

$$A_2^{\oplus 4}: \quad T = \mathbb{Z}_3^{\oplus 4}; \quad q(a, b, c, d) = 2(a^2 + b^2 + c^2 + d^2)/3.$$

$$R = \{(0000), (01x_1), (02y_1), (10x_2), (20y_2),$$

$$(11z), (21w), (12u), (22v)\},$$

where  $x_i, y_i \in \{1, 1\}, (1, 2), (2, 1), (2, 2)\}$ ,  $x_i + y_i = (0, 0)$ ,  $z, w, u, v \in \{(01), (02), (10), (20)\}$ , such that  $x_1 + x_2 = z$ ,  $x_1 + y_2 = w$ ,  $y_1 + x_2 = u$ , and  $y_1 + y_2 = v$ . One can choose  $x_1$  arbitrarily, and then any  $x_2$  with exactly one entry different from  $x_1$ . Then the  $y_i$ 's, and  $z, w, u, v$  are determined. There are eight such subgroups  $R$  of  $T$ .

$$A_1^{\oplus 8}: \quad T = \mathbb{Z}_2^{\oplus 8}; \quad q(\mathbf{a}) = 3 \sum a_i^2/4, \quad \text{where } \mathbf{a} = (a_1, \dots, a_8).$$

$\sum a_i^2 = 0$  defines a quadratic  $Q$  in  $\mathbb{P}_{\mathbb{F}_2}^7$ . This quadric is ruled by  $\mathbb{P}_{\mathbb{F}_2}^3$ 's in two different ways. Any of these linear spaces lift to subspaces of  $T$  of rank 4; these are the possible  $R$ 's. To be more specific, partition  $\{1, \dots, 8\}$  into pairs  $\{i_1, i_5\}$ ,  $\{i_2, i_6\}$ ,  $\{i_3, i_7\}$ , and  $\{i_4, i_8\}$ . Any such partition determines an isotropic subgroup  $R$  as  $R = \{\mathbf{x} \in \mathbb{F}_2^8 \mid x_{i_j} + x_{i_{j+1}} = 1 \text{ for } j = 1, \dots, 4\}$ . For example, if one chooses each  $i_k = k$ , then one obtains the  $R$  whose  $\mathbb{F}_2$ -basis is

$$(1, 0, 0, 0, 0, 1, 1, 1),$$

$$(0, 1, 0, 0, 1, 0, 1, 1),$$

$$(0, 0, 1, 0, 1, 1, 0, 1),$$

and

$$(0, 0, 0, 1, 1, 1, 0).$$

All others are gotten by permuting the columns; there are 105 such  $R$ 's.

*Remark III.4.* It is perhaps worth remarking that for each of the 15 root lattices above, one can construct a numerically elliptic surface which gives rise to it. See Section IV.

We are now able to prove Theorem 0.1:

*Proof of 0.1.* By Corollary II.6,  $H_F(g) = (1/|T_F|) \chi(G_F)$  for every  $F$  and every  $g$  in  $T_F$ . Moreover,  $|T_F| = s_F$ ,  $|R| = (\prod_F s_F)^{1/2}$  (by the Nikulin theory), and  $d([E^\wedge]_F) = m/m_F$ , and since  $\chi(G_F) = s_F^{1/2} Q_F$ , we have coefficient of  $t^{d([E^\wedge]_F)}$  in  $H_F$  (torsion part of  $[E^\wedge] + r_F$ ) = coefficient of  $t^{m/m_F}$  in  $s_F^{-1/2} Q_F$ , independent of  $r$ . Hence  $|\mathcal{E}| = |R| \prod_F (s_F^{-1/2} Q_F)_{m/m_F} = \prod_F (Q_F)_{m/m_F}$  as claimed. ■

*Remark III.5.* In general, it is clear from Theorem III.1 that to compute the number of exceptional curves  $|\mathcal{E}|$  in any case it suffices to know the following data:

- The reducible fibers  $F$  of the numerically elliptic surface. This information is used to label the vertices of the corresponding graphs, and to give projections onto the various torsion parts  $T_F$  of the  $\mathcal{A}_F^*$ 's.
- One  $(-1)$ -class  $[E^\wedge]$ , written in terms of the generators set up for the  $T_F$ 's in Section II.
- The subgroup  $R = [K^\perp \wedge / \Gamma^\wedge]$  of  $\bigoplus T_F$ .
- The relevant coefficients of the power series  $H_F$ .

Of course, in some situations one can get away with less; in particular, in Theorem 0.1 only the fibers and the power series are necessary. Moreover, the  $R$ 's can be determined from fairly meager data in many cases: the difference of any two exceptional curves gives a class in  $K^\perp$ , hence in  $R$ , and if one has enough such elements one can pin  $R$  down fairly quickly. Also, List III.3 shows that if the fibers are  $E_7$  and  $A_1$ , then  $R$  is already determined since only one  $R$  is possible. Finally, if one only knows the fibers and  $m$ , and no other information, one can at least give a finite list of possibilities for  $|\mathcal{E}|$  by making the calculation for all possible  $R$ 's and all possible cosets of those  $R$ 's (the class  $[E^\wedge]$  really only contributes in the formula of Theorem III.1 to the  $R$ -coset). And in fact this is not so complicated; it turns out that all  $R$ 's are conjugate under an automorphism of  $\bigoplus T_F$  and one can get all the possibilities by trying all cosets of just one  $R$ .

*Remark III.6.* We will close this section by explicitly showing how to determine the classes in  $\mathcal{E}$  themselves, instead of simply their number. Let  $\delta_i = v_i - \mu_i v_0$ ; each  $\delta_i$  is in the  $\mathbb{Q}$ -span of the  $v_i^\wedge$ , with  $i \geq 1$ . Note that  $[\delta_i] = g_i$ . Suppose that one  $(-1)$ -curve  $E$  is given. Write  $E^\wedge = \sum p_i v_i$  with  $p_i \geq 0$ . The set  $\mathcal{E}^\wedge$  is of course  $\{mv_0 + \sum q_i \delta_i \mid q_i \geq 0, \sum q_i \mu_i \leq m, \sum (p_i - q_i) g_i \in R\}$ . To find  $\mathcal{E}$  instead of  $\mathcal{E}^\wedge$ , we must know  $E$  and the  $(-2)$ -curves. Then we simply plug in and get all the exceptional curves as the set  $\{E + \sum (q_i - p_i) w_i + (\frac{1}{2})[1 + (E + \sum (q_i - p_i) \delta_i)^2] K_X\}$ , where we have written  $w_i$  for the element in the  $\mathbb{Q}$ -span of the  $v_i$  such that  $w_i^\wedge = \delta_i$ . If one desires to have the classes of  $\mathcal{E}$  in terms of an exceptional configuration, one must now express the  $(-2)$ -curves in that way.

We use this method in Example V.6.

#### IV. EXISTENCE

One can arbitrarily prescribe data  $\{\mathcal{A}, R, \delta \in \mathcal{A}^*\}$  (here  $\delta = [E^\wedge]$  if the data comes from a surface  $X$ ) and compute a putative number of  $(-1)$ -curves. But it is natural to ask:

*Question IV.1.* When does a surface exist with prescribed numerical data  $\{\mathcal{A}, R, \delta\}$ , when  $\mathcal{A}$  is an extremal set of fibers?

We will give a partial answer to this question in this section. Recall that after choosing a multiplicity one component of each fiber  $F$ , we may write  $\delta = \sum_F (n_F, g_F)$ , where  $g_F \in T_F$  and  $n_F \in \mathbb{Z}$ . Moreover, by Remark II.7, each  $n_F = m$  except possibly for one  $F$ , and in that case  $n_F = 1$ ; this case occurs only if the  $F$  with  $n_F = 1$  is the multiple fiber. We call the data  $\{\mathcal{A}, R, \delta\}$  *uniform* if  $n_F = m$  for every  $F$ ; this corresponds to either a Jacobian surface or a non-Jacobian surface whose multiple fiber is irreducible. It is the case of uniform data for which we will answer Question IV.1. Note that in any case the multiplicity  $m$  is well defined from the data  $\{\mathcal{A}, R, \delta\}$ .

To begin, fix uniform extremal data  $\{\mathcal{A}, R, \delta\}$ . For each  $F$  recall the degenerate lattice  $(\mathcal{A}_F)$  with basis  $\mathcal{A}_F$ , and define  $\bar{L}_F = (\mathcal{A}_F)/\text{rad}(\mathcal{A}_F)$ . By Nikulin's theory,  $R$  determines an intermediate lattice  $\bar{M}$  between  $\bigoplus_F \bar{L}_F$  and  $(\bigoplus_F \bar{L}_F)^*$ , which is abstractly isomorphic to an  $E_8$  lattice. Form the lattice  $M = \bar{M} \oplus \mathbb{Z}I$  by introducing an additional basis vector  $I$  which is isotropic and orthogonal to  $\bar{M}$ ;  $M$  is abstractly isomorphic to an  $E_9$  lattice.

Choose an inverse image  $e$  in  $\mathcal{A}^*$  of  $\delta$ . Define  $i_e: (\mathcal{A}) \rightarrow M = \bar{M} \oplus \mathbb{Z}I$  by  $i_e(x) = (x \bmod \text{rad}(\mathcal{A}), e(x)I)$ . We claim that  $i_e$  is independent of  $e$ , up to an automorphism of  $M$ . To see this, let  $e$  and  $e'$  be two lifts of  $\delta$  to  $\mathcal{A}^*$ . Then  $r^\wedge = e - e' \in (\mathcal{A}^\wedge)$ ; if one defines  $\alpha: M \rightarrow M$  by  $\alpha(v) = v + (r \cdot v)I$ , then  $\alpha$  transports  $i_{e'}$  to  $i_e$  and is a lattice automorphism. This justifies denoting  $i_e$  by  $i_\delta$ , which we will henceforward do.

Define  $\mathcal{G}$  to be the quotient  $M/\text{Im}(i_\delta)$ ; we call  $\mathcal{G}$  the *group associated to the data  $\{\mathcal{A}, R, \delta\}$* . Since we assume that  $\mathcal{A}$  is an extremal set of fibers,  $\mathcal{G}$  has rank 1, and is an extension of  $R$  by  $\mathbb{Z}/m\mathbb{Z}$ . If  $\{\mathcal{A}, R, \delta\}$  comes from a surface  $X$ , then  $\mathcal{G} \cong K_X^\perp/(\mathcal{A}^0)$ .

**DEFINITION IV.2.** The data  $\{\mathcal{A}, R, \delta\}$  is *split* if  $\mathcal{G} \cong R \oplus \mathbb{Z}/m\mathbb{Z}$ , i.e., if the extension  $\mathcal{G}$  of  $R$  by  $\mathbb{Z}/m\mathbb{Z}$  splits.

The structure of  $\mathcal{G}$  is, as we will see, important for deciding the answer to Question IV.1, and so it is useful to have a simple criterion for given data to be split. This we now provide.

**PROPOSITION IV.3.** *Let  $X$  be an extremal surface with uniform data  $\{\mathcal{A}, R, \delta\}$ . Then  $\{\mathcal{A}, R, \delta\}$  is split if and only if  $X$  has a  $(-1)$ -curve  $E$  such that, for every reducible fiber  $F$  of  $X$ ,  $E$  meets  $F$  only at a single, multiplicity one component of  $F$ .*

*Proof.* We may assume  $m \geq 2$ . Assume first that  $E$  as above is given. For each  $F$ , number the components of  $F$  as in Section II so that the multiplicity one component  $v_0$  is the one meeting  $E$ , and let  $\mathcal{A}'$  be the set of

other components. Then  $(\mathcal{A})$  is generated by the elements  $\mathcal{A}'$  orthogonal to  $E$  and by  $mK_X$ ; but  $K_X^\perp = E^\perp \oplus \mathbb{Z}K_X$ , so that

$$\mathcal{G} = K_X^\perp / (\mathcal{A}) = (E^\perp / \mathcal{A}') \oplus (\mathbb{Z} \cdot K_X / \mathbb{Z} \cdot mK_X) = R \oplus \mathbb{Z}/m\mathbb{Z}.$$

Conversely, assume that the data  $\{\mathcal{A}, R, \delta\}$  for  $X$  splits, and choose a splitting  $K^\perp / (\mathcal{A}) \rightarrow \mathbb{Z}/m\mathbb{Z}$ . Let  $\Omega$  be the kernel of the composition  $K^\perp \rightarrow \mathbb{Z}/m\mathbb{Z}$ . Since  $\Omega / (\Omega \cap \mathbb{Z} \cdot K_X) = \Omega / (m\mathbb{Z} \cdot K_X)$  is isomorphic to  $E_8$ , there is a splitting  $\Omega \cong E_8 \oplus \mathbb{Z} \cdot mK_X$ . Therefore, abstractly,  $\Omega$  is isomorphic to an  $E_9$  lattice.

In this situation there is a functional  $l_1$  in  $\Omega^*$  such that for every  $F$   $l_1(x_F) = 1$  for a unique weight one vertex  $x_F$  in  $\mathcal{A}_F$ , and  $l_1(v) = 0$  for other vertices  $v$  of  $F$ . To see this geometrically, pass to the Jacobian surface  $\bar{X}$  of  $X$ .  $\bar{X}$  has a section  $E$ , and  $E^\wedge$  is such a functional for  $K_{\bar{X}}^\perp$ . Simply transport this functional to  $l_1$  via an isomorphism between  $\Omega$  and  $K_{\bar{X}}^\perp$  which preserves  $\mathcal{A}$ . (One can exhibit the desired  $l_1$  using pure lattice theory also, à la the arguments of Section III.)

Let  $\Omega_0 = \ker(l_1)$ ; then  $\Omega \cong \Omega_0 \oplus \mathbb{Z} \cdot mK_X$  and  $\Omega_0$  is isomorphic to an  $E_8$  lattice. Moreover, regarding  $l_1$  as being in  $(\Omega_0 \oplus \mathbb{Z} \cdot mK_X)^*$  we have  $l_1(mK_X) = -1$ .

Since  $\Omega \cong \Omega_0 \oplus \mathbb{Z} \cdot mK_X \subseteq K_X^\perp$ , we must have  $K_X^\perp = \Omega_0 \oplus \mathbb{Z} \cdot K_X$ ; hence  $ml_1$  extends to a functional  $l$  on  $K_X^\perp$  such that  $l(K_X) = -1$ , and still  $l(x_F) = m$ , but  $l(v) = 0$  for  $v \in \mathcal{A}_F - \{x_F\}$ . Using the unimodularity of  $K^\perp / K$ , one can see that the image  $M_1^\wedge$  of  $M_1$  in  $(K^\perp)^*$  is exactly the set  $\{l \in (K^\perp)^* \mid l(K) = -1\}$ . Hence there is a  $(-1)$ -class  $E$  such that  $l = E^\wedge$ . The assumptions above imply that  $E^\wedge \in \mathcal{A}_+$ , so that  $E$  represents a  $(-1)$ -curve. ■

This splitting criterion can easily be reformulated so as not to depend on the existence of  $X$ ; one simply replaces the existence of  $E$  by the existence of the appropriate functional.

Note that, given a surface  $X$ , the data  $\{\mathcal{A}, R, \delta\}$  is not determined;  $R$  depends on a choice of weight one component (and a numbering of the components) in each fiber, and  $\delta$  depends on the choice of a  $(-1)$ -class. We can to some extent normalize the data in the uniform split case:

**COROLLARY IV.4.** *Let  $X$  be an extremal surface with uniform split data. Then  $R \subset \mathcal{A}^*$  can be taken arbitrarily, and  $\delta \in \mathcal{A}^*$  can be taken so that the torsion part of  $\delta$  is trivial.*

*Proof.* Let  $E$  be the  $(-1)$ -curve of the previous proposition. Then after appropriate renumbering of  $\mathcal{A}$ ,  $E$  meets only component  $v_0$  of each reducible fiber. Hence  $\delta$  is as desired. To see that  $R$  may be taken arbitrarily, simply note that all possible  $R$ 's are conjugate under lattice automorphisms of  $\mathcal{A}^*$  preserving  $\mathcal{A}$ , and these fix  $\delta$ . ■

We can now give our partial answer to IV.1:

**THEOREM IV.5.** (a) *Suppose  $X$  is extremal and has an irreducible fiber  $mC$ ,  $m \geq 1$ , where  $C$  is anti-canonical. Then  $K^\perp/(\Delta)$  is isomorphic to a subgroup of  $\text{Pic}^0(C)$ .*

(b) *Let  $\{\Delta, R, \delta\}$  be uniform extremal data such that the associated group  $\mathcal{G}$  is isomorphic to a subgroup of  $\text{Pic}^0(C)$  for some irreducible cubic curve  $C \subset \mathbb{P}^2$ . Then there is a numerically elliptic surface  $X$  inducing the data  $\{\Delta, R, \delta\}$ .*

*Proof.* (a) Assume first that  $\Delta \neq A_1^{\oplus 8}$ . Consider the restriction  $\pi: \text{Pic}(X) \rightarrow \text{Pic}(C)$ ; note that  $(\Delta) \subset \ker(\pi)$  so that  $K^\perp/(\Delta)$  maps to  $\text{Pic}^0(C)$ . Suppose that  $K^\perp/(\Delta)$  does not embed into  $\text{Pic}^0(C)$ ; then there is an intermediate lattice  $\Gamma$  between  $(\Delta)$  and  $K^\perp$ ,  $\Gamma \neq (\Delta)$ , with  $\pi(\Gamma) = 0$ . One checks that for  $(\Delta) \neq A_1^{\oplus 8}$ , any intermediate lattice is a root lattice, and so  $\Gamma$  has a  $(-2)$ -class  $r$  not in  $(\Delta)$ . Riemann-Roch now implies that either  $r$  or  $-r$  is effective. But any effective element in  $K^\perp$  is a linear combination of  $(-2)$ -curves, so this is impossible since  $r \notin (\Delta)$ .

Suppose finally that  $\Delta = A_1^{\oplus 8}$ . Then  $X$  is quasi-elliptic and the characteristic must be 2, so  $C$  must be of additive type, and in particular  $\text{Pic}^0(C)$  has only 2-torsion. Moreover, the data must split: otherwise the image of  $K_X$  in  $K_X^\perp/(\Delta)$  would be twice another element, hence  $K_X$  goes to zero under  $\pi$ , forcing  $X$  to be Jacobian, in which case the data is trivially split. Thus  $K^\perp/(\Delta) \cong (\mathbb{Z}/2\mathbb{Z})^t$ , where  $t = 8$  or 9 (depending on whether  $X$  is Jacobian). Hence certainly  $K^\perp/(\Delta)$  is isomorphic to a subgroup of  $\text{Pic}^0(C)$ , even though the induced map  $K^\perp/(\Delta) \rightarrow \text{Pic}^0(C)$  may not be injective.

(b) As in the discussion at the beginning of the section, the data  $\{\Delta, R, \delta\}$  determines an inclusion  $i_\delta: (\Delta) \rightarrow M$  and a quotient  $\pi: M \rightarrow \mathcal{G}$ , with  $\ker(\pi) = \text{Im}(i_\delta)$ . Choose a set of simple roots  $r_0, \dots, r_8$  of the  $E_9$  lattice  $M$  with respect to which  $\Delta$  is a set of positive roots. Now  $\pi: M \rightarrow \mathcal{G}$  induces  $f: M \rightarrow \text{Pic}^0(C)$ , since by hypothesis  $\mathcal{G} \subset \text{Pic}^0(C)$ . Pick a smooth point  $q_1$  of  $C$  and inductively define  $\{q_i \mid 2 \leq i \leq 9\}$  by  $\mathcal{O}_C(q_i) \cong \mathcal{O}_C(q_{i-1}) \otimes f(-r_{9-i})$ ; set  $\mathcal{L} = \mathcal{O}_C(q_1 + q_2 + q_3) \otimes f(r_8)$ . Now  $\mathcal{L}$  is very ample, giving an embedding of  $C$  into  $\mathbb{P}^2$ . Blowing up the images of the  $q_1, \dots, q_9$  we obtain a numerically elliptic surface  $X$  such that  $g: \text{Pic}(X) \rightarrow \text{Pic}(C)$  restricts to  $K^\perp \cong M$  as  $f$ . Now  $\Delta$  is a set of positive roots of  $K^\perp$  which generates  $\ker(\pi)$  so  $\Delta$  is indeed the set of  $(-2)$ -curves. Using the enumeration of  $\Delta$  as given in the data  $\{\Delta, R, \delta\}$ ,  $X$  induces the same  $R$  and  $\delta$  by construction. ■

## V. EXAMPLES

**EXAMPLE V.1.** The simplest example is of a surface  $X$  with an  $E_8$  fiber  $F$ . If  $F$  is multiple then  $X$  has precisely one exceptional curve. If  $F$  is not

a multiple fiber but the fibration on  $X$  has multiplicity  $m \geq 1$ , then the number of exceptional curves on  $X$  is the coefficient  $a_m$  of the  $t^m$  term of the Taylor series of  $[(1-t^6)(1-t^5)(1-t^4)^2(1-t^3)^2(1-t^2)^2(1-t)]^{-1}$ . So, for example,  $a_1 = 1$ ,  $a_2 = 3$ ,  $a_3 = 5$ , and  $a_6 = 27$ . These results follow immediately from Theorem 0.1, and the existence of an  $X$  for every  $m \geq 1$  follows from Theorem IV.5.

EXAMPLE V.2. If  $X$  is an extremal surface with  $\Delta = E_8$ ,  $A_8$ ,  $D_8$ ,  $E_7 \oplus A_1$ , or  $A_7 \oplus A_1$ , then there is only one possible  $R$  to use, so having a  $(-1)$ -class represented by some known  $\delta \in \Delta^*/(\Delta^\wedge)$ , the number of  $(-1)$ -curves on  $X$  is already determined. On the other hand, suppose  $F = E_6 \oplus A_2$  and  $\delta = (3, \beta) \oplus (3, \beta)$ . Using the two possible  $R$ 's in this case we find that there are either 68 or 66 exceptional curves on  $X$ . Both situations occur by Theorem IV.5.

EXAMPLE V.3. To give an example in greater detail, suppose  $X$  has reducible fibers  $F$  and  $G$  of types  $D_5$  and  $A_3$  and that  $X$  has an exceptional curve  $E$  such that (in the notation of Lemma II.1)  $E^\wedge = (v_0 + v_2 + v_5) \oplus (v_2) \in \Delta_F^* \oplus \Delta_G^*$ . Then by Table II.4 we have  $[E^\wedge] = (4, \beta^3) \oplus (1, \beta^2) \in (\Delta_F^*) \oplus (\Delta_G^*)$ . In this example the  $A_3$  fiber must, by II.7, be the multiple one, and the multiplicity is 4. There are two possible  $R$ 's to use here, one generated by  $(\beta, \beta)$ , the other by  $(\beta, \beta^3)$ . In the formula of Theorem III.1, the term corresponding to the fiber  $A_3$  is the coefficient of  $t^1$  in  $H_F(g)$  for various  $g$ 's; this coefficient is 1 for all  $g$ , and therefore Theorem 0.1 simplifies to

$$\begin{aligned} |\mathcal{E}| &= \sum_{r \in R} (\text{coefficient of } t^4 \text{ in } H_{D_5}(\text{torsion part of } [E^\wedge] + r_{D_5})) \\ &= \sum_{i=0}^3 (\text{coefficient of } t^4 \text{ in } H_{D_5}(\beta^i)) \\ &= 18 + 12 + 16 + 12 = 58, \text{ independently of which } R \text{ is chosen.} \end{aligned}$$

EXAMPLE V.4. The cases  $D_4 \oplus A_1^{\oplus 4}$  and  $A_1^{\oplus 8}$  are remarkable for only occurring for quasi-elliptic fibrations and only in characteristic 2. If  $X$  is a non-Jacobian quasi-elliptic surface with either of these two configurations of reducible fibers then the multiplicity of the fibration is 2. If the multiple fiber is irreducible, then by Section IV,  $K^\perp/\oplus_F \Gamma(F)$  splits and hence we can compute  $|\mathcal{E}|$  by choosing  $\delta = 0$  and choosing  $R$  arbitrarily. The result is  $|\mathcal{E}| = 133$  if  $\Delta = D_4 \oplus A_1^{\oplus 4}$  and  $|\mathcal{E}| = 481$  if  $\Delta = A_1^{\oplus 8}$ , and both actually occur. If the multiple fiber is reducible (a situation which definitely does occur for both configurations [HL]), then we get the following results by checking all possible choices of  $\delta$  and  $R$ . For  $D_4 \oplus A_1^{\oplus 4}$ , with  $D_4$  multiple,

$|\mathcal{E}|$  is either 40 or 41. If one of the  $A_1$ 's is multiple, then  $|\mathcal{E}|$  is 72 or 81. For  $A_1^{\oplus 8}$ , we get 270 or 297. Whether all of these actually occur we do not know.

**EXAMPLE V.5.** We now consider an example “from scratch.” Let  $C$  be a smooth conic in the plane, and let  $L_1, L_2$ , and  $L_3$  be three distinct tangent lines to  $C$ . Consider the pencil generated by  $3C$  and  $2(L_1 + L_2 + L_3)$ ; it has nine base points, three each at the tangent points. Upon resolving the base points of the pencil one obtains an elliptic fibration with one singular fiber of type  $E_6$  (the transform of  $3C$ ) and one of type  $A_2$  (the transform of the three lines). The  $A_2$  fiber has multiplicity 2, and we see three exceptional curves immediately: the last blowup at each tangent point produces a  $(-1)$ -curve. In this case Theorem 0.1 applies, and says that  $|\mathcal{E}| = (Q_{E_6})_2 \cdot (Q_{A_2})_1 = (9/\sqrt{3}) \cdot (1/\sqrt{3}) = 3$ . Therefore there are no other exceptional curves. This example illustrates 0.1, has a multiple reducible fiber, and we can “see” every curve with negative self-intersection on the surface.

**EXAMPLE V.6.** Our final example also starts from scratch, but is a bit more ambitious than the previous example. Let  $C$  be a smooth plane cubic, with a flex point  $p$  (taken to be the origin of the group law on  $C$ ), and the three nontrivial points  $q_1, q_2$ , and  $q_3$  of order 2. (The odd numbering is so that a fiber will be labeled properly at the end.) Let  $L$  be the flex line at  $p$ . Let  $v_7, v_2$ , and  $v_3$  be the tangents to  $C$  at  $q_1, q_2$ , and  $q_3$ , and let  $v_4$  be the line through the  $q_i$ . Note that  $v_7, v_2$ , and  $v_3$  all pass through  $p$ . Consider the pencil of sextics generated by  $2C$  and  $2v_4 + 2v_7 + v_2 + v_3$ . There are nine base points to the pencil, two each at  $q_2, q_3$ , and  $p$ , and three at  $q_7$ . Let us resolve the pencil rather explicitly.

First blow up the order two points  $q_i$ ; this produces  $v_5$  over  $q_7$ ,  $v_0$  over  $q_2$ , and  $v_1$  over  $q_3$ . Now blow up  $p$  twice: this produces first a curve  $v_8$ , then a curve  $B$ . At this point the pencil is generated by  $2C$  (writing  $C$ , etc. for the proper transform) and  $2v_4 + 2v_7 + v_2 + v_3 + 2v_5 + v_0 + v_1 + 2v_8$ . There is left one base point at each  $q_i$ ; blowing up  $q_7$  to produce  $E$ ,  $q_2$  to produce  $R_2$ , and  $q_3$  to produce  $R_3$  resolves the pencil and gives an elliptic fibration over  $\mathbb{P}^1$  with a double fiber ( $2C$ ) and one  $D_8$  fiber (the  $v_i$ 's). We see five exceptional curves:  $L, B, E, R_2$ , and  $R_3$ . The components of the  $D_8$  fiber are labeled properly to use the notations of Section II.

With  $F = D_8$ ,  $T_F \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , and the image of  $[E^\wedge]$  in  $T_F$  is  $[v_6 - 2v_0] = g_6 = 1$  (Table II.4). For amusement, the other exceptional curves have the following images in  $T_F$ :  $L \mapsto 1$ ,  $B \mapsto 1$ ,  $R_2 \mapsto \gamma$ , and  $R_3 \mapsto \gamma$ . Since  $R_2 - E \in K^\perp$ ,  $[(R_2 - E)^\wedge]$  is in  $R$ ; this is the element  $\gamma$ , so  $R$  must be  $\{1, \gamma\}$  (the other choice would be  $\{1, \beta\gamma\}$ ). Since  $d[E^\wedge]_{D_8} = 2$ , we have  $|\mathcal{E}| = (\text{coefficient of } t^2 \text{ in } H_{D_8}(1)) + (\text{coefficient of } t^2 \text{ in } H_{D_8}(\gamma)) = 7 + 2 = 9$ .

Where are the other four? This is not so clear at the outset, but some calculations show that the classes

$$\begin{aligned} G_1 &= v_0 + 2R_2 - L - K_X, & G_2 &= v_1 + 2R_3 - L - K_X, \\ G_3 &= L - v_0 - K_X, & G_4 &= L - v_1 - K_X \end{aligned}$$

all represent exceptional curves.  $G_1$  is the proper transform of a conic in  $\mathbb{P}^2$  tangent to  $C$  to order 4 at  $q_7$ , and tangent to  $C$  to order 2 at  $q_3$ . Similarly,  $G_2$  is the proper transform of a conic tangent to  $C$  to order 4 at  $q_7$  and to order 2 at  $q_2$ .

There is an order 2 automorphism of  $X$ , “reflecting” the  $D_8$  fiber about its middle component  $v_6$ .  $G_3$  and  $G_4$  are the images of  $G_1$  and  $G_2$  under this involution. An alternate description of  $G_3$  and  $G_4$  is to blow the surface down to  $\mathbb{P}^2$  in a different way, by blowing down  $E$ ,  $R_2$ ,  $R_3$ ,  $L$ ,  $v_4$ ,  $v_2$ ,  $v_3$ ,  $v_7$ , and  $v_6$ ; then  $G_3$  and  $G_4$  descend to conics similarly situated as  $G_1$  and  $G_2$  were using the original way to blow down. In this original way,  $G_3$  is the proper transform of a quartic in  $\mathbb{P}^2$ , which: has a tacnode at  $p$ , with tangent line  $L$ , meeting  $C$  four times at  $p$ ; meets  $C$  four times at  $q_7$ ; has a double point at  $q_2$ ; and is tangent to  $C$  at  $q_3$ .  $G_4$  is similar, exchanging the roles of  $q_2$  and  $q_3$ .

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