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Smoothing Cusp Singularities of Small Length

Robert Friedman[1] and Rick Miranda[2]

[1] Department of Mathematics, Columbia University, New York City, NY 10027, USA
[2] Department of Mathematics, Colorado State University, Fort Collins, CO 80523, USA

Dedicated to T.S.M., with deepest respect

0. Introduction

After the rational double points, perhaps the most tractable class of normal surface singularities is that of the minimally elliptic singularities of Laufer [7]. Such a singularity $(\bar{V}, p)$ is characterized by the properties that it is Gorenstein, and $R^1\pi_*\mathcal{O}_V \cong \mathbb{C}$, where $\pi: V \to \bar{V}$ is a resolution of the analytic germ of the singularity $(\bar{V}, p)$. Minimally elliptic singularities with reduced fundamental cycle in the minimal resolution fall into three broad classes: the simple elliptic singularities, the Dolgachev singularities $D_{p,q,r}$, and the cusp singularities. The versal deformation space of a simple elliptic singularity is well understood, by work of Pinkham and Looijenga; in particular, it is easy to describe which of them are smoothable. A considerable amount of recent work has centered on $D_{p,q,r}$ singularities with $\mathbb{C}^*$-action; smoothability for these has also been worked out by Wahl, Looijenga, and Pinkham.

In this paper, we consider the existence of smoothings of cusp singularities. It is shown that the existence of a smoothing is equivalent to a purely combinatorial statement concerning the existence of certain configurations of rational surfaces, given in Sect. 2. Our motivation has been the recent work in the degenerations of K3 surfaces, notably the classification of Kulikov and Persson-Pinkham. After some discussion of the combinatorics involved, we verify a conjective of Looijenga on the existence of smoothings of cusp singularities in special cases. As will no doubt be clear to the reader, the combinatorial problem involved in checking Looijenga's conjecture in the general case using our methods is rather daunting.

Using different techniques, Henry Pinkham and the first author have verified Looijenga's conjecture for almost all cusps of length less than or equal to three. The remaining cases are shown to be smoothable in Sect. 5, so that the two techniques combined verify the conjecture for all cusps of length $\leqq 3$.

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1. Preliminaries

Let $(\bar{V}, p)$ be the germ of a cusp singularity. Thus, if $\pi: V \to \bar{V}$ is the minimal resolution, the exceptional divisor

$$\pi^{-1}(p) = D = \sum_{i=1}^{r} D_i$$

is a cycle of rational curves D_i meeting transversally. By convention, D_i meets $D_{i\pm 1}$ transversally (where we use subscripts mod r), and a cycle of length one is an irreducible nodal rational curve. We will always denote by r [or $r(D)$ if necessary] the length of the cycle. Further, let

$$m = m(D) = \begin{cases} 2 & \text{if } D^2 = -1 \\ -D^2 & \text{otherwise} \end{cases}$$

denote the multiplicity of (V, p), and set

$$d_i = -D_i^2 \quad \text{for each} \quad i = 1, \dots, r.$$

The negative definiteness of the intersection matrix for the components of D is then equivalent to the conditions

$$\begin{matrix} \text{(i)} & d_i \geqq 2 & \text{for every } i \\ \text{(ii)} & d_j \geqq 3 & \text{for some } j \end{matrix} \Bigg\} \quad \text{if} \quad r \geqq 2$$

or just: $d_1 \geqq 1$ if $r = 1$.

The self-intersection D^2 can then be written

$$-D^2 = \begin{cases} d_1 & \text{if } r = 1 \\ \sum (d_i - 2) & \text{if } r \geqq 2, \end{cases}$$

so that

$$m(D) = \begin{cases} 2 & \text{if } D^2 = -1 \\ d_1 & \text{if } r = 1 \text{ and } d_1 \geqq 2 \\ \sum (d_i - 2) & \text{if } r \geqq 2 \text{ and } D^2 \leqq -2. \end{cases}$$

The cycle of integers $(d_1, \dots, d_r)$ determines the analytic type of the germ of the cusp singularity $(\bar{V}, p)$. For this reason, we will abuse notation and use the letter D to denote

a) the cycle of integers $(d_1, \dots, d_r)$,

b) the divisor D, the cycle of rational curves $\sum D_i$, with $-D_i^2 = d_i$, and

c) the germ of the cusp singularity $(\bar{V}, p)$ whose resolution is the cycle of curves D,

and we will use the cycle of integers $(d_1, \dots, d_r)$ when we need to be specific.

It will be convenient for some purposes to represent a cusp D not by the cycle of integers but by the following:

Notation (1.1). The $2 \times k$ matrix $\begin{pmatrix} a_1 \dots a_k \\ b_1 \dots b_k \end{pmatrix}$ denotes a cusp D with components D_i having the following self-intersections:

(a) There are k components $D_{j_1}, \ldots, D_{j_k}$ with $D_{j_i}^2 = -a_i$, and $a_i \geqq 3$.

(b) Separating D_{j_i} and $D_{j_{i+1}}$ there are b_i components with self-intersection -2; $b_i \geqq 0$.

Here we set $D_{j_{k+1}} = D_{j_1}$.

For example, $\begin{pmatrix} 3 & 4 & 4 \\ 1 & 0 & 2 \end{pmatrix}$ is the cusp whose cycle of integers is (3, 2, 4, 4, 2, 2). If $r(D)=1$, we will alternately use the notation $\begin{pmatrix} a_1 \\ 0 \end{pmatrix}$ or (a_1) when $D=D_1$ and $D_1^2 = -a_1$; here $a_1 \geqq 1$.

The following is now immediate.

Lemma (1.2). *Let D be the cusp $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$. Then*

(1.2.1) $$r(D) = \sum (b_i + 1),$$

(1.2.2) $$-D^2 = \begin{cases} a_1 & \text{if } r(D)=1 \\ \sum(a_i - 2) & \text{if } r(D) \geqq 2. \end{cases}$$

Hence $D^2 = -1$ if and only if D is either $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ b \end{pmatrix}$ for some $b \geqq 1$

(1.2.3) $$m(D) = \begin{cases} 2 & \text{if } D^2 = -1 \\ a_1 & \text{if } r(D)=1 \text{ and } a_1 \geqq 2 \\ \sum(a_i - 2) & \text{if } r(D) \geqq 2 \text{ and } D^2 \leqq -2. \end{cases}$$

Definition (1.3). Let D be the cusp $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$. The *dual cusp* D' to D is $\begin{pmatrix} a_i' \\ b_i' \end{pmatrix}$, where

$$a_i' = b_i + 3 \quad \text{and} \quad b_i' = a_{i+1} - 3$$

unless $r(D)=1$ or $D^2 = -1$.

If $D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $D' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

If $D = \begin{pmatrix} a \\ 0 \end{pmatrix}$ with $a \geqq 2$, then $D' = \begin{pmatrix} 3 \\ a-1 \end{pmatrix}$.

If $D = \begin{pmatrix} 3 \\ b \end{pmatrix}$ with $b \geqq 1$, *then* $D' = \begin{pmatrix} b+1 \\ 0 \end{pmatrix}$.

The following lemma is an immediate consequence of the above definition.

Lemma (1.4). *Let D be a cusp, and D' its dual. Then*

(1.4.1). *The dual of D' is D*

(1.4.2) $$r(D') = -D^2 = \begin{cases} 1 & \text{if } D = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 3 \\ b \end{pmatrix}, \quad b \geqq 1 \\ m(D) & \text{otherwise}, \end{cases}$$

(1.4.3) $$-D'^2 = \begin{cases} 1 & \text{if } D = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad a \geqq 1 \\ r(D) & \text{otherwise}. \end{cases}$$

Hence $D'^2 = -1$ *if and only if* $D = \begin{pmatrix} a \\ 0 \end{pmatrix}$ *for some* $a \geqq 1$

$$m(D') = \begin{cases} 2 & \text{if } D = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad a \leqq 1 \\ r(D) & \text{otherwise}. \end{cases} \tag{1.4.4}$$

We next describe a certain class of surfaces of Type VII_0 in Kodaira's list, the *Inoue-Hirzebruch* surfaces. These are surfaces V which have the following properties:

(1.5.1) $\mathcal{O}_V(-K_V) = \mathcal{O}_V(D + D')$, where D is a negative definite cycle of rational curves, D' is the dual cycle as in (1.3), and D and D' are disjoint.

(1.5.2) The only curves on V are the components of D and D'.

(1.5.3) Setting $h^{p,q} = \dim H^q(V, \Omega_V^p)$, then

$$h^{p,q} = \begin{cases} 1 & \text{if} \quad (p,q) = (0,0), (2,2), (0,1), \text{ or } (2,1) \\ r(D) + r(D') & \text{if} \quad (p,q) = (1,1) \\ 0 & \text{otherwise}. \end{cases}$$

(1.5.4) Every cusp D occurs on some Inoue-Hirzebruch surface.

For the construction, see [4, 3]. Briefly, V is the resolution of the completion $\bar{\bar{V}}$ of a non-compact surface which is the quotient of $\mathbb{H} \times \mathbb{C}$ by a discrete group associated to a real quadratic irrationality ω ($\mathbb{H}$ is the upper half-plane). $\bar{\bar{V}}$ has two dual cusp singularities whose resolutions yield the divisors D and D'.

Finally, we recall some facts about Type III degenerations which will motivate the constructions of the next section. Let Δ be the unit disk in $\mathbb{C}$ and $\pi: X \to \Delta$ a degeneration of K3 surfaces over Δ. By the semi-stable reduction theorem, we may assume that X is smooth and $X_0 = \pi^{-1}(0)$ is a reduced divisor with (local) normal crossings. By the theorem of Kulikov and Persson-Pinkham, if all components of X_0 are algebraic we may assume in addition that the global canonical divisor K_X of X is trivial. In this situation, the central fibers X_0 have been classified, and there are three types [6]. In the Type III case, X_0 has the following form:

(1.6.1) $X_0 = \sum V_i$, and if $\tilde{V}_i$ is the normalization of V_i, then each $\tilde{V}_i$ is a rational surface. If D_i is the inverse image on $\tilde{V}_i$ of the double curves of X_0 on V_i, then D_i is a cycle of rational curves. Moreover, the dual graph Γ of X_0 is a triangulation of S^2.

(1.6.2) $D_i \in |-K_{\tilde{V}_i}|$ for each i.

(1.6.3) If D_{ij} is an irreducible double curve joining V_i to V_j (i may equal j if $V_i = V_j$ meets itself), then

$$(D_{ij}^2)_{\tilde{V}_i} + (D_{ij}^2)_{\tilde{V}_j} = \begin{cases} 0 & \text{if} \quad D_{ij} \text{ is a nodal curve on } \tilde{V}_i \text{ or } \tilde{V}_j \\ -2 & \text{otherwise}. \end{cases}$$

This is usually referred to as the *triple point formula.*

The main result of [1] is that (1.6.1)–(1.6.3) are the *only* combinatorial restrictions on central fibers of Type III degenerations of K3 surfaces, i.e., we can always smooth a central fiber X_0 with the desired combinatorial description subject only to (1.6.1)–(1.6.3). This result is the motivation for the next section.

2. Deformation Theory

After a brief description of Looijenga's results, we state the combinatorial problem alluded to in the introduction, and prove that its solution is equivalent to the existence of a smoothing of a cusp singularity.

In the notation of Sect. 1, let $\bar{\bar{V}}$ be the singular Inoue-Hirzebruch surface with two dual cusps D and D'; let the corresponding singular points of $\bar{\bar{V}}$ be p and p'. Looijenga proves the

Theorem (2.1) [8]. *$\bar{\bar{V}}$ has a universal deformation which is semi-universal for the (disconnected) germ of the surface singularities $(\bar{\bar{V}}, p, p')$.*

In particular, suppose that $(\bar{\bar{V}}, p)$ is smoothable. By choosing a deformation $\bar{\bar{\pi}}: \bar{\bar{X}} \to \Delta$ with $\bar{\bar{X}}_0 = \bar{\bar{V}}$ which globalizes a given local smoothing of $(\bar{\bar{V}}, p)$, we obtain a family of surfaces $\bar{\bar{X}}_t$ with a unique singular point p'_t, $t \neq 0$; moreover, $(\bar{\bar{X}}_t, p'_t)$ is analytically isomorphic to $(\bar{\bar{V}}, p')$ and we may simultaneously resolve the surfaces $\bar{\bar{X}}_t$, $t \neq 0$, and $\bar{\bar{V}} = \bar{\bar{X}}_0$ at p'. This produces a degeneration $\bar{\pi}: \bar{X} \to \Delta$ where $\bar{X}_t$ is the resolution of $\bar{\bar{X}}_t$ for $t \neq 0$, and $\bar{X}_0$ is the (partial) resolution $\bar{V}$ of $\bar{\bar{V}}$ with one singular point p and the dual cycle of rational curves D'. The surfaces X_t, $t \neq 0$, contain D' as an anticanonical cycle. It is easy to show that $\bar{X}_t$ is, in fact, a smooth rational surface [8, (2.8)]. Hence:

Corollary (2.2). *If the cusp D is smoothable, then the dual cycle D' sits as an anticanonical divisor on a smooth rational surface.*

Looijenga's Conjecture (2.3) [8, (2.11)]. *Conversely, if D' is an anticanonical divisor on a smooth rational surface, then the cusp D is smoothable.*

We now fix some notation which will be used throughout the rest of this section. Assume

$$X_0 = \bigcup_{i \geq 0} V_i$$

is a surface with (local) normal crossings satisfying:

(2.4.1) The dual graph Γ of X_0 is a triangulation of S^2.

(2.4.2) V_0 is an Inoue-Hirzebruch surface as in 1.5.

(2.4.3) The normalization $\tilde{V}_i$ of V_i, $i > 0$, is a smooth rational surface.

(2.4.4) If D_0 is the double curve of X_0 on V_0, then $D_0 = D$ is one component of the anticanonical divisor on V_0.

(2.4.5) For $i > 0$, let D_i be the inverse image of the double curves of X_0 on $\tilde{V}_i$; then $(\tilde{V}_i, D_i)$ is an *anticanonical pair*, i.e., D_i is a reduced cycle of rational curves on $\tilde{V}_i$, and $D_i \in |-K_{\tilde{V}_i}|$.

(2.4.6) The irreducible double curves D_{ij} of X_0 satisfy the triple point formula (1.6.3).

The main result of this section is then:

Theorem (2.5). *A surface X_0 as in (2.4) exists if and only if the cusp singularity D is smoothable.*

The main point of the proof will be to show that the existence of X_0 as in (2.4) implies that the variety with (local) normal crossings X_0 is smoothable, in such a way that the total space of the smoothing X is itself smooth. This part of the proof will consist almost entirely of references to [1], where the corresponding statement is proved when *all* components V_i of X_0 are rational. Since our variety X_0 has one component V_0 which is an Inoue-Hirzebruch surface, some vary minor modifications in the argument must be made. We will only write down the objects involved in the proof and state the special lemmas needed in our case, indicating where necessary the differences in the analysis.

Let $T^0_{X_0}$ and $T^1_{X_0}$ be the cotangent sheaves of Lichtenbaum-Schlessinger and $\mathbb{T}^0_{X_0}$, $\mathbb{T}^1_{X_0}$ their global counterparts. In our case,

$$T^i_{X_0} = \underline{\mathrm{Ext}}^i(\Omega^1_{X_0}, \mathcal{O}_{X_0}) \quad \text{and} \quad \mathbb{T}^i_{X_0} = \mathrm{Ext}^i(\Omega^1_{X_0}, \mathcal{O}_{X_0}),$$

where $\Omega^1_{X_0}$ is the sheaf of Kähler differentials on X_0.

A variety with local normal crossings X_0 with singular locus $Q \subsetneqq X_0$ is said to be *d-semi-stable* if $T^1_{X_0} = \mathcal{O}_Q$. For a surface, this condition implies the triple point formula, which is topological in nature, but has more subtle analytic consequences as well.

Lemma (2.6). *If an X_0 exists as in* (2.4), *then an X'_0 exists, with the same Hirzebruch-Inoue component and double curve D_0, which is d-semi-stable.*

Proof. Identical to that of (5.14) of [1].

If X_0 is a d-semi-stable surface, and $n: \tilde{X}_0 \to X_0$ is the normalization map, then by [1], (3.2) and (3.5), there is an intrinsically defined subsheaf

$$\Lambda^1_{X_0} \subseteqq n_* \Omega^1_{\tilde{X}_0}(\log \tilde{Q})$$

(where $\tilde{Q}$ is the normalization of Q) and a resolution

$$0 \to \Omega^1_{X_0}/\tau_{X_0} \to \Lambda^1_{X_0} \to n_* \mathcal{O}_{\tilde{Q}} \to n_* \mathcal{O}_{\tilde{T}} \to 0$$

(where $\tilde{T} = T$ is the set of the triple points of X_0). Here τ_{X_0} is the torsion part of $\Omega^1_{X_0}$, and the natural map

$$\check{\Omega}^1_{X_0} \to \Omega^1_{X_0}/\tau_{X_0}$$

is an isomorphism. The role of $\Lambda^1_{X_0}$ in deformation theory is explained by the following exact sequences. Choose a generating section $\xi \in H^0(T^1_{X_0})$ and, via Lie bracket, consider the map $[\cdot, \xi]: T^0_{X_0} \to T^1_{X_0}$. Then

$$0 \to S_{X_0} \to T^0_{X_0} \xrightarrow[{[\cdot,\xi]}]{} T^1_{X_0} \to 0$$

is exact, where $\check{S}_{X_0} = \Lambda^1_{X_0}$.

Lemma (2.7). *With X_0 as in* (2.4), *and d-semi-stable*, $H^0(X_0, \Lambda^1_{X_0}) = 0$.

Proof. Identical to that of (5.9) of [1], starting with V_0 and the negative definite cycle D_0 of double curves on it in place of the non-hexagonal component used in (5.9) of [1].

Lemma (2.8).

(2.8.1) $H^2(T^0_{X_0})=0.$

(2.8.2) The natural map $\mathbb{T}^1_{X_0}\to H^0(T^1_{X_0})$ is surjective.

(2.8.3) The natural map $H^1(T^0_{X_0})\otimes H^0(T^1_{X_0})\to H^1(T^1_{X_0})$ is surjective.

Proof. First, using the resolution [1, (1.5)]

$$0\to\Omega^1_{X_0}/\tau_{X_0}\to n_*\Omega^1_{\tilde X_0}\to n_*\Omega^1_{\tilde Q}\to 0$$

and $H^0(V_0,\Omega^1_{V_0})=0$ (1.5.3), we obtain $H^0(\Omega^1_{X_0}/\tau_{X_0})=0$. By Serre duality, $H^2(T^0_{X_0})$ is dual to $H^0((\Omega^1_{X_0}/\tau_{X_0})\otimes\omega_{X_0})$, where ω_{X_0} is the dualizing sheaf. But, by contruction,

$$n^*\omega_{X_0}|_{V_0}=\mathcal{O}_{V_0}(-D')$$

and

$$n^*\omega_{X_0}|_{V_i}=\mathcal{O}_{V_i}\quad\text{for}\quad i>0,$$

and hence

$$H^0(F\otimes\omega_{X_0})\subsetneqq H^0(F)$$

for any torsion-free sheaf F. Thus $H^2(T^0_{X_0})=0$, implying (2.8.1). The Ext spectral sequence immediately gives (2.8.2). As for (2.8.3), as in (5.9) of [1], we must show that $H^2(S_{X_0})=0$, or equivalently, $H^0(\Lambda^1_{X_0}\otimes\omega_{X_0})=0$. Again, this is immediate from the vanishing of $H^0(\Lambda^1_{X_0})$. Q.E.D.

Lemma (2.9). *There exists a smooth threefold X and a proper flat map $\pi:X\to\Delta$ such that $\pi^{-1}(0)=X_0$ (as schemes).*

Proof. The proof of (5.10) of [1] applies, essentially unchanged.

Remark (2.10). The general fiber X_t of π is a rational surface.

The existence part of (2.5) now follows froma lemma due to Sheperd-Barron [9]:

Lemma (2.11). *The divisor $\sum_{i\geqq 1} V_i$ of X is contractible, and, if $\bar X$ denotes the contraction, there is a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & \bar X \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\bar\pi} & \\ & \Delta & \end{array}$$

Moreover, the map $\bar\pi:\bar X\to\Delta$ is flat, and exhibits a smoothing of $\bar V_0$ (which is V_0 contracted along $D_0=D$).

Remark (2.12). This lemma follows from the consideration of the map $f:X_0\to\bar V_0$ defined by: $f|_{V_0}$ is the contraction of D, and $f|_{V_i}$, $i>0$, is the map sending V_i to the singular point p of $\bar V_0$. One checks that $R^if_*\mathcal{O}_{X_0}=0$ for $i>0$, and uses standard results from deformation theory.

For the converse part of (2.5), assume that the cusp D is smoothable. Let $\bar V_0$ be the Inoue-Hirzebruch surface containing D, with D contracted to the cusp singularity p. By (2.1), there is a flat family $\bar\pi:\bar X\to\Delta$ with $\bar X_t$ smooth and

$\bar{V}_0 = \bar{\pi}^{-1}(0)$. By blowing up the singular point p of $\bar{V}_0$ in $\bar{X}$ and by making the necessary base changes, arrange $\pi: X \to \Delta$ semi-stable, with one component of X_0 being the resolution V_0 of $\bar{V}_0$. One can run through the argument of Kulikov and Persson-Pinkham, applied to the components of X_0 which occur with strictly maximal multiplicity in K_X; the point here is that the proper transform V_0 of $\bar{V}_0$ is a resolution of a minimally elliptic singularity with reduced fundamental cycle in the minimal resolution, and hence can never occur with strictly maximal multiplicity. Thus, by Kulikov's analysis, we may assume X_0 satisfies (2.4.2)–(2.4.6).

Now the dual graph Γ triangulates a closed surface. Let $e(\Gamma)$ be the Euler number. Recalling that X_t is rational and $q(V_0)=1$, an easy argument shows that

$$1 = \chi(\mathcal{O}_{X_t}) = \chi(\mathcal{O}_{X_0}) = e(\Gamma) - 1,$$

so that $e(\Gamma)=2$ and Γ triangulates S^2, so that X_0 satisfies (2.4.1) also. Q.E.D.

Remark (2.13). If D sits on a rational surface as an anticanonical divisor (which will not always be the case), we can play the same game as above with the appropriate rational surface S replacing the Inoue-Hirzebruch surface V_0. In this case, the smoothings will be K3 surfaces.

The following lemma shows that, in (2.4), it suffices to construct the disjoint collection of surfaces $\{\tilde{V}_i\}$, subject to the appropriate combinatorial restrictions.

Lemma (2.14). *Let $\{\tilde{V}_i\}$ be a collection of smooth surfaces, $\{D_{ij}\}$ a collection of smooth curves on $\tilde{V}_i$ and $\{\varphi_{ij}: D_{ij} \to D_{ji}\}$ a collection of isomorphisms satisfying:*

(2.14.0) $D_i = \bigcup_j D_{ij}$ *is a divisor with normal crossings,*

(2.14.1) $\varphi_{ij}^{-1} = \varphi_{ij}$,

(2.14.2) $p_{ijk} \in D_{ij} \cap D_{ik} \Leftrightarrow \varphi_{ij}(p_{ijk}) \in D_{ji} \cap D_{jk}$,

(2.14.3) $\varphi_{ik} \circ \varphi_{ij}(p_{ijk}) = \varphi_{ik}(p_{ijk})$.

Then there is a unique structure of a variety with normal crossings on the topological space

$$X_0 = \coprod \tilde{V}_i / x \in D_{ij} \sim \varphi_{ij}(x)$$

which is compatible with the inclusions $\tilde{V}_i \subseteq X_0$. A similar statement is true if the φ_{ij} are arranged to give, topologically, only local normal crossings.

Proof. We shall only sketch the proof of this elementary statement. As the question is local, assume $X_0 = \tilde{V}_i \cup \tilde{V}_j \cup \tilde{V}_k$ and let $n: \coprod \tilde{V}_i = \tilde{X}_0 \to X_0$ be the obvious map. With D_{ij}, p_{ijk} as in (2.14), we define a subsheaf

$$\mathcal{O}_{X_0} \subseteq n_* \mathcal{O}_{\tilde{X}_0}$$

by the recipe: if sections of $n_*\mathcal{O}_{\tilde{X}_0}$ at p are of the form $f=(f_1, f_2, f_3)$, $f \in \mathcal{O}_{X_0,p} \Leftrightarrow f_s \circ \varphi_{rs}(x) = f_r(x)$, $x \in D_{rs}$ $(r, s \in \{i,j,k\})$.

Alternatively, the sections of $\mathcal{O}_{X_0}$ are continuous functions on X_0 which are holomorphic on $\tilde{V}_i$. We leave to the reader the local calculation which identifies the ringed space $(X_0, \mathcal{O}_{X_0})$ locally with the germ of $\{(z_1, z_2, z_3) \in \mathbb{C}^3;\ z_1 z_2 z_3 = 0\}$. Q.E.D.

Combining (2.5) with (2.14), we obtain the following corollary to (2.5):

Proposition (2.15). *Let $D=D_1+\ldots+D_r$ be a smoothable cusp, with $-(D_1^2)\geqq 3$ and $-(D_j)^2\geqq 3$ for some $j\neq 1$. Then $C=C_1+\ldots+C_r$ is smoothable, where $(C_1^2)=(D_1^2)+1$ and $(C_i^2)=(D_i^2)$, $i\geqq 2$.*

Proof. By the argument for the converse half of (2.5), there is a semistable degeneration $\pi: X\to \Delta$ with $X_0=\pi^{-1}(0)$ as in (2.4) for the cusp D. Let V_0 be the Inoue surface and V_1 the component of X_0 meeting V_0 along D_1. Replace V_0 by V_0' = the Inoue surface for C, and V_1' by V_1 blown up at a smooth point of $D_1\subsetneqq V_1$ (with obvious modifications if $V_1=V_0$). If $V_i'=V_i$ for $i\neq 0, 1$, then the collection $\{V_i'\}$ yields the correct combinatorial configuration for the cusp C in (2.5). Q.E.D.

Note that (2.15) is not an adjacency relation between cusps. It would be amusing to verify directly that the dual of C sits as an anticanonical divisor on a rational surface.

3. Charge

Let V be a smooth surface, and let D be a reduced effective cycle of rational curves on V. (D need not be negative definite here.) Let $r(D)$ be the number of irreducible components of D; $r(D)\geqq 1$ and $r(D)=1$ when D is a single irreducible nodal curve.

Definition (3.1). Let V and D be as above. The *charge* of (V, D), denoted by $Q(V, D)$, is the quantity

$$Q(V,D)=12-D^2-r(D).$$

If we write

$$D=\sum_{j=1}^{r(D)} D_j,$$

then

$$Q(V,D)=\begin{cases} 11-D_1^2 & \text{if } r(D)=1\\ 12-\sum_{j=1}^{r(D)}[(D_j^2)+3] & \text{if } r(D)\geqq 2.\end{cases}$$

We will mainly be concerned with the case where V is rational and D is an anticanonical divisor of V, i.e., (V, D) is an anticanonical pair. In this case

$$Q(V,D)=12-K_V^2-r(D).$$

Notice that $Q(V, D)$ depends only upon $r(D)$ and the D_j^2's, so that if we represent the divisor D by the cycle of integers $(d_1, \ldots, d_r)$, where $r=r(D)$ and $d_j=-D_j^2$, then $Q(V, D)$ depends only on the cycle of integers. We will sometimes use $Q(D)$ for $Q(V, D)$ when no confusion can result.

It will be useful to understand the anticanonical pairs (V, D) where V is a minimal model $\mathbb{P}^2$ or $\mathbb{F}_N$; the proof of the following lemma is left to the reader.

Lemma (3.2). *Let (V, D) be an anticanonical pair, with V isomorphic to $\mathbb{P}^2$ or $\mathbb{F}_N$. Then (V, D) is one of the following:*

V	$(-d_1, \ldots, -d_r)$	$Q(V, D)$
$\mathbb{P}^2$	$(1,1,1)$	0
$\mathbb{P}^2$	$(1,4)$	1
$\mathbb{P}^2$	(9)	2
$\mathbb{F}_N$	$(-N,0,N,0)$	0
$\mathbb{F}_N$	$(-N,0,N+2)$	1
$\mathbb{F}_N$	$(-N,N+4)$	2
$\mathbb{F}_0, \mathbb{F}_2$	$(2,2)$	2
$\mathbb{F}_1$	$(1,3)$	2
$\mathbb{F}_1$	$(1,0,1)$	1
$\mathbb{F}_0, \mathbb{F}_1, \mathbb{F}_2$	(8)	3

The behavior of Q under blowups and blowdowns is given by the following two elementary lemmas.

Lemma (3.3). *Let (V, D) be an anticanonical pair, let p be a point on D, and let $\pi: \tilde{V} \to V$ be the blowup at p, with exceptional divisor E.*

(3.3.1) *If p is a smooth point of D, and $\tilde{D}$ is the proper transform of D on $\tilde{V}$, then $(\tilde{V}, \tilde{D})$ is an anticanonical pair, and $Q(\tilde{V}, \tilde{D}) = Q(V, D) + 1$.*

(3.3.2) *If p is a double point of D, and $\tilde{D}$ is the proper transform of D on $\tilde{V}$, then $(\tilde{V}, \tilde{D} + E)$ is an anticanonical pair, and $Q(\tilde{V}, \tilde{D} + E) = Q(V, D)$.*

Lemma (3.4). *Let (V, D) be an anticanonical pair. Let E be an exceptional curve on V, let $\pi: V \to \bar{V}$ be the blowdown of E, and let $\bar{D} = \pi_*(D)$. Then either*

(3.4.1) *E is not a component of D, there is a unique component D_j of D which meets E once transversally, and $Q(\bar{V}, \bar{D}) = Q(V, D) - 1$ or*

(3.4.2) *E is a component of D and $Q(\bar{V}, \bar{D}) = Q(V, D)$.*

Moreover, in both cases $(\bar{V}, \bar{D})$ is an anticanonical pair.

The proofs of the above are trivial consequences of the definition of Q, the blowup formula for the canonical bundle, and the equation

$$E \cdot K_V = -1$$

where E is an exceptional curve on V; we leave them to the reader.

Lemma (3.5). *Let (V, D) be an anticanonical pair. Then*

$$Q(V, D) \geqq 0.$$

Proof. By Lemma (3.3), the charge Q cannot decrease upon blowing up; so it suffices to prove the result when V is a minimal model $\mathbb{P}^2$ or $\mathbb{F}_N$. For this the classification of Lemma (3.2) suffices. Q.E.D.

A glance at the list of Lemma (3.2) shows that if V is a minimal model, and (V, D) is an anticanonical pair, then $0 \leqq Q(V, D) \leqq 3$; hence, by Lemma (3.3), if (V, D) is an arbitrary anticanonical pair, then $Q(V, D)$ is approximately equal to the number of blowups made at smooth points of the anticanonical cycles between V and a minimal model. This can be made more precise; we will only need the following relatively crude estimate:

Lemma (3.6). *Let (V, D) be an anticanonical pair, and let $E_1, \ldots, E_k$ be a pairwise disjoint set of k exceptional curves on V which are not components of D. Then $Q(V, D) \geqq k$.*

Proof. Since the E_i's are all disjoint, they can all be blown down, to obtain an anticanonical pair $(\bar{V}, \bar{D})$. By Lemma (3.4), $Q(\bar{V}, \bar{D}) = Q(V, D) - k$, and by Lemma (3.5), $Q(\bar{V}, \bar{D}) \geqq 0$. This gives the result. Q.E.D.

The following is the principle of "conservation of charge" for Type III degenerations of K3 surfaces. Let $X \to \Delta$ be such a degeneration, and write the special fiber X_0 as

$$X_0 = \sum_i V_i.$$

Let $\tilde{V}_i$ be the normalization of V_i, let D_i be the double curve of X_0 on V_i, and let $\tilde{D}_i$ be the inverse image of D_i on $\tilde{V}_i$. [The pairs (V_i, D_i) and $(\tilde{V}_i, \tilde{D}_i)$ are equal unless V_i has self-intersection in X.] Then, by (1.6.1) and (1.6.2), each $(\tilde{V}_i, \tilde{D}_i)$ is an anticanonical pair.

Proposition (3.7). *Let $X \to \Delta$ be a Type* III *degeneration of* K3 *surfaces as above. Then*

$$\sum_i Q(\tilde{V}_i, \tilde{D}_i) = 24.$$

Proof. Let v be the number of components V_i of X_0, e be the number of irreducible double curves D_{ij}, and f be the number of triple points of X_0. Since v is the number vertices, e the number of edges, and f the number of faces of the dual graph Γ of X_0, we have $v - e + f = 2$ by Euler's formula. Let r_i be the number of components of $\tilde{D}_i$. Then $\sum_i r_i = 2e$ since each double curve lies on two surfaces. Moreover, since Γ is a triangulation, $2e = 3f$. Now compute:

$$\begin{aligned}
\sum_{i=1}^{v} Q(\tilde{V}_i, \tilde{D}_i) &= \sum_{i=1}^{v} (12 - K_{\tilde{V}_i}^2 - r_i) \\
&= \sum_{\substack{i \text{ such that} \\ r_i = 1}} (11 - D_{iji}^2) + \sum_{\substack{i \text{ such that} \\ r_i \geqq 2}} \left(12 - \sum_{k=1}^{r_i} [\tilde{D}_{ijk}^2 + 3]\right) \\
&= 12v - \sum_{i,j} \tilde{D}_{ij}^2 - 3 \sum_i r_i + 2m,
\end{aligned}$$

where m is the number of components V_i with $r_i = 1$.

The quantity $\sum_{i,j} \tilde{D}_{ij}^2$ counts both terms $\tilde{D}_{ij}^2$ and $\tilde{D}_{ji}^2$

and so

$$\sum_{i,j} \tilde{D}_{ij}^2 = \sum_{\substack{\text{double curves} \\ D_{ij} \subset X_0}} \tilde{D}_{ij}^2 + \tilde{D}_{ij}^2 = -2e + 2m$$

by (1.6.3). Hence

$$\begin{aligned} \sum_i Q(\tilde{V}_i, \tilde{D}_i) &= 12v + 2e - 3(2e) \\ &= 12v - 4e \\ &= 12(2 + e - f) - 4e \\ &= 24 + 8e - 12f \\ &= 24. \quad \text{Q.E.D.} \end{aligned}$$

Remarks (3.8).

(3.8.1) If one component V_0 of X_0 is a Hirzebruch-Inoue surface, and X_0 is as in (2.4), then the same calculation as above shows that in this case

$$\sum_i Q(\tilde{V}_i, \tilde{D}_i) = 24$$

also. In particular, if $Q(V_0, D_0)$ is high, then severe restrictions are placed on the other components V_i, by Lemma (3.6).

(3.8.2) A calculation similar to the above can be made for a Type III degeneration of any $\kappa = 0$ surface. The result is:

$$\sum_i Q(\tilde{V}_i, \tilde{D}_i) = 12e(\Gamma).$$

where $e(\Gamma)$ is the topological Euler characteristic of the dual graph Γ.

(3.8.3) Using only Euler's formula, one can easily prove the formula

$$\sum_i (6 - r_i) = 12$$

for a Type III degeneration as above, which can sometimes be useful.

4. Cusps with Rational Duals

In this section, we will primarily be concerned with the case where D is a negative definite cycle of rational curves on a surface V, i.e., D is the resolution of a cusp singularity.

Lemma (4.1).

(4.1.1) *Let D be the cusp $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$, then*

$$Q(V, D) = \begin{cases} 11 + a_1 & \text{if} \quad r(D) = 1 \\ 12 + \sum_i (a_i - b_i - 3) & \text{if} \quad r(D) \geq 2. \end{cases}$$

(4.1.2) *If D' is the dual to D, then*

$$Q(V',D')=\begin{cases} 13-a & \text{if} \quad D=\begin{pmatrix} a \\ 0 \end{pmatrix}, \quad a \geqq 1 \\ 12+b & \text{if} \quad D=\begin{pmatrix} 3 \\ b \end{pmatrix}, \quad b \geqq 1 \\ 12+\Sigma(b_i-a_i+3) & \text{if} \quad D=\begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{otherwise.} \end{cases}$$

Hence in all cases we have $Q(D)+Q(D')=24$.

Proof. These statements follow immediately from Lemma (1.2), Definition (1.3), Lemma (1.4), and the Definition (3.1) of $Q(D)$. Q.E.D.

Remark (4.2). The above formula $Q+Q'=24$ and the formula of Proposition (3.7) suggest that there may be a relationship between the combinatorics of an X_0 as in (2.4) (with one compact a Hirzebruch-Inoue surface with the cusp D) and the dual cusp D'. Looijenga's conjecture (2.3) is only qualitative; one might hope for some more precise correlation, given Theorem (2.5).

Lemma (4.3). *Let (V, D) be an anticanonical pair, and assume that D is negative definite, so that D is the resolution of a cusp singularity. Then $Q(V, D) \geqq 3$.*

Proof. Since V is a rational surface, the rank of the Neron-Severi group $NS(V)$ of V is $10-K_V^2=10-D^2$. Since D is negative definite, and $NS(V)$ contains a positive class, we must have $r(D) \leqq \operatorname{rank} NS(V)-1$, or $9-D^2-r(D) \geqq 0$. Hence $Q(V, D)=12-D^2-r(D) \geqq 3$. Q.E.D.

Definition (4.4). Let D be a cusp. We say that D is *rational* if there exists a rational surface V on which D is an anticanonical divisor, i.e., such that (V, D) is an anticanonical pair. If D is a cusp such that the dual D' to D is rational, we say that *D has a rational dual.*

Theorem (4.5). 1) *If D has a rational dual, $Q(D) \leqq 21$.*

2) *Conversely, let D be a cusp with $r(D) \leqq 3$ and $Q(D) \leqq 21$. Then D has a rational dual, except in the following cases:* (4, 11), (7, 8), (2, 4, 12), (2, 8, 8), (3, 3, 12), (3, 4, 11), (3, 7, 8), (4, 4, 10), (4, 6, 8), (4, 7, 7), (5, 5, 8).

We break the proof of (4.5) up into several steps.

Lemma (4.6). *If D has a rational dual, then $Q(D) \leqq 21$.*

Proof. Combine (4.1.2) with Lemma (4.3). Q.E.D.

Remark. This lemma follows from a theorem of Wahl [11], where the condition $Q(D) \leqq 21$ is restated as $m(D) \leqq r(D)+9$.

We will now classify those cusps D with $r(D) \leqq 3$ which have rational duals. We begin with the $r=1$ case.

Proposition (4.7). *Let D be a cusp with $r(D)=1$. Then D has a rational dual if and only if $Q(D) \leqq 21$.*

Proof. Since $r(D)=1$, $D=\begin{pmatrix}a\\0\end{pmatrix}$ for some $a\geqq 1$. In this case $Q(D)=11+a$, so $Q(D)\leqq 21$ when $a\leqq 10$. Assume $a\leqq 10$. If $a=1$, $D'=\begin{pmatrix}1\\0\end{pmatrix}$, which is clearly rational: blow up 10 smooth points on a nodal cubic in $\mathbb{P}^2$ to obtain (V', D'). If $a\geqq 2$, then $D'=\begin{pmatrix}3\\a-1\end{pmatrix}$. To construct a rational surface V' with D' as an anticanonical divisor, take a nodal cubic C in the plane and blow up the node $a-1$ times; this produces a cycle of a curves $C_0, \ldots, C_{a-1}$, where C_0 is the proper transform of C and $C_1, \ldots, C_{a-1}$ are the $a-1$ exceptional curves. We have $(C_0^2)=7-a$, $(C_1^2)=-1$, and $(C_i^2)=-2$ for $i\geqq 2$. Finally, blow up one smooth point on C_1 and $10-a$ smooth points on C_0. This gives $\begin{pmatrix}3\\a-1\end{pmatrix}$, with the proper transform of C_0 having self-intersection -3. Q.E.D.

The situation with cusps of length two is not so simple:

Proposition (4.8). *Let D be a cusp with $r(D)=2$.*

(i) *If $Q(D)\leqq 20$, then D has a rational dual.*

(ii) *If $Q(D)=21$, then D is one of the following cusps:* (2, 13), (3, 12), (4, 11), (5, 10), (6, 9), (7, 8).

(iii) *The cusps* (2, 13), (3, 12), (5, 10), *and* (6, 9) *have rational duals.*

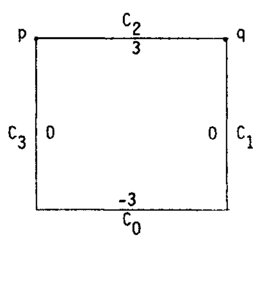
(iv) *The cusps* (4, 11) *and* (7, 8) *do not have rational duals.*

Proof. First notice that $r(D)=2$ if and only if D is either $\begin{pmatrix}a\\1\end{pmatrix}$ or $\begin{pmatrix}a_1 & a_2\\0 & 0\end{pmatrix}$. If $D=\begin{pmatrix}a\\1\end{pmatrix}$, then $Q(D)=8+a$; if $D=\begin{pmatrix}a_1 & a_2\\0 & 0\end{pmatrix}$, then $Q(D)=6+a_1+a_2$. This proves (ii).

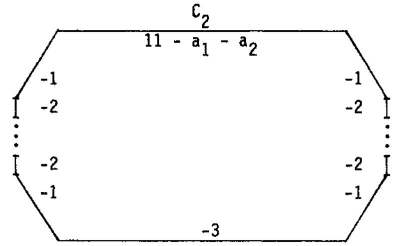
Assume $D=\begin{pmatrix}a\\1\end{pmatrix}$, with $4\leqq a\leqq 13$. Then $D'=\begin{pmatrix}4\\a-3\end{pmatrix}$. To produce an anticanonical pair (V', D'), we argue as in the proof of the previous proposition. Take C to be a nodal cubic in $\mathbb{P}^2$, and blow up the node $a-3$ times. This gives a cycle $C_0, C_1, \ldots, C_{a-3}$, with C_0 the proper transform of C, $(C_0^2)=9-a$, $(C_1^2)=-1$, and $(C_i^2)=-2$, for $i\geqq 2$. Now blow up one smooth point on C_1 and $13-a$ points on C_0, to get D'.

If $D=\begin{pmatrix}3\\1\end{pmatrix}$, $D'=\begin{pmatrix}4\\0\end{pmatrix}$, which can be obtained from $\mathbb{P}^2$ by blowing up 13 smooth points on a nodal cubic.

Assume now that $D=\begin{pmatrix}a_1 & a_2\\0 & 0\end{pmatrix}$, so that $D'=\begin{pmatrix}3 & 3\\a_1-3 & a_2-3\end{pmatrix}$, and $Q(D)\leqq 20$, so that $a_1+a_2\leqq 14$. Assume first that a_1 and a_2 are both at least 4. Start with $\mathbb{F}_3$ and the anticanonical square

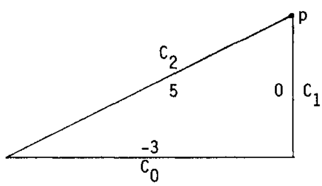


Blow up a_1-4 times at p and a_2-4 times at q to produce

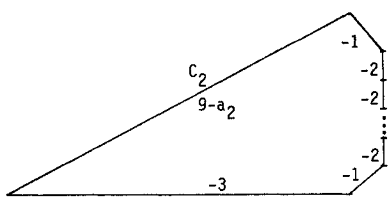


where the two fibers of the original ruling in this anticanonical cycle have lengths a_1-3 and a_2-3. (If either a_i is equal to 4, no blowups are made and the fiber is smooth.) Now blow up one point on each of the (-1)-curves in the fibers [or two points on the (0)-curve if $a_i=4$] and also blow up the proper transform of the original positive section C_2 $14-a_1-a_2$ times at smooth points. This produces (V', D').

Assume secondly that $a_1=3$ and $4 \leqq a_2 \leqq 12$. Start with $\mathbb{F}_3$ and the anticanonical triangle



Blow up a_2-4 times at p as above to produce



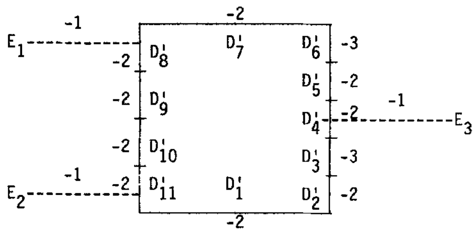
Finally, blow up one point on each of the (-1) curves in the displayed fiber [or two points on the (0)-curve if $a_2=4$] and also blow up the proper transform of C_2 $12-a_2$ times. This produces (V', D'). Notice that this construction also exhibits a rational dual for the $Q=21$ cusp $\begin{pmatrix} 3 & 12 \\ 0 & 0 \end{pmatrix}$.

Finally, assume $a_1=a_2=3$. Then $D'=\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}$ and can be obtained from the anticanonical pair ($\mathbb{P}^2$, line + conic) by blowing up smooth points.

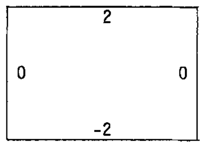
This proves (i).

To finish the proof of (iii), we must exhibit the rationality of the duals of $\begin{pmatrix} 5 & 10 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 6 & 9 \\ 0 & 0 \end{pmatrix}$. For these we will simply draw the surface V' on which D' sits, with the exceptional curves required to blow the surface down to a minimal model.

For $\begin{pmatrix}5 & 10\\ 0 & 0\end{pmatrix}$: $D'=\begin{pmatrix}3 & 3\\ 2 & 7\end{pmatrix}$

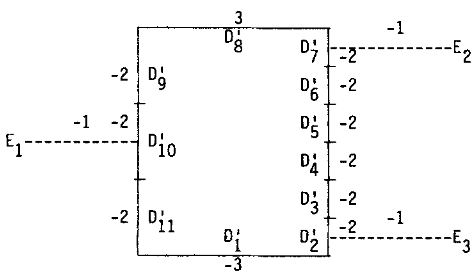


Blow down E_1, E_2, E_3, D'_3, D'_4, D'_5, D'_6, D'_8, D'_9, and D'_{10} to obtain

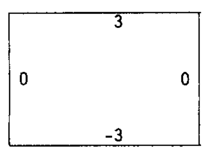


on $\mathbb{F}_2$.

For $\begin{pmatrix}6 & 9\\ 0 & 0\end{pmatrix}$: $D'=\begin{pmatrix}3 & 3\\ 3 & 6\end{pmatrix}$

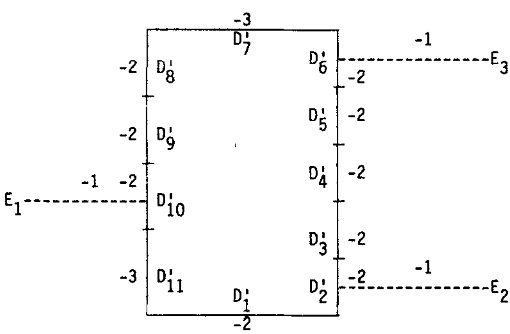


Blow down E_1, E_2, E_3, D'_3, D'_4, D'_5, D'_6, D'_7, D'_9, and D'_{10} to obtain

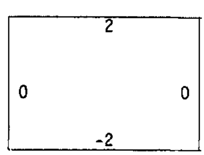


on $\mathbb{F}_3$

or, for $\begin{pmatrix}6 & 9\\ 0 & 0\end{pmatrix}$:



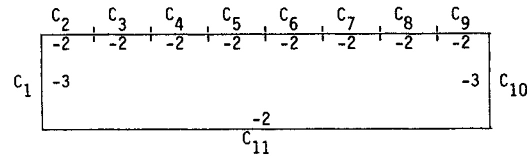
Blow down E_1, E_2, E_3, D'_3, D'_4, D'_5, D'_6, D'_8, D'_9, and D'_{10} to obtain



on $\mathbb{F}_2$.

This proves (iii).

Unfortunately, our proof of (iv) is rather ad hoc. Let us first take up the case of the $\begin{pmatrix} 4 & 11 \\ 0 & 0 \end{pmatrix}$ cusp, whose dual is the cusp $\begin{pmatrix} 3 & 3 \\ 1 & 8 \end{pmatrix}$:



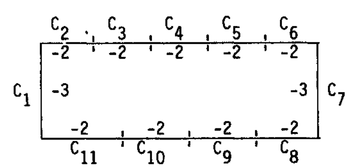
Assume this cusp is rational. Then there exists an exceptional curve E on the surface V; visibly E is not a component of the cusp, so E meets a unique C_i transversally in one point. Let us analyze the possibilities for C_i.

If E meets C_1 or C_{10}, then blowing down E produces a rational cusp with $Q=2$; this is impossible, by Lemma (4.3).

If E meets C_{11}, blowing down E and then C_{11} produces a cycle of 10 (-2)-curves. This is not minimal, so there must be a second exceptional curve F meeting one of the 10 curves. Then one can blow down F and 9 of the 10 (-2)-curves; however, this produces a nodal curve C with $(C^2)=10$. Since $C \in |-K|$, the resulting rational surface has $K^2=10$, a contradiction.

Hence E must meet $C_2, C_3, \ldots,$ or C_9; by symmetry we may assume C meets $C_2, C_3, C_4,$ or C_5. We may further assume that E meets C_j, $2 \leqq j \leqq 5$, and that there is no exceptional curve on the surface meeting C_i, for $2 \leqq i < j$. If so, we may blow down $E, C_j, C_{j+1}, \ldots, C_9$, producing a cycle consisting of the images of the curves $C_{10}, C_{11}, C_1, C_2, \ldots, C_{j-1}$. This is a non-minimal cycle by Lemma (3.2); however, by assumption, there are no exceptional curves meeting $C_2, \ldots, C_{j-1}$ and by the previous argument, there are none meeting C_1, C_{10}, or C_{11}. This contradiction proves that $\begin{pmatrix} 3 & 3 \\ 1 & 8 \end{pmatrix}$ is not rational.

Finally, assume that $\begin{pmatrix} 7 & 8 \\ 0 & 0 \end{pmatrix}$ has a rational dual, i.e., that $\begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}$ is rational:



Let E be an exceptional curve on the surface V which contains $\begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}$; as above, E must meet some component C_i of the cusp.

If E meets C_1 or C_7, blowing down E produces a cusp with $Q=2$, contradicting Lemma (4.3).

If E meets C_8, then $C_7+3C_8+2C_9+C_{10}+3E$ form a fiber of a ruling on V, for which C_6 and C_{11} must be sections. Hence C_1, C_2, C_3, C_4, and C_5 are components of a fiber of this ruling. They do not support a complete fiber, so there is a second exceptional curve F on V meeting one of these five curves. By the above, F meets

C_2, C_3, C_4, or C_5. If F meets C_2, then $C_1+3C_2+2C_3+C_4+3F$ is a full fiber, not containing C_5, a contradiction. If F meets C_3, $C_2+2C_3+C_4+2F$ is a full fiber, and if F meets C_4, $C_3+2C_4+C_5+2F$ is a full fiber, which are contradictions. Hence F must meet C_5. But now C_1, C_2, C_3, C_4, C_5, and F still do not support a complete fiber, so there must be a third exceptional curve G; by the above, G must also meet C_5. Now $F+G+C_5$ is a full fiber, a contradiction. Hence E does not meet C_8, and by symmetry, E does not meet C_{11} either.

If E meets C_9, then $C_8+2C_9+C_{10}+2E$ form a fiber of a ruling on V, for which C_7 and C_{11} are sections. Hence C_1, C_2, C_3, C_4, C_5, and C_6 are components of a fiber of this ruling. They do not support a full fiber, so there is an exceptional curve F, which by the previous argument cannot meet C_1. Using an analysis as above, one can easily see that if F meets C_2, C_3, C_4, or C_5, there is a full fiber supported on only a proper subset of the components; hence F must meet C_6. But, as above, C_1, C_2, C_3, C_4, C_5, C_6, and F still do not support a full fiber, so there is a third exceptional curve G, which must then meet C_6. Now $F+G+C_6$ forms a full fiber, a contradiction. Hence E cannot meet C_9 or, by symmetry, C_{10}.

Now the argument proceeds as for the previous cusp. We may assume that E meets C_j, with $2\leqq j\leqq 4$, and that there is no exceptional curve meeting C_7, C_8, C_9, C_{10}, $C_1,\dots,C_{j-1}$. If so, we may blow down $C_j, C_{j+1},\dots,C_6$ to produce a cycle consisting of C_7, C_8, C_9, C_{10}, $C_1,\dots,C_{j-1}$; by Lemma (3.2), this cycle is not minimal, but by assumption there are no exceptional curves. This contradiction proves that $\begin{pmatrix}3 & 3\\ 4 & 5\end{pmatrix}$ is not rational and completes the proof of Proposition (4.8). Q.E.D.

Finally, we have the following result for cusps of length three. Its proof is similar to that of the previous proposition, and we leave it to the reader.

Proposition (4.9). *Let D be a cusp with $r(D)=3$.*

(i) *If $Q(D)\leqq 20$, then D has a rational dual.*

(ii) *If $Q(D)=21$, then D has a rational dual if and only if D is one of the following:* (2,2,14), (2,3,13), (2,5,11), (2,6,10), (2,7,9), (3,5,10), (3,6,9), (4,5,9), (5,6,7), (6,6,6).

(iii) *The $Q=21$, $r=3$ cusps which do not have rational duals are the following:* (2,4,12), (2,8,8), (3,3,12), (3,4,11), (3,7,8), (4,4,10), (4,6,8), (4,7,7), (5,5,8).

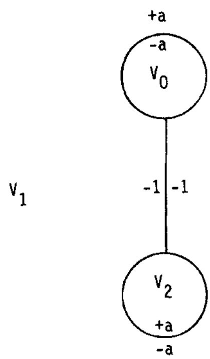
5. The Special Fibers of Smoothings of Cusps of Small Length

In this section, we will exhibit smoothings of all the cusp singularities of length one and two which have rational duals, verifying Looijenga's conjecture in these cases. We will also be able to smooth some cusps of length three. The method will be that outlined in Sect. 2; we will construct a Type III degeneration of rational surfaces whose special fiber contains one component (the Inoue surface) which has the resolution of the cusp to be smoothed on it, as the double curve.

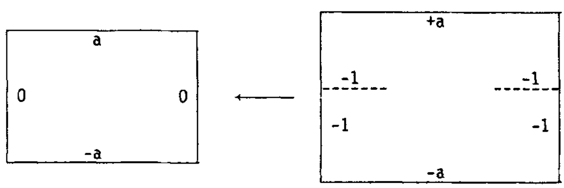
If V_i is a component of the special fiber X_0 of a given Type III degeneration, we will write $Q(V_i)$ for $Q(\tilde{V}_i.\tilde{D}_i)$ as in (3.7).

The Smoothings of Length one Cusps

By Proposition (4.7), if $r(D)=1$, then D has a rational dual if and only if $(D^2) \geqq -10$. Set $a=-(D^2)$. For $1 \leqq a \leqq 9$, use the following special fiber X_0:

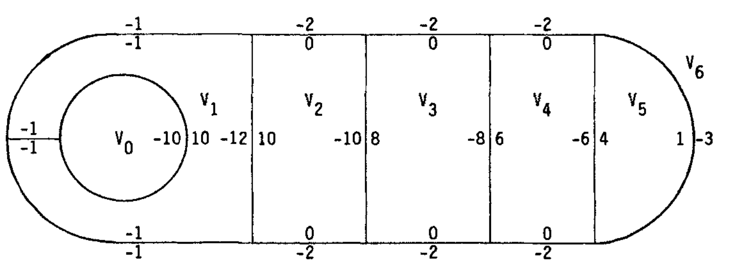


The component V_0 is the Inoue surface with double curve D. The component V_2 is the blowup of a $\mathbb{P}^2$ at $9-a$ points of a nodal cubic. The component V_1 is the blowup of $\mathbb{F}_a$ at two smooth points on different fibers,

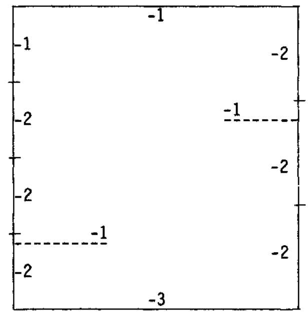


then glued to itself by joining the two (-1)-curves. $Q(V_0)=11+a$, $Q(V_1)=2$, and $Q(V_2)=11-a$ in this degeneration.

For $a=10$, we use the following X_0:



V_0 is the Inoue surface. V_1 is a two-fold blowup of $\mathbb{F}_{10}$ at two points of the negative section, then glued to itself as shown. $V_2 \cong \mathbb{F}_{10}$, $V_3 \cong \mathbb{F}_8$, and $V_4 \cong \mathbb{F}_6$. V_5 is a $\mathbb{P}^2$ with double curves a line and a conic. V_6 is a blowup of $\mathbb{F}_3$,



with the last two non-double exceptional curves shown above. Here $Q(V_0)=21$, $Q(V_5)=1$, and $Q(V_6)=2$.

This completes the analysis of the cusps of length one. Before we discuss those of length two, we need a

Lemma (5.1). *Let (V, D) be an anticanonical pair with $r(D)=2$. Write $D=D_1+D_2$. Then either*

(i) $(D_1^2)+(D_2^2)\leqq 4$, *or*

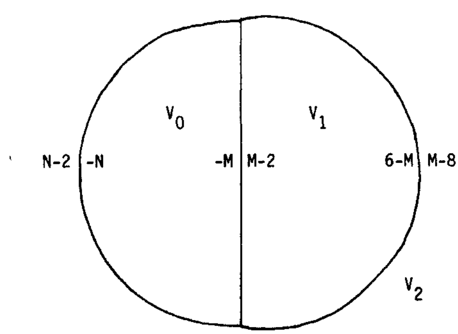
(ii) $(D_1^2)=1$, $(D_2^2)=4$, *and* $V\cong \mathbb{P}^2$, *with* D_1 *a line and* D_2 *a conic.*

Conversely, given two integers a_1 *and* a_2 *such that* $a_1+a_2\leqq 4$, *there is an anticanonical pair* (V, D) *with* $r(D)=2$, *such that* $(D_1^2)=a_1$ *and* $(D_2^2)=a_2$.

Proof. The first statement follows directly from the classification in Lemma (3.2) of the minimal models. The second statement follows from the existence of (V, D) where $r(D)=2$ and $(D_1^2)+(D_2^2)=4$ for any (D_1^2). [If either (D_1^2) or (D_2^2) is $\leqq 0$, use $(\mathbb{F}_N, (-N, N+4))$; if both are positive, use $(\mathbb{F}_1, (1, 3))$ or $(\mathbb{F}_0, (2, 2))$.] Q.E.D.

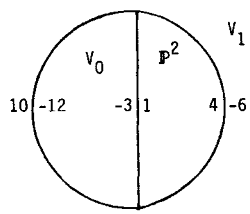
The Smoothings of the Cusps of Length Two

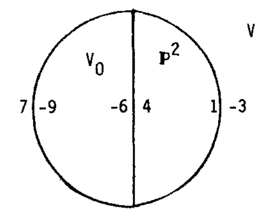
Let us begin with the cusps D with $r(D)=2$ and $Q(D)\leqq 20$, which all have rational duals, by Proposition (4.8). If $D=D_1+D_2$, with $(D_1^2)=-N$ and $(D_2^2)=-M$, then M and N are both at least two, and $Q(D)\leqq 20$ if and only if $N+M\leqq 14$. Use the following X_0:



Here $Q(V_0)=N+M+6$, $Q(V_1)=2$, and $Q(V_2)=16-N-M$. Again V_0 is the Inoue surface. V_1 is $\mathbb{F}_{6-M}$ if $M \geqq 4$ and is $\mathbb{F}_{M-2}$ if $M=2$ or 3. By Lemma (5.1), V_2 exists (as an anticanonical pair, with its two double curves) if $(N-2)+(M-8) \leqq 4$, i.e., $N+M \leqq 14$.

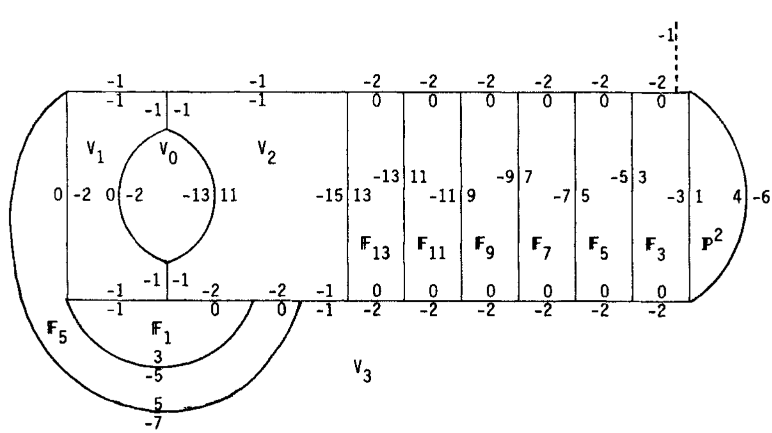
To smooth two of the four remaining $Q=21$ cusps, we may use the same type of degeneration, exploiting the existence of ($\mathbb{P}^2$, line+conic); the cusps $\begin{pmatrix} 3 & 12 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 6 & 9 \\ 0 & 0 \end{pmatrix}$ may be smoothed by using the following X_0's:





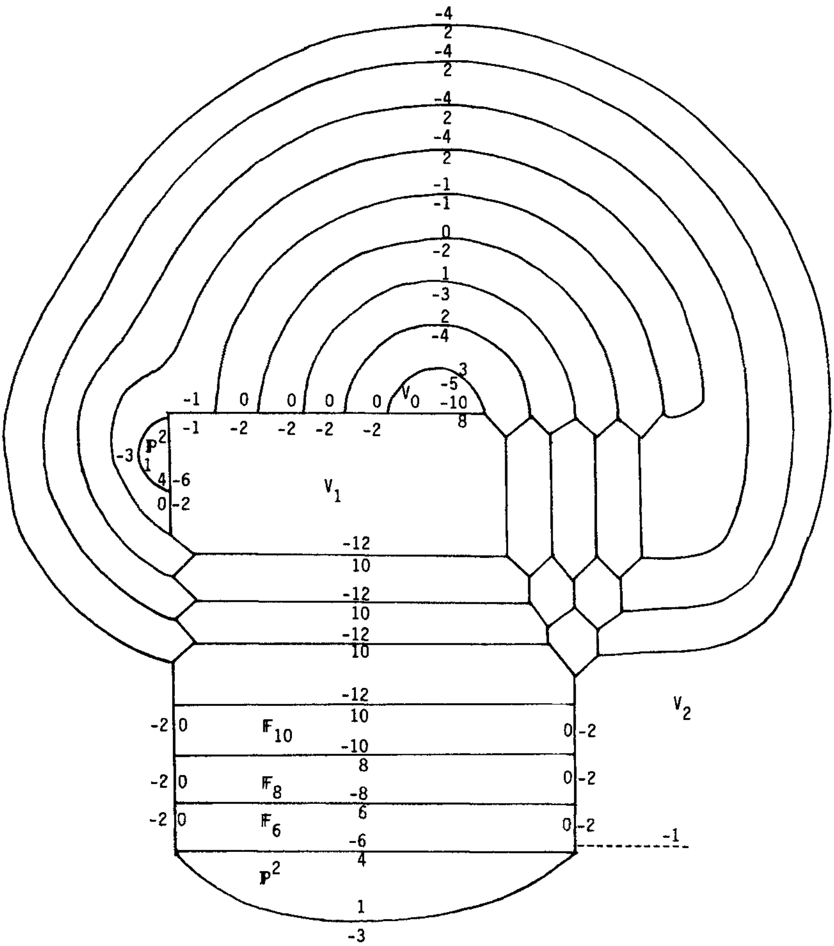
In the $\begin{pmatrix} 3 & 12 \\ 0 & 0 \end{pmatrix}$ case, the surface V_1 is $\mathbb{F}_6$; in the $\begin{pmatrix} 6 & 9 \\ 0 & 0 \end{pmatrix}$ case, the surface V_1 is $\mathbb{F}_3$. Both have $Q=2$.

The last two cases involve much more elaborate special fibers X_0. For the cusp $\begin{pmatrix} 13 \\ 1 \end{pmatrix}$, use the following:

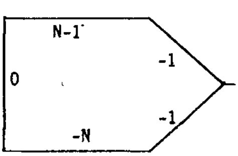


V_0 is the Inoue surface. V_1 is a two-fold blowup of $\mathbb{F}_2$, and V_2 is a four-fold blowup of $\mathbb{F}_{15}$. The other surfaces are as marked, except for V_3; V_3 is a 14-fold blow-up of $\mathbb{F}_6$, where the only exceptional curve which is not a double curve is shown above (meeting the $\mathbb{F}_3$). Here $Q(V_0)=21$, $Q(\mathbb{F}_1)=Q(\mathbb{P}^2)=Q(V_3)=1$, and $Q\,(\text{others})=0$.

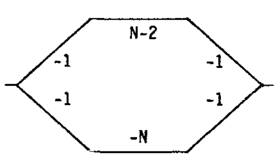
Finally, for the cusp $\begin{pmatrix} 5 & 10 \\ 0 & 0 \end{pmatrix}$, we have



Again V_0 is the Inoue surface. V_1 is an eight-fold blowup of $\mathbb{F}_{10}$; V_2 is an eight-fold blowup of $\mathbb{F}_3$ (with one non-double exceptional curve, shown meeting the $\mathbb{F}_6$). All other surfaces are either $\mathbb{P}^2$, $\mathbb{F}_n$, or have five or six double curves and are of the form



or



All unmarked double curves have self-intersection -1 (on both surfaces). Here $Q(V_0)=21$, $Q(\mathbb{P}^2)=1$, and $Q(V_2)=1$.

We have therefore verified Looijenga's conjecture for cusps of length 1 or 2:

Theorem (5.2). *Let $(\bar{V}, p)$ be the germ of a cusp singularity with resolution D, and assume $r(D) \leqq 2$. If D has a rational dual, then $(\bar{V}, p)$ is smoothable.*

The Smoothings of Certain Cusps of Length Three

Combining this result with a special case of a theorem of Karras [5] and Wahl [10] we can prove the smoothability of all cusps D of length three and $Q \leqq 20$. The result for $r(D) = 3$ is the

Proposition (5.3). *Let $D = D_1 + D_2 + D_3$ be a cusp of length 3. Then D deforms to the cusp C, where $C = C_1 + C_2$ is of length 2, $(C_1^2) = (D_1^2)$, and $(C_2^2) = (D_2^2) + (D_3^2) + 2$.*

Proposition (5.4). *Let D be a cusp of length 3, with $Q(D) = 20$. Then D is smoothable.*

Proof. It suffices to show that D deforms to a smoothable cusp C of length 2. By Proposition (5.3), if $Q(D) = 20$, then $Q(C) = 21$, so that C is smoothable if and only if C is either (2, 13), (3, 12), (5, 10), or (6, 9). Therefore, $D = (d_1, d_2, d_3)$ is smoothable if and only if one of the d_i's is either 2, 3, 5, 6, 9, 10, 12, or 13; a simple check of all triples (d_1, d_2, d_3) with $d_1 + d_2 + d_3 = 17$ [which is equivalent to $Q(D) = 20$] and $d_i \geqq 2$ show that in all cases there is a d_i equal to one of the above numbers. Equivalently, not all the d_i can be chosen from among $\{4, 7, 8, 11\}$ to achieve $d_1 + d_2 + d_3 = 17$. Q.E.D.

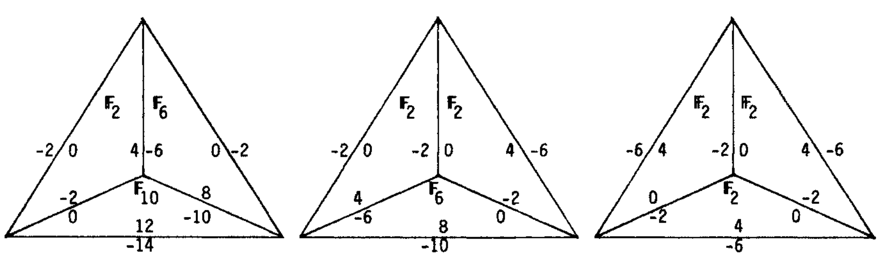
If $Q(D) < 20$, let C be as in (5.3). Then $Q(C) \leqq 20$, so, by (5.2), C is smoothable. Since D deforms to C, D is smoothable as well. Alternatively, we could deduce this from (5.4) and (2.15). Summarizing:

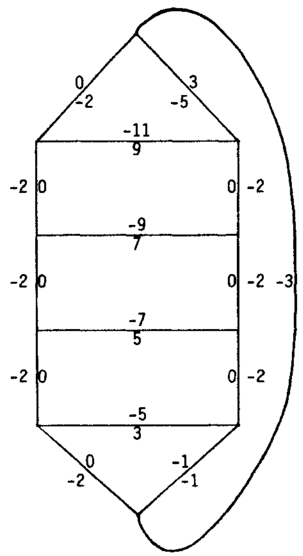
Theorem (5.5). *Let $(\bar{V}, p)$ be the germ of a cusp singularity D, with $r(D) = 3$ and $Q(D) \leqq 20$. Then $(\bar{V}, p)$ is smoothable.*

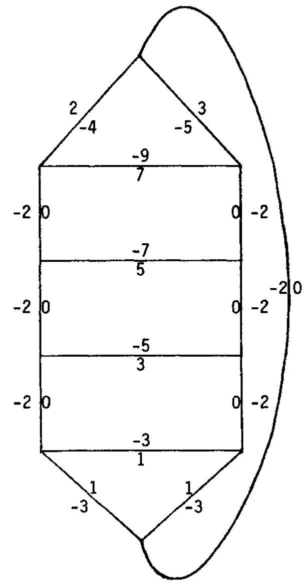
Using the methods of this paper, we have only been able to exhibit smoothings of a few $Q = 21$ cusps of length three.

Proposition (5.6). *The cusps* (2, 2, 14), (2, 6, 10), *and* (6, 6, 6) *are smoothable. The cusps* (2, 5, 11) *and* (4, 5, 9) *are smoothable.*

Proof.





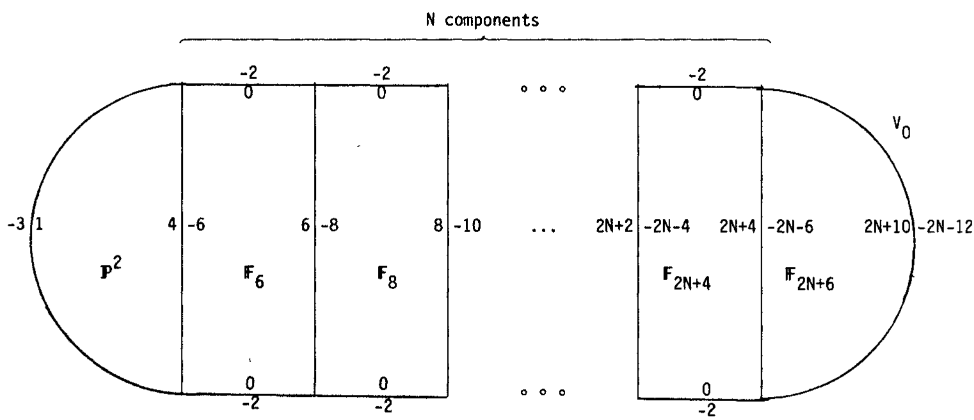


We leave the verification to the reader. Q.E.D.

Although the methods of this paper become quickly intractible if the length of the cusp is large, there are some special fibers X_0 which can be constructed with an arbitrarily large number of components, producing some interesting examples of smoothable cusps. We will just mention one such construction.

Proposition (5.7). *The* $Q=21$ *cusp* $\begin{pmatrix} 3 & 2N+12 \\ N & N \end{pmatrix}$, $N \geqq 0$, *is smoothable.*

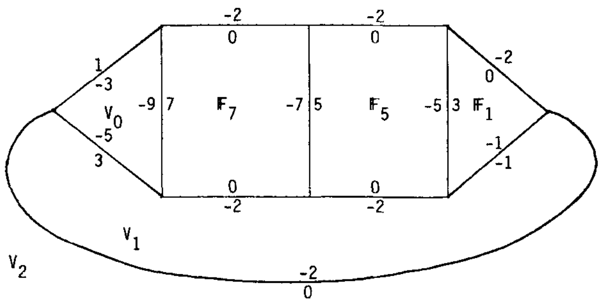
Proof. Use the special fiber



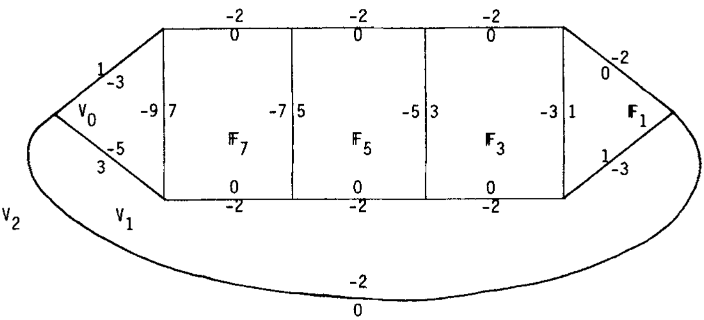
where V_0 is the Inoue surface with the given $\begin{pmatrix} 3 & 2N+12 \\ N & N \end{pmatrix}$ cusp. Here $Q(\mathbb{P}^2)=1$, $Q(\mathbb{F}_{2N+6})=2$, and $Q(V_0)=21$. Q.E.D.

6. Some Concluding Remarks

It is a theorem of Wahl [11] that the dimension of a smoothing component of the versal deformation space of a germ of a cusp singularity D is $22-Q(D)$, if one exists. Hence when $Q(D)\leqq 20$ and D is smoothable, one expects that the different directions of smoothing would imply the existence of distinct combinatorial data producing the special fiber X_0 as in (2.4). We see this empirically in many cases; one example is the cusp $(3,5,9)$ which has $Q=20$, and two different X_0's enhibiting a smoothing, which are not equivalent by base change or birational modifications:



$Q(V_1)=Q(\mathbb{F}_1)=1$
$Q(V_2)=2$
$Q(V_0)=20$



$Q(V_1)=2$
$Q(V_2)=Q(\mathbb{F}_1)=1$
$Q(V_0)=20$

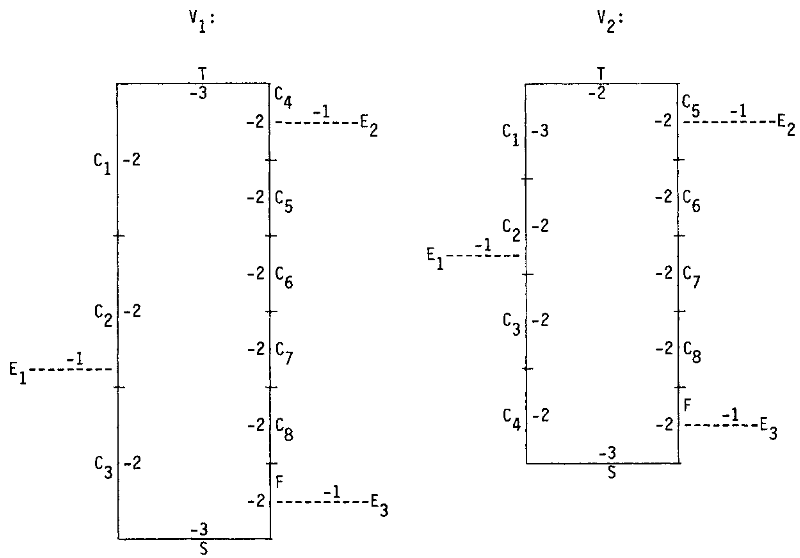
For a $Q=21$ cusp, which is smoothable, each smoothing component is one-dimensional. If there is only one smoothing component, one would expect that all special fibers X_0 exhibiting a smoothing would be base changes of a "primitive" X_0. It can be checked whether a given X_0 is the base change of another special fiber (see [2] for details) and in all cases the special fibers X_0 which we have produced for the smoothing of $Q=21$ cusps in Sect. 5 are *not* base changes of any other X_0.

If a $Q=21$ cusp had more than one smoothing component, one would expect distinct special fibers X_0 exhibiting the different smoothings, which were not related by base change or birational modifications. We have no examples of this phenomenon as yet, but we make the following

Conjecture (6.1). *Let D be a cusp, and D' its dual cusp. Assume $Q(D)=21$. Then the number of smoothing components in the versal deformation space of D is the number of non-isomorphic anticanonical pairs (V, D').*

[Two anticanonical pairs (V_1, D_1) and (V_2, D_2) are isomorphic if there is an isomorphism $f: V_1 \to V_2$ of the surfaces, such that $f^*D_2 = D_1$.]

We know of several examples of this phenomenon. Let D be the (6,9) cusp, whose dual is $D' = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}$. Then D' sits on two different blowups of $\mathbb{F}_3$ as an anticanonical divisor, as depicted below:



In both cases, the Neron-Severi group of V_i is generated by S, E_1, E_2, E_3, C_1 $C_2, \ldots, C_8$, and the rank $NS(V_i) = 12$. Since $r(D') = 11$, and D' is negative definite, there is exactly one primitive vector G_i (up to sign) which is orthogonal to all components of D' in both cases. A computation shows that

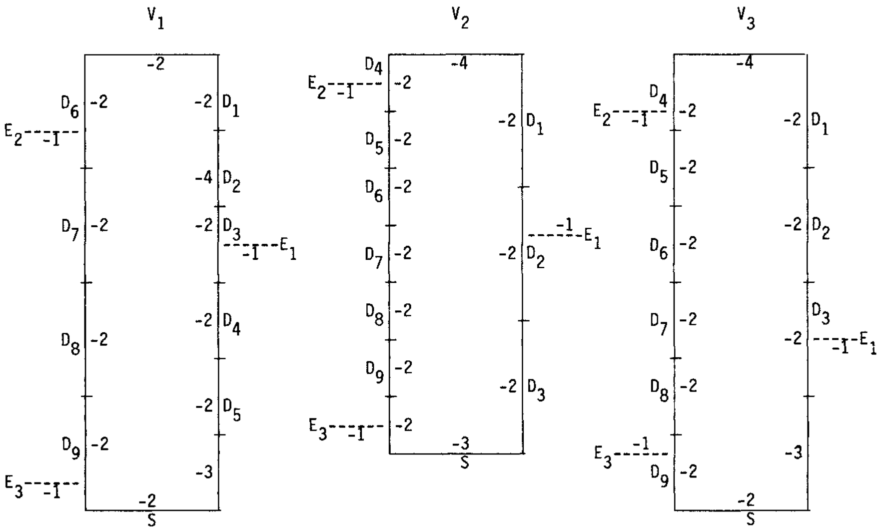
$$G_1 = 2S + 9E_1 - 6E_2 - E_3 + 5C_1 + 10C_2 + 6C_3 - 5C_4 - 4C_5 - 3C_6 - 2C_7 - C_8$$

and

$$G_2 = 12S + 80E_1 - 35E_2 - 5E_3 + 28C_1 + 84C_2 + 60C_3 + 36C_4 - 28C_5 - 21C_6 - 14C_7 - 7C_8.$$

We see that $G_1^2 = 2$ and $G_2^2 = 50$; moreover, if L is the lattice generated by the components of D', we have $|\det L| = 50$. Therefore, if the image of L in $NS(V_i)$ is the sublattice L_i, then L_2 is embedded primitively in V_2, and L_1 is embedded with index 5. (I.e., $L_2 = L_2^{\perp\perp}$ and $[L_1^{\perp\perp} : L_1] = 5$.) In particular, this proves that (V_1, D') is not isomorphic to (V_2, D'). It would be very interesting to find an alternative special fiber X_0 which exhibits a different smoothing of the (6,9) cusp than the one produced in Sect. 5, and also to be able to decide, given the special fiber X_0, which anticanonical pair is being produced as the general fiber. If it is the case that (V_1, D') is the special fiber, then there would be $\mathbb{Z}/5\mathbb{Z}$-torsion in the first homology group of the Milnor of this smoothing.

A more complicated example is afforded by the (2, 6, 10) cusp. In matrix notation, this cusp is $D=\begin{pmatrix}6 & 10\\ 0 & 1\end{pmatrix}$, and its dual is $D'=\begin{pmatrix}3 & 4\\ 7 & 3\end{pmatrix}$, which occurs on three distinct anticanonical pairs (V_i, D'), $i=1,2,3$:



As in the previous example, the lattice L generated by the components of D' has corank one in the Neron-Severi group of V_i in each case. Let G_i be the primitive vector in $NS(V_i)$ orthogonal to L. $NS(V_i)$ is generated by $S, E_1, E_2, E_3, D_1, D_2, \dots, D_9$ for each i (note that these letters refer to different curves, depending on i) and a computation shows that

$$G_1 = 9D_1 + 18D_2 + 63D_3 + 42D_4 + 21D_5 - 9D_6 - 20D_7 - 31D_8 - 42D_9 + 66E_1 + 2E_2 - 32E_3 - 21S,$$

$$G_2 = 30D_1 + 60D_2 + 36D_3 - 30D_4 - 25D_5 - 20D_6 - 15D_7 - 10D_8 - 5D_9 + 54E_1 - 35E_2 - 7E_3 + 12S,$$

and

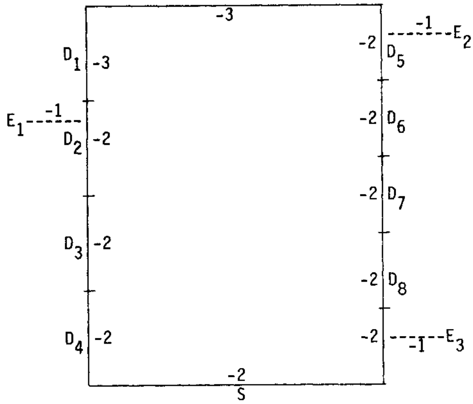
$$G_3 = D_1 + 2D_2 + 3D_3 - D_4 - 2D_5 - 3D_6 - 4D_7 - 5D_8 - 6D_9 + 4E_1 - 4E_3 - 3S.$$

One can see that $G_1^2 = G_2^2 = 100$ and $G_3^2 = 4$; since $|\det L| = 100$, L is embedded primitively in $NS(V_1)$ and in $NS(V_2)$, but has index 5 in $L^{\perp\perp}$ in $NS(V_3)$. In particular, $(V_1, D') \not\cong (V_3, D') \not\cong (V_2, D')$ as anticanonical pairs. To see that $(V_1, D') \not\cong (V_2, D')$, assume that they are isomorphic. Then the curve E_2 on V_1 must also appear as an exceptional curve E on V_2, which must meet the curve D_6 on V_2, once transversally, and be disjoint from the other components of D'. These conditions force E to be

$$E = xG_2 + 10D_1 + 20D_2 + 12D_3 - 10D_4 - 9D_5 - 8D_6 - 6D_7 - 4D_8 - 2D_9 + 18E_1 - 11E_2 - 2E_3 + 4S$$

for some integer x. However, the condition that $E^2 = -1$ is equivalent to $50x^2 + 39x + 7 = 0$, which has no integral solutions. Hence no such E exists on V_2 and $(V_1, D') \not\cong (V_2, D')$.

As a final example, consider the (3, 12) cusp D, whose dual is the cusp $D' = \begin{pmatrix} 3 & 3 \\ 0 & 9 \end{pmatrix}$. There is only one anticanonical pair (V, D') up to isomorphism, shown below:



The classes of S, E_1, E_2, E_3, $D_1, \ldots, D_8$ generate $NS(V)$ and the primitive vector G orthogonal to the lattice L generated by the components of D' is $G = 4D_1 + 12D_2 + 9D_3 + 6D_4 - 4D_5 - 3D_6 - 2D_7 - D_8 + 11E_1 - 5E_2 - 2E_3 + 3S$. Here $G^2 = 2$, and $|\det L| = 32$, so that L has index 4 in $L^{\perp\perp}$ in $NS(V)$, and there is no anticanonical pair (V, D') such that L is embedded primitively in $NS(V)$. In this case, therefore, there is in fact torsion in the first homology of the Milnor fiber, and indeed (as pointed out to us by Wahl) the fundamental group of the Milnor fiber is of order 4.

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References

1. Friedman, R.: Global smoothings of varieties with normal crossings (to appear)
2. Friedman, R.: Base change, automorphisms, and stable reduction of Type III K3 surfaces. In: The birational geometry of degenerations. Progr. Math. **29** (1983) (to appear)
3. Hirzebruch, F.: Hilbert modular surfaces. Ens. Math. **19** 183–281 (1973)
4. Inoue, M.: New surfaces with no meromorphic functions. II. In: Complex analysis and algebraic geometry. Baily, W.L., Shioda, T. (eds.). Cambridge: Cambridge University Press 1977
5. Karras, U.: On pencils of elliptic curves and deformations of minimally elliptic singularities. Math. Ann. **247**, 43–65 (1980)
6. Kulikov, V.: Degenerations of K3 surfaces and Enriques' surfaces. Math. USSR Izv. **2**, 957–989 (1977)
7. Laufer, H.: On minimally elliptic singularities. Am. J. Math. **99**, 1257–1295 (1977)
8. Looijenga, E.: Rational surfaces with an anti-canonical divisor. Ann. Math. **114**, 267–322 (1981)
9. Shepherd-Barron, N.: Extending polarizations on families of K3 surfaces. In: The birational geometry of degenerations. Progr. Math. **29** (1983) (to appear)
10. Wahl, J.: Elliptic deformations of minimally elliptic singularities. Math. Ann. **253**, 241–262 (1980)
11. Wahl, J.: Smoothings of normal surface singularities. Topology **20**, 219–246 (1981)

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