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General Components of the Noether-Lefschetz Locus and their Density in the Space of all Surfaces

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1. Introduction

Let $\mathbb{P}^3$ be the projective space of dimension 3 over an algebraically closed field k of any characteristic. We denote by $\Sigma(d)$ the projective space of dimension

$$N(d) = \binom{d+3}{3} - 1$$

whose points correspond to all surfaces of degree d in $\mathbb{P}^3$; and by $S(d)$ the open subset of $\Sigma(d)$ whose points correspond to smooth surfaces. The Noether-Lefschetz theorem says that if $d \geqq 4$, for a general point s in $S(d)$ the corresponding surface S has Picard group $\mathrm{Pic}(S) \cong \mathbb{Z}$ generated by $\mathcal{O}_S(1)$. More precisely one can define for $d \geqq 4$ the *Noether-Lefschetz locus* $\mathrm{NL}(d) \subset S(d)$ to be the set of points s corresponding to surfaces S such that $\mathrm{Pic}(S)$ is not generated by $\mathcal{O}_S(1)$. Then the Noether-Lefschetz theorem asserts that $\mathrm{NL}(d)$ is a countable union of proper irreducible closed subvarieties of $S(d)$.

Let now V be an irreducible component of $\mathrm{NL}(d)$ whose codimension in $S(d)$ we will denote by $c(V)$. Then, at least over the complex numbers, one has the inequalities

$$d - 3 \leqq c(V) \leqq p_g(d), \qquad (\oplus)$$

where

$$p_g(d) = \binom{d-1}{3}$$

is the geometric genus of any smooth surface of degree d in $\mathbb{P}^3$ (see [CGGH], [G]). Of the two inequalities contained in $(\oplus)$ the one

$$c(V) \leqq p_g(d) \qquad (\otimes)$$

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is very easy to understand: it simply says that it is at most $p_g(d)$ conditions on the moduli of a surface for an integral cycle on that surface to be algebraic: we require the vanishing on that cycle of all the integrals of the holomorphic 2-forms of the surface. It is also natural to expect that for most cycles these conditions should be independent, or, in other words, that for *most* components V of NL(d) the equality should hold in ($\otimes$). For this reason we will call such components of NL(d) the *general* components, calling other components *special*. Our purpose in this paper is to sketch a first attempt of description of the configuration in $S(d)$ of the general components of the Noether-Lefschetz locus; in particular we will prove the following *density theorem*:

(*) **Theorem.** *For any* $d \geqq 4$ *the Noether-Lefschetz locus* NL(d) *contains infinitely many general components; and the union of these components is Zariski dense in* $S(d)$.

The proof uses the degeneration techniques employed in [GH] to give an algebro-geometric proof of the Noether-Lefschetz theorem. It is based on an inductive argument (the theorem is known to be true for $d=4$, but also in this case we give a new proof of it) both to prove the existence of infinitely many general components of NL(d) and to show that their closures in $\Sigma(d)$ intersect the closed set $R(d)=\{$points of $\Sigma(d)$ corresponding to surfaces containing a plane as a component$\}$ in closed subvarieties of $R(d)$ whose union is dense in $R(d)$. Since $R(d)$ is not a divisor, this would not be enough for the proof of Theorem (*). But we are in fact able to prove also that the directions in which our general components of NL(d) approach $R(d)$ are dense in the set of all normal directions to $R(d)$ in $\Sigma(d)$. In other words we first blow up $\Sigma(d)$ along $R(d)$ (the geometry of this blow up is described in Sect. 2 below) and then verify that the intersection of the closures of our general components of NL(d) with the exceptional divisor are dense in the exceptional divisor itself (this is worked out in Sects. 3, 4).

Over the complex numbers one can, of course, look for a more refined density theorem, to the effect that NL(d) is dense in $S(d)$ in the natural topology. This result does hold and it is in fact a consequence of Theorem (*), as was pointed out to us by Green. We very briefly report on Green's idea of the proof in Sect. 5.

To conclude this introduction, we note that there are still many interesting (and naive) questions about the Noether-Lefschetz locus for which we have no answer. First about the general components of the Noether-Lefschetz locus: does our inductive construction give, at least in principle, all of them? What is the Picard group of a surface corresponding to a general point of one of our general components? (Of course, one might ask the same question for any component of NL(d).) One may suspect the answer to the last question should be $\mathbb{Z}^2$ (and in this direction seem to point recent results by Angelo Lopez [L]), but we have not been able to prove it. But the most interesting questions concern, in our opinion, the special components of NL(d). In particular, is their number finite? Is it possible to describe and classify them in some way? It is may be worthwhile to recall, in this circle of ideas, the following:

Codimension $d-3$ Conjecture (see [CGGH]): *If* $d>4$, *the only component of* NL(d) *of codimension* $d-3$ *is the set of points of* $S(d)$ *corresponding to surfaces containing a line.*

As a matter of fact, no component V of $\mathrm{NL}(d)$ of codimension strictly smaller than $2d-7$ is known if $d \geqq 5$ except the one corresponding to surfaces containing a line. There is a component of codimension $2d-7$, corresponding to surfaces containing a conic; but no one as yet has exhibited any other component of codimension $2d-7$. So the codimension $d-3$ conjecture mentioned above seems to be only the first one of a whole chain of possible conjectures, each one included in the other, of which the second could be the following:

Codimension $2d-7$ Conjecture. *If $d>4$, the only components of $\mathrm{NL}(d)$ of codimension at most $2d-7$ are given either by the set of points of $S(d)$ corresponding to surfaces containing a line (a component of codimension $d-3$) or by the set of points of $S(d)$ corresponding to surfaces containing a conic (a component of codimension $2d-7$).*

The techniques employed in the proof of Theorem (*), we believe, could prove to be of some use also in the analysis of these conjectures.

2. Blowing up the Locus of Surfaces Containing a Plane and Pulling back the Universal Family

In this paragraph we will provide some technical tools useful in the proof of the density theorem. The first point is to give a geometric interpretation for the exceptional divisor of the blow up of $\Sigma(d)$ along the locus $R(d)$ of surfaces containing a plane. More precisely, we let $R_U(d) \subset R(d)$ be the locus of points corresponding to surfaces of the type $S_0 = T \cup P$, where T is a smooth surface of degree $d-1$ and P is a plane such that the curve $C = T \cap P$ is smooth. As above, we denote by $S(d)$ the locus of points corresponding to smooth surfaces and set $U(d) = R_U(d) \cup S(d)$ (observe that $U(d)$ is an open neighborhood of $R_U(d)$, i.e., a small deformation of a surface in $R_U(d)$ is either smooth or again in $R_U(d)$). Clearly $R_U(d)$ is smooth, of codimension

$$r(d) = \binom{d+2}{2} - 3$$

in $U(d)$. Let now

$$p : \tilde{U}(d) \to U(d)$$

be the blow up of $U(d)$ along $R_U(d)$. We want to prove that for any point s_0 in $R_U(d)$ corresponding to a reducible surface $S_0 = T \cup P$ the points in $p^{-1}(s_0)$ are in a natural way in one to one correspondence with the divisors of the complete linear series $|\mathcal{O}_C(d)|$ on the curve $C = T \cap P$. Let us make this statement more precise.

First we notice that there is a smooth family of curves

$$\begin{array}{c} \mathscr{C} \subset R_U(d) \times \mathbb{P}^3 \\ \Big\downarrow \pi \\ R_U(d) \end{array}$$

such that for any point s_0 corresponding to a reducible surface $S_0 = T \cup P$ the fibre of π over s_0 is just the curve $C = T \cap P$. Since $\mathscr{C}$ is a family of curves in $\mathbb{P}^3$, we do have the line bundle $\mathcal{O}_{\mathscr{C}}(1)$ on $\mathscr{C}$ and we can then consider the sheaf $\mathscr{E} = \pi_* \mathcal{O}_{\mathscr{C}}(d)$ on $R_U(d)$ which clearly is locally free of rank $r(d)$. On the other hand we can also consider another locally free sheaf of the same rank $r(d)$ on $R_U(d)$, namely the normal sheaf $\mathscr{N}_{R_U, U}$ of $R_U(d)$ in $U(d)$. Now we define a map

$$\varphi : \mathscr{N}_{R_U, U} \to \mathscr{E}$$

of vector bundles in the following way. Let s_0 be, as usual, a point in $R_U(d)$. A normal vector to $R_U(d)$ at s_0 can be assigned by giving the class, modulo $T_{R_U(d)}, s_0$, of a tangent vector $\mathbf{v}$ to $\Sigma(d)$ at s_0. If S_0 is defined by the equation $hg = 0$, where h is the linear form defining P and g the polynomial of degree $d-1$ whose zero locus is T, then $\mathbf{v}$ corresponds to a first order deformation of S_0 defined by an equation of the type $hg + \varepsilon f = 0$ where $\varepsilon^2 = 0$ and f is a homogeneous polynomial of degree d. But now the restriction of f to the curve C defined by $h = g = 0$, gives us a section in $H^0(C, \mathcal{O}_C(d))$ which we define to be the image under φ of the class of $\mathbf{v}$. The map φ is well defined and turns out to be an isomorphism. In fact, with the above notation, $\mathbf{v}$ determines the zero section in $H^0(C, \mathcal{O}_C(d))$ if and only if one has $f = ah + bg$ where a, b are forms of degrees $d-1$ and 1 respectively. But in this case $\mathbf{v}$ is a tangent vector to $R_U(d)$ corresponding to the first order deformation $(h + \varepsilon b)(g + \varepsilon a) = 0$.

Summarizing the above analysis, we can rephrase the statement about the fibres of p in the following way:

Proposition 1. *The exceptional divisor E of the blow up $p : \tilde{U}(d) \to U(d)$ is naturally isomorphic to $\mathbb{P}(\mathscr{E})$.*

In view of this proposition it will be natural to look at points x in E as pairs (S_0, X), where S_0 is a reducible surface $T \cup P$ with $C = T \cap P$ smooth and X is a divisor in $|\mathcal{O}_C(d)|$.

The second and final point of this paragraph will be the smoothing, at least over a suitable open subset of $\tilde{U}(d)$, of the pull back of the universal family of surfaces. We start in fact by considering the universal family of surfaces of degree d in $\mathbb{P}^3$

$$\begin{array}{c} \mathscr{F}(d) \subset \Sigma(d) \times \mathbb{P}^3 \\ \downarrow \\ \Sigma(d). \end{array}$$

Since we have the morphism $p : \tilde{U}(d) \to U(d) \subset \Sigma(d)$ we can pull this family back via p. Now, while $\mathscr{F}(d)$ is smooth, the total space $\mathscr{F}(d)^*$ of this new family will not be; but the singularities of $\mathscr{F}(d)^*$ are easy to describe. First we observe that the cartesian square

$$\begin{array}{ccc} \tilde{\mathscr{C}} & \longrightarrow & \mathscr{C} \\ {\scriptstyle \tilde{\pi}} \downarrow & & \downarrow {\scriptstyle \pi} \\ E & \xrightarrow{p} & R_U(d) \end{array}$$

defines a smooth family of curves on E. Clearly there is a divisor $\mathscr{D}$ in $\tilde{\mathscr{C}}$ with the property that for each point x in E corresponding to a pair (S_0, X) where X is a divisor on the curve $\tilde{\pi}^{-1}(x) = \pi^{-1}(p(x))$, $\mathscr{D}$ restricts to X on $\tilde{\pi}^{-1}(x)$. Moreover we see that there is a commutative diagram

$$\begin{array}{ccc} \tilde{\mathscr{C}} & \xrightarrow{\gamma} & \mathscr{F}(d)^*|_E \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow \\ E & \longrightarrow & E \end{array}$$

with γ a closed embedding. Then $\mathscr{D}'$, the image of $\mathscr{D}$ via γ, is exactly the singular locus of $\mathscr{F}(d)^*$ (see, for instance, the analysis of Sect. 2 in [GH]). Now, in order to get rid of the singularities of $\mathscr{F}(d)^*$ we would like to blow it up along $\mathscr{D}'$. Before this, and in order to make things simpler, we first consider the maximal open subset A of $\tilde{U}(d)$ such that $A \cap E$ is the open subset E_0 of E over which $\mathscr{D}' \to E$ is étale (these are just the points such that the divisor X consist of $d(d-1)$ distinct points). We also denote by $\mathscr{F}$ the pull back of $\mathscr{F}(d)^*$ over A and by D the pull back of $\mathscr{D}'$ over A. Finally now, following again the analysis of [GH], we can see that:

(i) the singular locus of $\mathscr{F}$ is exactly D, and the tangent cones to $\mathscr{F}$ at the points of D are rank four quadrics;

(ii) blowing up $\mathscr{F}$ along D we get a smooth variety $\mathscr{F}$ and a commutative diagram

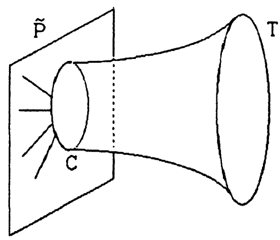
$$\begin{array}{ccc} \tilde{\mathscr{F}} & \longrightarrow & \mathscr{F} \\ \downarrow & & \downarrow \\ A & \longrightarrow & A, \end{array}$$

(iii) moreover the exceptional divisor $\tilde{D}$ in $\tilde{\mathscr{F}}$ can be realized, via the map $\tilde{D} \to D$, as a rank 4 quadric bundle over D. Precisely if x is a point of E_0, corresponding to a pair (S_0, X), where $S_0 = T \cup P$ and $X = p_1 + \ldots + p_{d(d-1)}$ is a divisor on $C = T \cap P$ formed by $d(d-1)$ distinct points in the linear series $|\mathcal{O}_C(d)|$, then the fibre of $\mathscr{F} \to A$ at x consists of a surface reducible in $d(d-1)+2$ components $\tilde{T}, \tilde{P}, Q_1, \ldots, Q_{d(d-1)}$. Here $\tilde{T}, \tilde{P}$ are the blow ups of T, P respectively at the points $p_1, \ldots, p_{d(d-1)}$ and $Q_1, \ldots, Q_{d(d-1)}$ are rank 4 quadrics; moreover $\tilde{T}$ and $\tilde{P}$ are glued along the strict transform (on both of them) of the curve C, and each quadric Q_i is glued to $\tilde{T}$ and $\tilde{P}$ along the exceptional divisors of the blow ups at p_i which are, on the quadric, two lines of different rulings;

(iv) there exists a blow down $\tilde{\mathscr{F}} \to \mathscr{G}$ of $\tilde{\mathscr{F}}$ along $\tilde{D}$ such that $\mathscr{G}$ is smooth, and a commutative diagram

$$\begin{array}{ccc} \tilde{\mathscr{F}} & \longrightarrow & \mathscr{G} \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \end{array}$$

such that, for any point x of E_0 (we use the same notation as above), the fibre S_x of $\mathscr{G} \to A$ over x is reducible in the two surfaces T and $\tilde{P}$, glued along the curve C and its strict transform on $\tilde{P}$ (which, abusing notation, we shall again denote by C); in other words, the effect of the map $\mathscr{F} \to \mathscr{G}$ is to blow down, in any fibre, the rulings of the quadrics Q_i containing the exceptional divisors of the surface $\tilde{T}$. The picture of the fiber of $\mathscr{G}$ over a point of E_0 is thus.



The end result of this process is thus a simultaneous small resolution of the singular points of the original family.

In conclusion we need to say a few words about some line bundles on $\mathscr{G}$. First notice that $A - E_0$ is isomorphic, via the restriction of p to A, to $S(d)$, so we shall actually identify $A - E_0$ with $S(d)$. Moreover the restriction of $\mathscr{G} \to A$ to $A - E_0$ can also be identified with the restriction of the universal family $\mathscr{F}(d) \to \Sigma(d)$ over $S(d)$. Therefore the existence of the line bundle $\mathcal{O}_{\mathscr{F}(d)}(1)$ and the smoothness of $\mathscr{G}$ yield the existence of a line bundle $\mathscr{L}$ extending the restriction of $\mathcal{O}_{\mathscr{F}(d)}(1)$ over $S(d)$.

Of course $\mathscr{L}$ is not unique because it is defined up to elements of the subgroup $\mathscr{P}$ of $\mathrm{Pic}(\mathscr{G})$ of line bundles with trivial restriction to the fibres of $\mathscr{G} \to A$. But the construction of $\mathscr{G}$ makes it clear that we can make it unique by imposing the condition that for any point x of E_0 whose fibre in $\mathscr{G} \to A$ is the surface $S_x = T \cup \tilde{P}$, $\mathscr{L}$ restricts on $T \cup \tilde{P}$ to the line bundle $\mathscr{L}_x$ which is $\mathcal{O}_T(1)$ on T and $\mathcal{O}_{\tilde{P}}(1)$, the pull back to $\tilde{P}$ of $\mathcal{O}_P(1)$, on $\tilde{P}$. On the other hand $\mathscr{P}$ is easy to describe: it contains $\mathrm{Pic}(A)$, viewed in a natural way as a subgroup of $\mathrm{Pic}(\mathscr{G})$, and is generated by $\mathrm{Pic}(A)$ and the line bundle $\mathscr{N}$ on $\mathscr{G}$ corresponding to the divisor described by all components T corresponding to surfaces of degree $d-1$ in the reducible fibres of $\mathscr{G} \to A$. Equivalently, $\mathscr{N}$ is the dual of the line bundle associated to the divisor formed by the components $\tilde{P}$ corresponding to planes in the reducible fibers of $\mathscr{G} \to A$. Note now that by the first description of $\mathscr{N}$, for any point x of E_0 the restriction $\mathscr{N}_x$ of $\mathscr{N}$ to $S_x = T \cup \tilde{P}$ restricts to the line bundle $\mathcal{O}_{\tilde{P}}(C) \cong \mathcal{O}_{\tilde{P}}(d-1) \otimes \mathcal{O}_{\tilde{P}}(-E_1 - \ldots - E_{d(d-1)})$ (here by E_i we mean the exceptional divisor of $\tilde{P}$ corresponding to the point p_i), and by the second description of $\mathscr{N}$ it restricts to the line bundle $\mathcal{O}_T(-C) \cong \mathcal{O}_T(-1)$ on T. Thus, if we twist the restriction $\mathscr{N}_x$ of $\mathscr{N}$ to $S_x = T \cup \tilde{P}$ with $\mathscr{L}_x$ we get the line bundle which is trivial on T and coincides with $\mathcal{O}_{\tilde{P}}(d) \otimes \mathcal{O}_{\tilde{P}}(-E_1 - \ldots - E_{d(d-1)})$ on $\tilde{P}$. So for any x in E_0 the line bundles $\mathscr{L}_x$ and $\mathscr{N}_x$ generate a subgroup $\mathscr{S}_x$ of $\mathrm{Pic}(S_x)$ that, by what we have seen above and the analysis in [GH], consists just of the line bundles on S_x that can be deformed onto nearby smooth surfaces to a multiple of the (hyper)plane bundle.

It is worth recalling here that the main point of the proof of the Noether-Lefschetz theorem in [GH] consists of showing that, for a general point x in E_0, $\mathscr{S}_x$ coincides with $\mathrm{Pic}(S_x)$. For any $x \in E_0$ we shall call any line bundles on S_x not in the group $\mathscr{S}_x$ *extra line bundles on* S_x.

3. Construction of General Components of the Noether-Lefschetz Locus (I)

In this paragraph and the next we will as promised construct infinitely many components of $\mathrm{NL}(d)$ having codimension $p_g(d)$ in $S(d)$ and show that they are Zariski dense. The construction consists of two steps, as follows:

Step 1. We describe an infinite collection of subvarieties of E_0 of codimension at most $p_g(d)$ in E_0, each of which is a "good candidate" to be a component of the intersection of E_0 with a general component of $\mathrm{NL}(d)$, in the sense that if V_0 is one of these subvarieties then the general fibre of $\mathscr{G}\to A$ over V_0 possesses at least one extra line bundle. Moreover we will see by induction that these subvarieties are dense in E_0.

Step 2. We prove that for each subvariety of codimension $p_g(d)$ found in step 1 the extra line bundle mentioned above can actually be deformed to some nearby smooth fibres of $\mathscr{G}\to A$.

To begin with step 1, let W be a component of $\mathrm{NL}(d-1)$ if $d\geqq 5$, and let W be $S(3)$ if $d=4$; let $\mathscr{W}$ be the restriction of the universal family $\mathscr{F}(d-1)$ over W. Then we may find a finite covering $W'\to W$ such that on the pull back $\mathscr{W}'$ of $\mathscr{W}$ to W' there is a line bundle $\mathscr{E}$ whose restriction to each fibre of $\mathscr{W}'\to W'$ is a line bundle not a power of the hyperplane bundle. Since there is a natural surjective morphism $E_0\to S(d-1)$ we can take the inverse image E_W of W. Then via the cartesian square

$$\begin{array}{ccc} E_{W'} & \longrightarrow & E_W \\ \downarrow & & \downarrow \\ W' & \longrightarrow & W \end{array}$$

we define the variety $E_{W'}$ whose points naturally correspond to triples $(S_0, X, \mathscr{E}_0)$ where, with the usual notation, $S_0=T\cup P$, $X=p_1+\ldots+p_{d(d-1)}$ is a divisor on $C=T\cap P$ formed by $d(d-1)$ distinct points in $|\mathcal{O}_C(d)|$, and $\mathscr{E}_0$ is a line bundle on S_0 not equal to $\mathcal{O}_{S_0}(k)$ for any k. Finally we will consider the étale covering $F_{W'}\to E_{W'}$ of degree $d(d-1)!$ whose fibres over points like $(S_0, X, \mathscr{E}_0)$ consist of all triples $(S_0, (p_{i_1}, \ldots, p_{i_{d(d-1)}}), \mathscr{E}_0)$, with $(i_1, \ldots, i_{d(d-1)})$ any permutation of $(1, \ldots, d(d-1))$.

Now, for any vector $\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_{d(d-1)})$ in $\mathbb{Z}^{d(d-1)}$ such that $\alpha_i\neq 0$ for any $i=1, \ldots, d(d-1)$, and any integer h define $F_{W'}(\boldsymbol{\alpha}, h)$ to be the variety consisting of all triples $(S_0, (p_1, \ldots, p_{d(d-1)}), \mathscr{E}_0)$ such that

$$\mathcal{O}_C(\alpha_1 p_1+\ldots+\alpha_{d(d-1)}p_{d(d-1)})\cong(\mathscr{E}_0)^{\otimes h}\otimes\mathcal{O}_C;$$

Let $F'_{W'}(\boldsymbol{\alpha}, h)$ be an irreducible component of $F_{W'}(\boldsymbol{\alpha}, h)$. Then pushing $F'_{W'}(\boldsymbol{\alpha}, h)$ down to E_W we find a closed subvariety $E_W(\boldsymbol{\alpha}, h)$ of the same dimension of $F'_{W'}(\boldsymbol{\alpha}, h)$. Moreover it is clear that for any point $x=(S_0, X, \mathscr{E}_0)$ in $E_W(\boldsymbol{\alpha}, h)$, the surface $S_x=T\cup\tilde{P}$ has some extra line bundles: precisely, for any point $(S_0, (p_1, \ldots, p_{d(d-1)}), \mathscr{E}_0)$ of $F'_{W'}(\boldsymbol{\alpha}, h)$ mapping to $(S_0, X, \mathscr{E}_0)$, we have the extra line bundle $\mathscr{E}_x$ that restricts to $(\mathscr{E}_0)^{\otimes h}$ on T and to $\mathcal{O}_{\tilde{P}}(\alpha_1 E_1+\ldots+\alpha_{d(d-1)}E_{d(d-1)})$ on $\tilde{P}$. We will denote by $c(E_W(\boldsymbol{\alpha}, h))$ the codimension of $E_W(\boldsymbol{\alpha}, h)$ in E_0.

Now we come to the main result in this paragraph:

Proposition 2. *For infinitely many vectors* $\boldsymbol{\alpha}$ *in* $\mathbb{Z}^{d(d-1)}$ *and integers* h *there exist components* $F'_{W'}(\boldsymbol{\alpha}, h)$ *of* $F_{W'}(\boldsymbol{\alpha}, h)$ *with*

$$c(E_W(\boldsymbol{\alpha}, h)) = c(W) + (d-2)(d-3)/2. \tag{+}$$

Moreover the union of the corresponding $E_W(\boldsymbol{\alpha}, h)$ *is Zariski dense in* E_W.

Proof. To begin with, we notice that (+) is equivalent to

$$c(F'_{W'}(\boldsymbol{\alpha}, h)) = (d-2)(d-3)/2, \tag{++}$$

where $c(F'_{W'}(\boldsymbol{\alpha}, h))$ is the codimension of $F'_{W'}(\boldsymbol{\alpha}, h)$ in $F_{W'}$. So first of all we have to prove that, for infinitely many $\boldsymbol{\alpha}$, $F_{W'}(\boldsymbol{\alpha}, h)$ has a component of codimension $(d-2)(d-3)/2$ in $F_{W'}$. Moreover we observe that there is a natural morphism $F_{W'} \to R_W(d)$, where $R_W(d)$ is the closed subvariety of $R_U(d)$ given as the inverse image of W via the natural map $R_U(d) \to S(d-1)$. Then it is sufficient to show that there are infinitely many $\boldsymbol{\alpha}$ and h for which there is an open subset $\mathscr{V}_{\boldsymbol{\alpha}}$ of $R_W(d)$ such that, for any point in $\mathscr{V}_{\boldsymbol{\alpha}}$, the fibre over that point of any component of $F_{W'}(\boldsymbol{\alpha}, h)$ has codimension $(d-2)(d-3)/2$ in the corresponding fibre of $F_{W'}$. But now this is a consequence of the following:

Claim. Let C be a smooth plane curve of degree $d-1$. Let $\mathscr{D} \subset |\mathcal{O}_C(d)|$ be the open subset of divisors consisting of $d(d-1)$ distinct points, and let $\mathscr{Y}$ be the $(d(d-1))!$-sheeted étale covering of $\mathscr{D}$ whose points correspond to the orderings of the divisors $D \in \mathscr{D}$. For any vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{d(d-1)})$ in $\mathbb{Z}^{d(d-1)}$ such that $\alpha_1 + \ldots + \alpha_{d(d-1)} = l$ let $\varphi_{\boldsymbol{\alpha}}$ be the morphism

$$\varphi_{\boldsymbol{\alpha}} : \mathscr{Y} \to \mathrm{Pic}^{(l)}(C)$$

defined by

$$\varphi_{\boldsymbol{\alpha}}(p_1, \ldots, p_{d(d-1)}) = \mathcal{O}_C(\alpha_1 p_1 + \ldots + \alpha_{d(d-1)} p_{d(d-1)}).$$

Then for infinitely many vectors $\boldsymbol{\alpha}$ the map $\varphi_{\boldsymbol{\alpha}}$ has maximal rank $g = g(C) = (d-2)(d-3)/2$ everywhere in $\mathscr{Y}$.

Assume for a moment the claim and let us see how the proposition follows. Let in fact $S_0 = T \cup P$ be a point of $R_W(d)$ and let $C = T \cap P$. By the claim we know there is some (and in fact infinitely many) $\boldsymbol{\beta}$ such that $\varphi_{\boldsymbol{\beta}}$ has maximal rank in $\mathscr{Y}$. Choose one such $\boldsymbol{\beta}$ and, for any integer h and line bundle L on C with $L^{\otimes h} \cong \mathcal{O}_C$, consider the composite map

$$\mathscr{Y} \longrightarrow \mathrm{Pic}^{(l)}(C) \longrightarrow \mathrm{Pic}^{(hl)}(C),$$

where the first map is $\varphi_{\boldsymbol{\beta}}$ and the second map takes $\mathscr{E}|_C \otimes L$ to $(\mathscr{E}|_C)^{\otimes h}$; this map is just $\varphi_{h\boldsymbol{\beta}}$ (independently of L, of course). By the density of torsion points on the Jacobian of C, there exist infinitely many h and L such that $\mathscr{E}|_C \otimes L$ is in the image of $\varphi_{\boldsymbol{\beta}}$, and hence infinitely many h such that $(\mathscr{E}|_C)^{\otimes h}$ is in the image of $\varphi_{h\boldsymbol{\beta}}$. We can thus find infinitely many positive integers h such that $\varphi_{h\boldsymbol{\beta}}^{-1}((\mathscr{E})^{\otimes h})$ is nonempty of pure codimension $g(C)$ in $\mathscr{W}$; and moreover the union, as h varies, of these inverse images is dense in $\mathscr{W}$. This procedure can now be repeated for any point of $R_W(d)$ and accordingly we find infinitely many components of $F_{W'}(\boldsymbol{\alpha}, h)$ for which (++) holds and whose union is dense in $F_{W'}$. The corresponding components $E_W(\boldsymbol{\alpha}, h)$ are therefore dense in E_W.

Finally we prove the claim, and for simplicity, we do this only over the complex numbers, the proof requiring for any algebraically closed field only slight modifications. Let $(p_1, \ldots, p_{d(d-1)})$ be a point in $\mathscr{Y}$. We denote by $r = d(d-1) - g$ the dimension of $\mathscr{Y}$ and by $x = (x_1, \ldots, x_r)$ coordinates on $\mathscr{Y}$ centered at $(p_1, \ldots, p_{d(d-1)})$. If $\omega_1, \ldots, \omega_g$ is a basis of the holomorphic differentials of C then for any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{d(d-1)})$ the local expression of $\varphi_{\boldsymbol{\alpha}}$ is

$$\varphi_{\boldsymbol{\alpha}}(\mathbf{x}) = \left(\ldots, \sum_{i=1}^{d(d-1)} \alpha_i \cdot \int_p^{p_i(x)} \omega_j, \ldots \right),$$

where we may identify $\mathrm{Pic}^{(l)}(C)$ with $J(C)$ and p is any given point of C. Let us now denote by $\mathbf{A}$ the $d(d-1) \times g$ matrix $(\omega_j(p_i))$ and by $\mathbf{B}$ the $r \times d(d-1)$ jacobian matrix $\partial(p_1, \ldots, p_{d(d-1)})/\partial(x_1, \ldots, x_r)$. Note that $\mathbf{B}$ has maximal rank r and $\mathbf{A}$, which is just the Brill-Noether matrix of $(p_1, \ldots, p_{d(d-1)})$ has maximal rank g as well (no holomorphic differential on C vanishes at all the p_i). Let $D(\boldsymbol{\alpha})$ be the $d(d-1)$-square diagonal matrix with entries $\alpha_1, \ldots, \alpha_{d(d-1)}$. Then the differential of $\varphi_{\boldsymbol{\alpha}}$ at $(p_1, \ldots, p_{d(d-1)})$ is the product $\mathbf{B} \cdot D(\boldsymbol{\alpha}) \cdot \mathbf{A}$ and we want to show that we can chose $\boldsymbol{\alpha}$ in infinitely many ways so to have this be of maximal rank g. In order to do this, let us consider a typical minor of order g of the product matrix $\mathbf{M}(\boldsymbol{\alpha}) = \mathbf{B} \cdot D(\boldsymbol{\alpha}) \cdot \mathbf{A}$, say the minor $M_{\mathbf{j}}(\boldsymbol{\alpha})$ determined by using rows $j_1, \ldots, j_g$. This minor may be written as

$$M_{\mathbf{j}}(\boldsymbol{\alpha}) = \sum_{\mathbf{i}} (\mathbf{B}_{\mathbf{j},\mathbf{i}} \cdot \mathbf{A}_{\mathbf{i}} \cdot \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_g}),$$

where $\mathbf{A}_{\mathbf{i}}$ is the minor of $\mathbf{A}$ determined by the rows of index $i_1, \ldots, i_g$, $\mathbf{B}_{\mathbf{j},\mathbf{i}}$ is the minor $\mathbf{B}$ using the rows of index $j_1, \ldots, j_g$ and columns of index $i_1, \ldots, i_g$, and the summation is taken over all multi-indices $\mathbf{i} = (i_1, \ldots, i_g)$ of order g of $(1, \ldots, d(d-1))$. Now $M_{\mathbf{j}}(\boldsymbol{\alpha})$ can be regarded as a homogeneous polynomial of degree g in $\boldsymbol{\alpha}$. This is certainly not divisible by $l - \sum_{i=1,\ldots,d(d-1)} \alpha_i$ unless

$$B_{\mathbf{j},\mathbf{i}} \cdot A_{\mathbf{i}} = 0$$

for any multi-index $\mathbf{i} = (i_1, \ldots, i_g)$. But this is not the case. In fact, there are multi-indices $i_1, \ldots, i_g$ for which $A_{\mathbf{i}} \neq 0$, and this means that the points $p_{i_1}, \ldots, p_{i_g}$ impose independent conditions on the canonical series. Since $|\mathcal{O}_C(d)|$ contains the canonical series, they give independent conditions to $|\mathcal{O}_C(d)|$ too, which in turn yields $B_{\mathbf{j},\mathbf{i}} \neq 0$. So for all $\boldsymbol{\alpha}$'s except those satisfying a nontrivial polynomial equation $M_{\mathbf{j}}(\boldsymbol{\alpha}) = 0$ the map $\varphi_{\boldsymbol{\alpha}}$ has rank g at $(p_1, \ldots, p_{d(d-1)})$; the claim follows by the quasi-compactness of U in the Zariski topology.

Step 1 is thus concluded. We remark here that, by the maximality of W in the above construction, the subvarieties $E_W(\boldsymbol{\alpha}, h)$ of E_0 we find are in fact components of the locus of $x \in E_0$ such that the surface S_x has extra line bundles. Also, before passing to step 2 we want to observe that, although we have, in the above, taken W to be a component of $\mathrm{NL}(d-1)$, we could as well have taken W to be equal to $S(d-1)$, getting with a few slight changes similar results. Of course, we are going to use in what follows only components for which $c(W) = p_g(d-1)$, and since

$$p_g(d) = p_g(d-1) + (d-2)(d-3)/2$$

we find $p_g(d)$-codimensional subvarieties $E_W(\boldsymbol{\alpha}, h)$ of E_0.

4. Construction of General Components of the Noether-Lefschetz Locus (II) and Proof of the Density Theorem

Step 2 essentially consists in proving a rather simple deformation theoretic result for the surfaces corresponding to points of the subvarieties $E_W(\alpha, h)$ of E_0 introduced in Sect. 3. As we have seen, for any x in $E_W(\alpha, h)$, the surface S_x has the extra line bundle $\mathscr{E}_x$. Notice now that the line bundle $\mathscr{M}_x = (\mathscr{L}_x)^{\otimes 2} \otimes \mathscr{N}_x$ has ample restrictions to both components of S_x. For any integer n we will denote by $\mathscr{E}_x^n$ the line bundle $\mathscr{E}_x \otimes \mathscr{M}_x^{\otimes n}$ on S_x. It is then very easy to see that there exists an integer N such that for any $n \geqq N$ and for any x in $E_W(\alpha, h)$, one has:

(i) $h^2(S_x, \mathscr{E}_x^n) = h^1(S_x, \mathscr{E}_x^n) = 0$; and

(ii) the general curve Γ in $|\mathscr{E}_x^n|$ is a reduced, local complete intersection on S_x.

Moreover if, as usual, $\mathscr{N}_{\Gamma, S_x}$ denotes the normal bundle of Γ in S_x, we have, by virtue of (i):

(iii) $h^1(\Gamma, \mathscr{N}_{\Gamma, S_x}) = h^2(S_x, \mathscr{O}_{S_x}) = p_g(d)$.

Now we recall that we have a family $\mathscr{G} \to A$, that $x \in E_W(\alpha, h) \subset A$, and that S_x is the fibre of $\mathscr{G} \to A$ over x. Therefore we can look at Γ as a closed subscheme of $\mathscr{G}$ and we can consider the dimension $h(\Gamma)$ of the Hilbert scheme parametrizing closed subschemes of $\mathscr{G}$ at the point corresponding to Γ. What we shall need is an estimate for $h(\Gamma)$. Precisely we want to prove the:

Proposition 3. $h(\Gamma) \geqq N(d) + h^0(\Gamma, \mathscr{N}_{\Gamma, S_x}) - p_g(d)$.

Proof. Consider the exact sequence of sheaves

$$0 \to \mathscr{N}_{\Gamma, S_x} \to \mathscr{N}_{\Gamma, \mathscr{G}} \to \mathscr{N}_{S_x, \mathscr{G}} \otimes \mathscr{O}_\Gamma \to 0$$

and the induced map

$$f: H^1(\Gamma, \mathscr{N}_{\Gamma, \mathscr{G}}) \to H^1(\Gamma, \mathscr{N}_{S_x, \mathscr{G}} \otimes \mathscr{O}_\Gamma).$$

Then, since Γ is locally a complete intersection, $H^1(\Gamma, \mathscr{N}_{\Gamma, \mathscr{G}})$ is known to be an obstruction space for the functor Hilb_Γ of infinitesimal deformations of Γ in $\mathscr{G}$ (see [S]); but we can see more, namely that $\operatorname{Ker} f$ is an obstruction space for Hilb_Γ and this will yield

$$h(\Gamma) \geqq h^0(\Gamma, \mathscr{N}_{\Gamma, \mathscr{G}}) - \dim \operatorname{Ker} f$$

whence the proposition, by (iii) above, since clearly

$$\mathscr{N}_{S_x, \mathscr{G}} \otimes \mathscr{O}_\Gamma \cong \mathscr{O}_\Gamma^{\dim A}$$

and $\dim A = N(d)$. In order to prove that $\operatorname{Ker} f$ is an obstruction space for Hilb_Γ we argue as follows. For any Artin ring A_0 and for any deformation Δ_0 of Γ in $\mathscr{G}$ parametrized by A_0 there is an obstruction map

$$o(\Delta_0): \mathrm{Ext}_k(A_0, k) \to H^1(\Gamma, \mathscr{N}_{\Gamma, \mathscr{G}})$$

(see [S], prop. (8.4)), and our assertion will follow if the composition of $o(\Delta_0)$ with f is zero.

To prove this we first quote a general fact, which says in effect that if we have a flat family of projective varieties $\pi: \mathscr{X} \to B$ and a connected subvariety $Y_0 \subset X_0 = \pi^{-1}(b_0)$ of a fiber of the family, any flat deformation of Y_0 in the total space $\mathscr{X}$ will continue to lie in a fiber of π, even infinitesimally. Specifically, this is the

Lemma. *Let $\pi: \mathscr{X} \to B$ and $Y_0 \subset X_0$ be as above. Let $\eta: \mathscr{Y} \to B'$ be a flat deformation of Y_0 in $\mathscr{X}$, i.e., a subscheme $\mathscr{Y} \subset \mathscr{X} \times B'$ such that the composition η of the inclusion with the projection to B' is flat, and such that $\eta^{-1}(b'_0) = Y_0$. Then there is a unique morphism $\varrho: B' \to B$ such that the inclusion $\mathscr{Y} \hookrightarrow \mathscr{X} \times B'$ factors though the inclusion $\mathscr{X} \times_B B'$.*

Proof. The basic observation here is simply that, since Y_0 is reduced, projective and connected and η is flat, we have

$$\eta_*(\mathcal{O}_{\mathscr{Y}}) = \mathcal{O}_{B'}.$$

Now, we may assume that $B = \operatorname{Spec} A$ and $B' = \operatorname{Spec} A'$ are affine. The composition

$$\mathscr{Y} \hookrightarrow \mathscr{X} \times B' \to \mathscr{X} \to B$$

of the inclusion with projection to $\mathscr{X}$ and thence to B induces a map

$$A = \Gamma(B, \mathcal{O}_B) \to \Gamma(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}) = \Gamma(B', \mathcal{O}_{B'}) = A'$$

which yields the desired map from B' to B.

(We would like to thank Angelo Vistoli for pointing out the simple proof of this fact).

In the present circumstances, this lemma says that our deformation Δ_0 of Γ in $\mathscr{G}$ will induce a deformation Δ'_0 of S_x in $\mathscr{G}$ parametrized by A_0; and so we get another obstruction map

$$o(\Delta'_0): \operatorname{Ext}_k(A_0, k) \to H^1(S_x, \mathcal{N}_{S_x, \mathscr{G}}).$$

From the exact sequence

$$0 \to \mathcal{N}_{S_x}(-\Gamma) \to \mathcal{N}_{S_x, \mathscr{G}} \to \mathcal{N}_{S_x, \mathscr{G}} \otimes \mathcal{O}_\Gamma \to 0$$

we also obtain the map

$$g: H^1(S_x, \mathcal{N}_{S_x, \mathscr{G}}) \to H^1(\Gamma, \mathcal{N}_{S_x, \mathscr{G}} \otimes \mathcal{O}_\Gamma)$$

and it is a matter of local computation (see [S], prop. (4.8)) to check that the composition of $o(\Delta_0)$ with f is equal to the composition of $o(\Delta'_0)$ with g. But since $\mathcal{N}_{S_x, \mathscr{G}} \cong \mathcal{O}_{S_x}$, one has

$$H^1(S_x, \mathcal{N}_{S_x, \mathscr{G}}) \cong H^1(S_x, \mathcal{O}_{S_x}) \cong (0)$$

and this proves the assertion.

As an immediate consequence of Proposition 3 we have the

Corollary 4. *If $c(W) = p_g(d-1)$ then there exists some component V of $NL(d)$ such that $c(V) = p_g(d)$ and $\bar{V} \cap E_0$ contains $E_W(\alpha, h)$ as a component (the closure of V is, of course, taken in A).*

Proof. Let x be a general point of $E_W(\alpha, h)$, let Γ be as above and let $\mathscr{H}$ be a component of maximal dimension of the Hilbert scheme of curves in $\mathscr{G}$ containing the point corresponding to Γ. Clearly there is a morphism $F: \mathscr{H} \to A$ that is proper; since the surfaces in question are regular, the fibres of F consist of a disjoint union of a finite number of projective spaces corresponding to linear systems on the fibres of $\mathscr{G} \to A$. (Indeed, by (i) above, these linear systems can be all assumed to be of the same dimension $h^0(\Gamma, \mathcal{N}_{\Gamma,S_x})$: by upper semi-continuity, for nearby curves Γ_t lying on nearby surfaces S_t we have $h^1(S_t, \mathcal{O}(\Gamma_t))=h^2(S_t, \mathcal{O}(\Gamma_t))=0$ and it follows that $h^0(S_t, \mathcal{O}(\Gamma_t))=h^0(S_x, \mathcal{O}(\Gamma))$. Then $F(\mathcal{H})$ is a closed subset of A and, by Proposition 3, we have

$$\dim F(\mathcal{H}) \geqq N(d)-p_g(d). \qquad (*)$$

Notice now that $F(\mathcal{H})$ contains $E_W(\alpha, h)$ and

$$\dim E_W(\alpha, h)=N(d)-p_g(d)-1. \qquad (**)$$

Now, all the points of $F(\mathcal{H})$ correspond to surfaces that support extra line bundles. On the other hand, as we have already remarked at the end of Sect. 3, $E_W(\alpha, h)$ is a component of the locus in E_0 of points corresponding to such surfaces; so clearly $E_W(\alpha, h)$ is a component of $F(\mathcal{H}) \cap E_0$. From before, we have that the codimension of $E_W(\alpha, h)$ in E_0 is equal to $p_g(d)$. It follows from this that the codimension of $F(\mathcal{H})$ in A is at least equal to this; since by (*) the codimension of $F(\mathcal{H})$ is at most $p_g(d)$ we may conclude that the codimension of $F(\mathcal{H})$ in A is exactly equal to $p_g(d)$.

It remains to see that $F(\mathcal{H})$ is indeed a component of the Noether-Lefschetz locus NL(d). But if it were not, the component V containing it would have to intersect E_0 in a subvariety including a component that contained $E_W(\alpha, h)$ and had larger dimension than $E_W(\alpha, h)$, but was still contained in the locus corresponding to surfaces S_x with extra line bundles; and this contradicts the fact that $E_W(\alpha, h)$ is a component of this locus.

We are now in position to complete the:

Proof of the Density Theorem. For $d=4$ the theorem directly follows from Corollary 4 and the last assertion of Proposition 2, since in this case $W=S(3)$ and $c(W)=p_g(3)=0$. For $d>4$ one can use induction, and the theorem is proved in the same manner applying Corollary 4 and Proposition 2.

Before concluding this section it is may be worth remarking the role played by the condition $c(W)=p_g(d-1)$, which essentially comes into the picture only at the very end of the argument, namely in Corollary 4, in order to deform the extra line bundle off the locus of reducible surfaces. The question remains (and an answer would be interesting in order to understand the special components of the Noether-Lefschetz locus) for which W's with $c(W)<p_g(d-1)$ and which (suitably defined!) $E_W(\alpha, h)$ a similar deformation theoretic result may be proved.

5. The Density in the Natural Topology over the Complex Numbers (Following Green)

As we said in the introduction, Green pointed out how the existence of just one component of codimension $p_g(d)$ of NL(d) yields, over the complex numbers, the density of NL(d) in $S(d)$ in the natural topology. Now we will briefly sketch Green's argument.

First we recall the existence of a real vector bundle $\mathcal{H}=\mathcal{H}^2(d)$ on $S(d)$ whose fibre at a point s corresponding to a surface S is the real vector space $H^2(S, \mathbb{R})$, and likewise of a complex vector bundle $\mathscr{H}_{\mathbb{C}}=\mathscr{H}\otimes\mathbb{C}$ with fiber $H^2(S,\mathbb{C})$ at s ($\mathscr{H}$ and $\mathscr{H}_{\mathbb{C}}$ are the real and complex vector bundles associated to the local system $R^2\pi_*\mathbb{Z}$, where $\pi:\mathscr{F}(d)\to S(d)$ is the universal surface as above). Note that the bundles $\mathscr{H}$ and $\mathscr{H}_{\mathbb{C}}$ are flat, so that in particular $\mathscr{H}_{\mathbb{C}}$ has the structure of holomorphic vector bundle. $\mathscr{H}$ has a subbundle $\mathscr{H}^{1,1}(d)$ of corank $2p_g(d)$ whose fibre at a point s corresponding to a surface S is $H^{1,1}(S)\cap H^2(S,\mathbb{R})$ and $\mathscr{H}_{\mathbb{C}}$ has subbundles $\mathscr{H}_{\mathbb{C}}^{i,j}$ for each $i+j=2$, with fibers $H^{i,j}(S)$; in terms of the holomorphic structure on $\mathscr{H}_{\mathbb{C}}$ the subbundles $\mathscr{H}_{\mathbb{C}}^{i,j}$ are not in general holomorphic, but the subbundles $\mathscr{H}_{\mathbb{C}}^{0,2}$ and $\mathscr{F}^1=\mathscr{H}_{\mathbb{C}}^{0,2}\oplus\mathscr{H}_{\mathbb{C}}^{1,1}$ are.

Now choose an open subset (in the natural topology) U of $S(d)$ over which $\mathscr{H}$ and $\mathscr{H}_{\mathbb{C}}$ may be trivialized so that $\mathbb{P}(\mathscr{H}_{|U})\cong U\times\mathbb{P}$ and $\mathbb{P}(\mathscr{H}_{\mathbb{C}|U})\cong U\times\mathbb{P}_{\mathbb{C}}$, where $\mathbb{P}$ and $\mathbb{P}_{\mathbb{C}}$ are a real and complex projective space of dimension $h^2(S,\mathbb{R})-1$ respectively. Moreover we have maps $G_U:\mathbb{P}(\mathscr{H}_{|U}^{1,1})\to\mathbb{P}$ and $G_{U,\mathbb{C}}:\mathbb{P}(\mathscr{F}_{|U}^1)\to\mathbb{P}_{\mathbb{C}}$ that are linear and injective on each fibre; and by the above remark $G_{U,\mathbb{C}}$ is in fact holomorphic. Suppose that U intersects some general component of $\mathrm{NL}(d)$. We then make the

Basic Claim. *The image of G_U contains some nonempty open subset $\mathscr{U}$ of $\mathbb{P}$.*

Proof. We prove first the corresponding statement for $G_{U,\mathbb{C}}$, namely that the image of $G_{U,\mathbb{C}}$ contains a nonempty open subset $\mathscr{U}_{\mathbb{C}}$ of $\mathbb{P}_{\mathbb{C}}$ meeting $\mathbb{P}\subset\mathbb{P}_{\mathbb{C}}$. To see this, observe that the hypothesis that U meets a general component of $\mathrm{NL}(d)$ means that there are rational points γ in $\mathbb{P}_{\mathbb{C}}$ such that the fibre of $G_{U,\mathbb{C}}$ over them has complex codimension equal to $\dim\mathbb{P}_{\mathbb{C}}$. Since $G_{U,\mathbb{C}}$ is holomorphic it follows that the image of $G_{U,\mathbb{C}}$ contains a neighborhood of γ.

The claim now follows if we observe that a point $\gamma\in\mathbb{P}=\mathbb{P}(H^2(S,\mathbb{R}))\subset\mathbb{P}_{\mathbb{C}}=\mathbb{P}(H^2(S,\mathbb{C}))$ that lies in the image of the map $G_{U,\mathbb{C}}$ – i.e., that lies in the subspace $H^{1,1}(S)\oplus H^{0,2}(S)\subset H^2(S,\mathbb{C})$ for some surface S corresponding to a point in U – must also lie in the conjugate subspace $H^{1,1}(S)\oplus H^{2,0}(S)\subset H^2(S,\mathbb{C})$ and hence in $H^{1,1}(S)$, that is, in the image of G_U. The image of G_U thus contains the intersection $\mathscr{U}$ of $\mathscr{U}_{\mathbb{C}}$ with $\mathbb{P}\subset\mathbb{P}_{\mathbb{C}}$.

Given the Claim, we see that the inverse image Q_U in $\mathbb{P}(\mathscr{H}^{1,1}(d)_{|U})$ via G_U of the rational points of $\mathbb{P}$ (not corresponding to the class of the hyperplane section) is dense in the natural topology, and so is the image of Q_U in U under the map $\mathbb{P}(\mathscr{H}^{1,1}(d)_{|U})\to U$. But this image is nothing else than $\mathrm{NL}(d)$, so $\mathrm{NL}(d)\cap U$ is dense in U in the natural topology.

In order to complete the argument it is sufficient to show that for any point x in $S(d)$ there is an open neighborhood U of x in the natural topology such that $\mathscr{H}^2(d)$ can be trivialized over U and the (consequently defined) map $G_U:\mathbb{P}(\mathscr{H}^{1,1}(d)_{|U})\to\mathbb{P}$ is of maximal rank somewhere. In order to see the sufficiency, we simply observe that if we take an open cover $\{U_i\}_{i\in\mathscr{I}}$ of $S(d)$ such that $\mathscr{H}^2(d)$ can be trivialized over U_i for any $i\in\mathscr{I}$, and such that the transition functions are constant non degenerate $h^2(S,\mathbb{R})\times h^2(S,\mathbb{R})$ square matrices with coefficients in $\mathbb{Z}$ (the structure group of $\mathscr{H}^2(d)$ can be always reduced to $GL(h^2(S,\mathbb{R}),\mathbb{Z})$), the analytic subsets $G(Ui)$ over which G_{U_i} fails to be of maximal rank patch together defining an analytic subset G of $\mathbb{P}(\mathscr{H}^{1,1}(d))$. This is proper because, as we have seen, $G(Ui)$ is whenever U_i intersects a general component of $\mathrm{NL}(d)$ and if any one $G(Ui)$ is proper they must all be.

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Note added in proof. The conjectures stated in the introduction of this paper have both been solved. The Codimension $d-3$ Conjecture has been proved by C. Voisin, Une précision concernant le théorème de Noether, Math. Ann. **280**, 605–611 (1988), and independently by M. Green, Components of maximal dimension in the Noether-Lefschetz locus, to appear in J. Differ. Geom. The Codimension $2d-7$ Conjecture was again proved by C. Voisin, first in the case $d=5$ (Composantes de lieû de Noether-Lefschetz en degré cinq, preprint) and then the general case (Composante de petite codimension du lieû de Noether-Lefschetz, preprint). Further developments on this subject have been recently announced to the first author of the present paper by Ch. Peskine (joint work with G. Ellingsrud).